# VII

# Valuation of Derivative Securities

The role of financial intermediaries lies in crafting derivative securities out of cash flows of securities supplied by the business sector of the economy. Generally, the assets acquired by financial institutions have their values determined by established capital markets; if no market value is available, assets can be valued in relation to the existing markets. However, the firm's liabilities become customers' private placement assets and their value must be calculated by the firm. Such calculation lies at the very heart of the actuarial profession. If the treatment of assets and liabilities is to become integrated, we must admit that the actuarial side should yield.

This is, of course, not an entirely new concept. Financial theory views insurance policies as financial instruments that are traded in the market and whose prices reflect supply and demand. Models of insurance taking that perspective are termed "financial pricing models." Borch (1974) and Buhlmann (1980, 1984) recognized early the role of supply and demand in determining price. The financial pricing models discussed in Chapter 1 are generally either *equilibrium models,* or *arbitrage models.*

The equilibrium approach looks at agents in the economy. They are assumed to be rational wealth maximizers, trading in the existing financial securities markets and being subject to their resource constraints. Equilibrium in the economy is obtained when no agent has any more incentive to trade, and the market clears. If agents' preferences can be modeled, one can derive prices of claims to cash flows. Both the Modern Portfolio Theory (Markowitz 1952, 1959) and the Capital Asset Pricing Model (developed in the 1960s by William Sharpe and, independently, by John Lintner and Jan Mossin) as well as their refinements, are equilibrium models.

From the perspective of this paper, however, the Arbitrage Pricing Model developed by Ross (1976) is of greater interest. Consider a very simple model of financial uncertainty, with a world of two periods: now and the future. We know the state of the world now, but tomorrow is uncertain, with $S$ being the set of possible states of the world. Recall the definition of an Arrow-Debreu security, and let $p_s$ with $s \in S$ be the price of an Arrow-Debreu security paying \$1 exactly in the state $s$. Consider $N$ arbitrary assets and $S$ portfolios constructed from these assets. Let $b_{ij}$ be the number of shares (units) of asset $i$, where $i = 1, 2, \ldots, N$ in a portfolio $j$, where $j = 1, 2, \ldots, S$, and $d_{ji}$ be the value of asset $i$, $i = 1, 2, \ldots, N$, in the state of the world $j$, $j = 1, 2, \ldots, S$. Consider the matrices $D = [d_{ji}]_{j=1,2,\ldots,S;i=1,2,\ldots,N}$, and $B = [b_{ij}]_{i=1,2,\ldots,N;j=1,2,\ldots,S}$, and let $I_{N\times N}$ be the unit $N \times N$ matrix. The market of these securities is said to be *complete* if it is possible to find $S$ portfolios, as defined above, such that:

$$D \cdot B = I_{N\times N}, \tag{7.1}$$

where the multiplication is the matrix multiplication. There must be exactly $S$ linearly independent assets in a solution. Such assets are called *primitive assets* or *primitive securities* (with any other assets termed *redundant*), and the market is complete if, and only if, it contains a set of primitive assets that can be combined into a set of portfolios replicating the Arrow-Debreu securities (and, effectively, Arrow-Debreu securities are available for trading). If we do have exactly $S$ primitive securities, then $D$ is a square matrix and $B = D^{-1}$. The price of an asset is a linear operator and the prices $p_s$ of Arrow-Debreu securities are given; thus, the vector $D^{-1} \cdot \vec{p}$ where $\vec{p} = [p_s]_{s=1,2,\ldots S}$, gives the prices of primitive securities. If the markets are complete, then a portfolio consisting of one of each Arrow-Debreu securities can be constructed, and it has a payoff independent of future states and the world. It is called a *riskless security.* Its price today is $\nu = p_1 + p_2 + \cdots + p_s$, and $r = \nu^{-1} - 1$ is the one-period riskless rate of return (with force of interest $\delta$ defined by $e^{\delta} = 1 + r$ used for continuous compounding).

As we can see, existence of prices of Arrow-Debreu securities (also termed *state contingent prices*) is essential for this methodology. The general equilibrium approach would have the state contingent prices determined by the model. Otherwise, they are taken as given, and such a framework is termed the *partial equilibrium model.* This work is not concerned with any particular model, or with model building, but rather with the implications of financial pricing of capital assets for the ALM of an insurance firm. Babbel and Merrill (1996) include an extensive analysis of applicable models.

No-arbitrage pricing theory is a partial equilibrium methodology. It assumes the Arrow-Debreu securities prices as given, and complete markets. Consider a one-period market with $S + 1$ assets, and $S$ future states of the world. Given the completeness of the market, one of the assets, say the $(S + 1)$-st, is redundant, and it can be replicated by a portfolio of the other assets, which can be then written as vectors $\vec{\nu}_1 = [b_{1i}]_{i=1,2,\ldots,S}, \ldots, \vec{\nu}_s = [b_{Si}]_{i=1,2,\ldots,S}$ (same notation as above).

Let $D$ the matrix with the same meaning as above (i.e., its columns are the payoffs of the primitive assets in corresponding states), and let $\vec{x} = [x_i]_{i=1,2,\ldots,s}$ be the vector defining the portfolio of primitive assets that replicates the $(S + 1)$-st asset, which can also be written as $\vec{\nu}_{S+1}$. Then

$$\vec{\nu}_{S+1} = x_1\vec{\nu}_1 + \cdots + x_S\vec{\nu}_S = \vec{x}^T \cdot D^{-1} \qquad (7.2)$$

(in terms of future payoffs). Since the price is a linear operator and the prices of primitive securities are given by $D^{-1} \cdot \vec{p}$, the price of the $(S + 1)$-st asset is

$$\vec{x}^T \cdot D^{-1} \cdot \vec{p}. \qquad (7.3)$$

Note that prices of all securities are linear combinations of prices of Arrow-Debreu securities. A more powerful statement can be made. In a complete market with no trading costs, no consideration for taxes, and a finite number of securities, the *Principle of No Arbitrage* (also called the *Principle of One Price*) states that any portfolio $\vec{x} = [x]_{i=1,2,\ldots,s}$ for which the future payoffs vector $\vec{x}^T - D^{-1}$ contains only nonnegative entries, and at least one positive entry, must have a positive price. This means that there are no "free lunches": a security that produces a future payoff requires a cash outlay for its purchase.

*The Fundamental Theorem of Asset Pricing* as discussed in Dybvig and Ross (1987, 1989) and Panjer (1998) states that the Principle of No Arbitrage is equivalent to the price of a security (in a finite market as above) with payoffs $d_1, d_2, \ldots, d_s$ in the states of the world $1, 2, \ldots, S,$ respectively, being given by the expression $d_1p_1 + d_2p_2 + \cdots + d_sp_s$, where $p_1, p_2, \ldots, p_s$ are the prices of Arrow-Debreu securities. Duffie (1996) provides a proof (not restricted to the finite space as presented here) and magnificent insight into the modern dynamic asset valuation theory.

Consider now an amount of a riskless security whose price today is 1. Because price is a linear operator, its value at the end of the period is $1/\nu = e^\delta = 1 + r$. Denote the quantity $p_s(1 + r)$ by $\theta_s$. Then

$$\sum_{s=1}^{S} \theta_s = 1, \qquad (7.4)$$

and the quantities $\theta_s$ can be regarded as probabilities of states $s \in S$. They are called *arbitrage probabilities, risk-neutral probabilities,* or *martingale probabilities.* Given this definition, the price of the security is

$$\nu(d_1\theta_1 + d_2\theta_2 + \cdots + d_s\theta_s), \qquad (7.5)$$

which can be interpreted as the expected present value, or *actuarial present value,* of the future payout of the security. The probabilities so produced should not be misconstrued as actual probabilities of the future states of the world. They represent probabilities in an abstract construct of a *risk-neutral world.* In general, without strong assumptions such as in this discussion (we assumed a finite, complete, frictionless market), they are not unique.

This statement can be generalized. Recall (see, e.g., Ross, 1996) that a *stochastic process* is a collection of random variables $\{X(t){:}t \in T\}$, with the index set $T$ typically referring to time (it may be a continuous time interval or a discrete set of, e.g., nonnegative integers).

A stochastic process $\{X(t){:}t \geq 0\}$ is said to have *stationary increments* if for any $s < t$ and $u > 0$, the probability distribution of $X(t - X(s)$ is the same as the probability distribution of $X(t + u) - X(s + u)$. A process $\{X(t){:}t \geq 0\}$ is said to have *independent increments* if for any $t_1 < t_2 \leq t_3 < t_4$ the random variables $X(t_2) - X(t_1)$ and $X(t_4) - X(t_3)$ are independent. A *filtration* in a probability space $\{\Omega, F, P\}$ (where $\Omega$ is the sample space, $F$ is the sigma-algebra of events, and $P$ is the probability measure) is a collection of sigma-algebras $\{F_t\}$ such that for $s < t$, $F_s \subset F_t$, $F$ is the $\sigma$-algebra generated by $\cup_{t\geq 0} F_t$, $F_t = \cap_{s>t} F_s$, and all sets of probability zero belong to $F_0$. A *martingale* is a stochastic process $\{X(t){:}t \geq 0\}$ such that:

(i) Each $X(t)$ is a random variable on the probability space $\{\Omega, F_t, P\}$.
(ii) $\{F_t\}$ is a filtration.
(iii) $E(X(t) \mid X(s), s < t) = X(s)$.

Condition (iii) is the key one. It says that the best guess (as represented by expected value) of the future value of the process is its current value. Since the other parts of the definition will hold for the processes discussed here automatically, for all practical purposes, one only needs to check condition (iii) to be satisfied that we do have a martingale to deal with. If the index set $T$ is the set of nonnegative integers, the condition (iii) can be simplified to say:

$$E(Z_{n+1} \mid Z_0, Z_1, \ldots, Z_n) = Z_n. \tag{7.6}$$

This is a particularly simple form of the definition of a martingale applicable to time-discrete processes (Ross 1996).

Let $S(t)$ be the value of a security at time $t > 0$, where $S(t)$ is assumed to be a stochastic process, and let the risk-free continuously compounded rate of interest be $\delta$. The generalized Fundamental Theorem of Asset Pricing says that, in a complete, frictionless (i.e., no transaction costs, no taxes, and no other impediments to trade) market, the absence of arbitrage is equivalent to the existence of a probability distribution such that, with respect to that probability distribution, $\{e^{-\delta t}S(t)\}$ is a *martingale*; that is, for each $u < t$, $E(e^{(u-t)\delta}S(t)) = S(u)$, and, in particular, $E(e^{-t\delta}S(t)) = S(0)$.

Harrison and Kreps (1979) provide the proof of this version of the Fundamental Theorem of Asset Pricing (see also Harrison and Pliska 1981), while Schachermeyer (1992) gives an elegant and self-contained proof of this in finite discrete time, using the Hilbert space methodology of functional analysis. Note that the effective meaning of this theorem is that, as Samuelson (1965) anticipated it: "properly anticipated prices fluctuate randomly."

To have a better perspective on the meaning of the above statements, consider a finite-state discrete-time security market model, with a complete and frictionless market and trades occurring only at the times 0, 1, 2, . . . . Let the risk-free interest rate at the time $t$ be $i(t)$, so that a riskless unit security bought at the time $t$ is worth $1 + i(t)$ at the time $t + 1$, regardless of the future state of the world. Let $V_j(t)$ be the value of the $j$-th primitive security at time $t$, and let $D_j(t)$ be the cash flow for that primitive security at time $t$ (with $V_j(t)$ being the ex-cash-flow, e.g., ex-dividend, value). Then $V_j(t)$ and $D_j(t)$ are random variables as seen from any time $s < t$. The Principle of No Arbitrage (cf. Panjer 1998, pp. 107–9) is equivalent to the existence of a probability measure under which

$$V_j(t) = E_t\left(\frac{1}{1 + i(t)}\,(V_j(t + 1) + D_j(t + 1))\right), \tag{7.7}$$

where $E_t$ denotes the expected value with respect to the aforementioned probability measure, and its subscript indicates that the expected value is taken with respect to information available at the time $t$. The probability measure is again called a *risk-neutral probability measure.* Equation (7.7) implies that, in general, the value at time 0 of a stochastic cash flow stream $\{D(t){:}t = 1, 2, \ldots\}$ is

$$E\left(\sum_{t=1}^{\infty} \frac{D(t + 1)}{(1 + i(0))(1 + i(1)) \cdots (1 + i(t))}\right). \tag{7.8}$$

Note that this form of valuation takes into account the contingent form of payments and, if values of primitive securities are known, can be used for pricing of various derivative instruments, such as options, mortgage-backed securities, and derivatives embedded in insurance policies. Furthermore, if $D(t) = 1$ for $t = t_n$, and 0 for all other values of $t$, then Equation (7.8) gives the valuation of the noncallable default-free zero-coupon bond maturing at the time $t = t_n$. Since such zero-coupon bonds are traded in the existing markets, and their values are given as $(1 + i_n)^{-1}$, with $i_n$ denoting the spot rate for $t = t_n$, we can then consider the expression (7.8) to be a function of various spot rates and derive the familiar duration and convexity measures as discussed in Chapters 2 and 3. If $V(\{i_n\})$ is (7.8) written as a function of spot rates, then for a small value of $h$,

$$\frac{V(\{i_n\}) - V(\{i_n + h\})}{hV(\{i_n\})}, \tag{7.9}$$

and

$$\frac{V(\{i_n + h\}) - 2V(\{i_n\}) + V(\{i_n - h\})}{h^2V(\{i_n\})} \tag{7.10}$$

are measures of duration and convexity, respectively, with respect to parallel shifts of the spot curve. They are called *option-adjusted duration* and *convexity,* because the valuation of the security considered takes into account all contingent claims (i.e., options and the like) embedded in it.

One can also write Equation (7.8) as

$$\sum_{\omega\in\Omega} \Pr(\omega) \left(\sum_{t=1}^{\infty} \frac{D(t + 1, \omega)}{(1 + i(0))(1 + i(1, \omega)) \cdots (1 + i(t, \omega))}\right), \tag{7.11}$$

where the events $\omega$ of the probability space $\Omega$ are specified, and each is identified with an interest rate *path,* or *scenario* $\{i(0), i(1, \omega), \ldots, i(t, \omega), \ldots\}$, and the cash flows along that path are $\{D(1, \omega), D(2, \omega),$

. . . , $D(t, \omega)$, . . .}. Tilley (1992) reviews the process of generating stochastic interest rate scenarios.

Formula (7.11) is also the basis for the widely used methodology for the valuation of various assets, especially fixed-income securities with implicit or explicit derivatives, such as mortgage-backed securities, caps and floors, options, swaps, etc. In practice, it is not possible to produce all elements of the probability space $\Omega$ for this type of valuation, and an appropriate sample of interest rate scenarios is chosen. This raises two important issues in producing such samples. First, if the scenarios used for the calculation are only a random sample from $\Omega$, then Equation (7.11) is merely a *point estimator* of the expected value sought and, as such, a random variable itself. Estimation of the probability that Equation (7.11) is sufficiently close to the actual value becomes an additional, and challenging, task.

Second, the probability space $\Omega$ is the *risk-neutral* space. Just producing some interest rate scenarios will not assure us that they are indeed members of that space. Given that this is an arbitrage-free valuation, we should be assured that no arbitrage (i.e., no riskless profit) is available in the sample of interest rate paths. Tilley (1992) discusses a procedure for assuring that. An alternative is possible in practice. If we produce a set of interest-rate paths $\Xi$, with $K$ being the number of its elements, then we can set

$$\text{Market Price of the Security} = \sum_{\omega\in\Xi}\frac{1}{K}\left(\sum_{t=1}^{\infty}\frac{D(t+1,\omega)}{(1+i(0)+s)(1+i(1,\omega)+s)\cdots(1+i(t,\omega)+s)}\right), \tag{7.12}$$

where $s$ is a "fudge factor" to attain the equality in Equation (7.12), and the market value of the security is assumed to be given (so this approach cannot be used for insurance liabilities, our main area of interest, but it can be used for marketable assets, and then for comparison with liabilities of similar structure). This "fudge factor" is called the *option-adjusted spread* (OAS) of the security. OAS is widely used in the investment area for the purpose of comparing relative value of securities—those assets offering higher option-adjusted spread are believed to be a better value.

Observe that, if the scenarios and cash flows produced for them properly account for credit risk, sensitivities of the cash flows to interest rates, liquidity of the security, and other derivatives embedded in the security, then the OAS adjustment should, theoretically, account only for the difference between the *risk-neutral world* and the interest rate path sample used. Therefore, the strategy of buying high OAS, given that diversifiable risk of the security has been accounted for, appears to be merely a strategy of buying assets with higher nondiversifiable risk. This is a rational strategy, if the risk is paid for, but it does not appear to be based on value. Of course, this reasoning is based on the efficiency of the market; proponents of value investing are generally nonbelievers in this area. However, from the perspective of an efficiency believer, value investing is always rewarded merely for taking on extra nondiversifiable risk. The pitfalls of option-adjusted spread analysis are very thoroughly addressed in the work of Babbel and Zenios (1992).

While our analysis implies that practical valuation of securities, including derivatives created by insurance firms, calls for the use of interest rate scenarios and some form of the formula (7.11), we should firmly keep in mind various technical assumptions underlying it, and that variations on these assumptions may force us to reexamine the methodology. Fitton and McNatt (1996) provide an excellent discussion of this issue. They distinguish between arbitrage-free modeling and equilibrium modeling, and separately between a risk-neutral probability measure and a realistic probability measure.

Recall that arbitrage-free pricing theory takes the prices of Arrow-Debreu securities as given, in particular the existing yield curve, and then complements them with some process of evolution of the yield curve, which serves to produce a random sample from the probability space under consideration. Such a process, however, is bound by adherence to existing market prices in order to be arbitrage-free. Thus, it does not attempt to emulate any dynamics of the yield curve and its underlying economic processes (such emulation is a defining characteristic of equilibrium models). The question of whether the reality is *perfectly* arbitrage-free may not have an affirmative answer because of factors such as trading costs, taxes, and other real-life considerations. In contrast, equilibrium models begin with some general idea of the interest rate process and accept existing market prices as possibly subject to statistical or other types of error.

The distinction between risk-neutral probabilities and realistic probabilities is also important. As pointed out previously, risk-neutral probabilities should not be misconstrued to mean realistic probabilities in some form existing in the markets. As shown by Fitton and McNatt (1996), the risk-neutral probability distribution results in a yield curve in which the spot for every term is equal to the expected return from investing at the short rate over the same term. In other words, term premium, which indeed is *risk premium* in the real world, does not exist in the risk-neutral world. However, valuing a security under a realistic probability distribution would require application of risk premium for additional (later) discount factors.

Fitton and McNatt (1996) also discuss the relevance of various combinations of the pairs: (1) arbitrage-free models/equilibrium models and (2) risk-neutral probability distribution/realistic probability distribution. They point out that arbitrage-free models lead, by definition, to risk-neutral valuation, and that the combination of a realistic probability distribution with an arbitrage-free model is of no value. However, the other combinations do possess special features of interest:

- If reliable market data is available, the pricing of existing securities, be it on the assets or liabilities balance sheet, should be done with arbitrage-free models using risk-neutral probabilities.
- If market data is unreliable or unavailable for current pricing, or in horizon pricing (where risk preferences may be of significance), one can combine an equilibrium model providing reasonable estimates for market prices with risk-neutral probabilities.
- However, in stress testing, reserve testing, or modeling of borrower behavior (e.g., mortgage-backed securities), where realistic risk preferences and term premium for the yield curve acquire significance, the authors suggest using equilibrium models with a realistic probabilitydistribution.

Fitton and McNatt (1996) also provide an interesting perspective on the insurance firm management process in their conclusions. Insurance policies features that contains derivative securities with respect to existing market securities, such as indexing provisions, minimum interest rate guarantees, crediting strategies for deferred annuities, must be priced using the arbitrage-free risk-neutral methodology. Even if the firm intended to charge more for these than their existing or derived market prices (assuming, of course, that the firm could do so in the face of competition), knowledge of the market price is a prerequisite to any product strategy. However, the insurance firm is also bound by regulatory requirements that impose stress-type constraints. Here, an analysis based on an equilibrium model with realistic probabilities seems most appropriate.

The firm may also find itself holding a portfolio of assets for which no market exists, for example, private issues. A combination of an equilibrium model with risk-neutral probabilities is the methodology of choice in this case.

Last, derivatives created by an insurer may be based on risks faced by the insured, which are nondiversifiable only to the insured but diversifiable per se. This is the area of most standard actuarial methodology but, given that securities issued for such a purpose (for example, life insurance policies) typically contain additional securities with market risks, such as interest rate guarantees, one must bring the modern financial methodology into the valuation.

Note that the modern era has brought with it certain insurance policies that contain only, or almost only, nondiversifiable market risk and, as such, very much resemble standard derivatives, for example, options and futures. Single Premium Deferred Annuity, as well as Guaranteed Investment Contract, are insurance-company-created, but marketable derivatives. As financial theory indicates, they should be valued appropriately.

Embrechts (1996) provides insight into the integration of actuarial and financial pricing of insurance. The classical actuarial approach views the loss as a random variable $X$, and derives the premium $\Theta$ based on one of the following principles:

- The expectation (expected value) principle: $\Theta = E(X) + \delta E(X)$.
- The variance principle: $\Theta = E(X) + \delta Var(X)$.
- The standard deviation principle: $\Theta = E(X) + \delta\sqrt{Var(X)}$.
- The semi-variance principle: $\Theta = E(X) + \delta E(\max(X - E(X), 0))$.
- The exponential principle: $\Theta = 1/\delta \ln(E(e^{\delta X}))$.
- The $(1 - \delta)$-quantile principle: $\Theta$ is the number such that $\Pr(X > \Theta) = \delta$.
- The Esscher principle: $\Theta = E(Xe^{\delta X})/E(e^{\delta X})$.

In all of the above, $\delta$ is a number chosen to meet a certain solvency margin, generally derived from a ruin estimate based on the distribution of the loss process. Embrechts (1996) proceeds to point out that the above approaches can all be associated with a financial pricing model. He also quotes from the standard actuarial mathematics text by Bowers et al. (1986), indicating the emergence of arbitrage-free pricing in a competitive insurance market:

> "In a competitive economy, market forces will encourage insurers to price short-term policies so that deviations of experience from expected value will behave as independent random variables. Deviations should exhibit no pattern that might be exploited by the insured or insurer to produce consistent gains. Such consistent deviations would indicate inefficiencies in the insurance market."

The key features of markets in which the models developed here apply are: completeness (i.e., availability of Arrow-Debreu securities for trading), frictionless trading (i.e., lack of barriers due to transaction costs, taxes, etc.), and absence of arbitrage (in insurance, this means markets competitive enough in terms of initial purchases, reinsurance, and assumption). If these conditions are met, financial pricing of insurance does become a reality. To the degree that certain portions of insurance policies have these features, financial pricing of them is a necessary condition for the insurance firm's existence. Given the unrelenting evolution of insurance markets towards meeting these conditions, it is easy to see why such dramatic changes have been happening in our field in the last 20 years.

Delbaen and Haezendock (1989; also see refinements by Sondermann 1991 and Embrechts 1996) analyze pricing of reinsurance in arbitrage-free markets. They begin with the standard risk process of the form:

$$X(N_t) = \sum_{k=1}^{N_t} X_k,\ 0 \le t \le T, \tag{7.13}$$

where the random variables $X_k$ are independent and identically distributed with the cumulative distribution function $F$, and $\{N_t\}$ is a homogeneous Poisson process (Ross 1996 and Bowers et al. 1986) with intensity $\lambda > 0$, so that we have:

$$N_t = \sup\{n \in N{:}T_1 + T_2 + \cdots + T_n \le t\}, \tag{7.14}$$

where the random variables $T_i$ are independent and identically distributed with the exponential distribution with parameter $\lambda$, and $T_i$ refers to the occurrence time of the $i$-th claim of size $X_i$. The processes $\{T_i\}$ and $\{X_i\}$ are assumed to be independent. Then $X(N_t)$ is a compound Poisson process (Bowers et al. 1986) whose cumulative distribution function has the form:

$$\sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \Pr(X_1 + X_2 + \cdots + X_k \le x). \tag{7.15}$$

If at this point one assumes that at each time $t$ the company can sell the remaining risk of the period $(t, T]$ for a premium $\Theta_t$, then the underlying price process has the form $S_t = \Theta_t + X(N_t)$. If the market for buying and selling the remaining risk meets the conditions of completeness, lack of friction and arbitrage, then we can conclude from the work of Harrison and Kreps (1979) that there exists a risk-neutral probability distribution $Q$ such that $\{e^{-\delta t}S(t)\}$ is a martingale with respect to $Q$. Furthermore, if it is assumed that $\Theta_t = \theta(T - t)$, where $\theta$ is a *premium density,* then it can be shown that, for certain $Q$, $\{X(N_t)\}$ remains a Poisson process. If the premium density is of the form $\theta_Q = E^Q(X(N_1)) = E^Q(X_1)E^Q(N_1)$, then the distribution $Q$ that preserves Poisson process has the form:

$$F_Q^{(\beta)}(x) = \frac{1}{E(e^{\beta(X_1)})} \int_0^x e^{\beta(y)}\, dF(y),\ x \ge 0 \tag{7.16}$$

where $\beta{:}[0, +\infty) \to R$ is a function such that $E(e^{\beta(X_1)})$ and $E(X_1 e^{\beta(X_1)})$ exist. Special choices of the function $\beta$ give rise to very familiar results. If $\beta$ is constant, $E^Q(N_1) = e^{\beta}\lambda$ and $E^Q(X_1) = E(X_1)$, which follows the expected value principle. If $\beta(x) = \ln(a + bx)$, with $b > 0$ and $a = 1 - bE(X_1) > 0$, then $E^Q(N_1) = \lambda$ and $E^Q(X_1) = E(X_1) + bVar(X_1)$, following the variance principle. Finally, if $\beta(x) = ax - \ln(e^{aX_1})$, with $a > 0$, then $E^Q(N_1) = \lambda$ and $E^Q(X_1) = E(X_1 e^{aX_1})/E(e^{aX_1})$, following the Esscher principle.

The *Esscher Transform* (Esscher 1932), in which the Esscher principle originates, gives rise to a novel perspective on arbitrage-free pricing developed by Gerber and Shiu (1994, 1996). Let $Y$ be a random variable, and let $h$ be a real number such that the expected value $E(e^{hY})$ exists. The random variable $e^{hY}/E(e^{hY})$ can be used (via the Radon-Nikodym derivative) to define a new probability measure. If $\psi$ is a measurable function, then with respect to this measure we have

$$E(\psi(Y);\, h) = \frac{E(\psi(Y)e^{hY})}{E(e^{hY})}. \tag{7.17}$$

Gerber and Shiu (1994, 1996) call this new measure the Esscher measure of parameter $h$. The probability distribution so obtained is called the Esscher Transform (or *exponential tilting* in some statistical literature).

Assume that a stochastic process $\{X(t{:}t \ge 0\}$ has independent and stationary increments, and additionally that $X(t) - \delta t$ assumes both positive and negative values, where $\delta$ is the risk-free force of interest. Consider

$$S(t) = S(0)e^{X(t)} \tag{7.18}$$

for $t \ge 0$. Samuelson (1965) and Parkinson (1977) provide a justification for Equation (7.18) being a model of stock prices. Assume also that the moment generating function

$$M(h, t) = E(e^{hX(t)}) \tag{7.19}$$

of $X(t)$ exists and that

$$M(h, t) = M(h, 1)^t. \tag{7.20}$$

Note that the condition (7.20) is actually not an assumption; it follows from the fact that $\{X(t{:}t \geq 0\}$ has independent and stationary increments. The process

$$\left\{\frac{e^{hX(t)}}{M(h,\,t)}\right\} = \{e^{hX(t)}M(h,\,1)^{-t}\} \tag{7.21}$$

is a positive (i.e., it takes on nonnegative values with probability one) martingale, and it can be used to define a change of probability measure, the so called *Esscher transform measure* of parameter $h$. The *risk-neutral Esscher measure* is the Esscher measure of parameter $h = h^*$, such that the process

$$\{e^{-\delta t}S(t)\} = \{S(0)e^{X(t)-\delta t}\} \tag{7.22}$$

is a martingale with respect to it.

As mentioned before, the risk-neutral probability measure producing arbitrage-free valuation is not unique. However, the Esscher Tranform, if it exists, allows for the creation of a unique pricing mechanism. Gerber and Shiu (1994, sect. 3.1) show that the classical Black-Scholes option pricing formula (Black and Scholes 1973) can be derived with the use of Esscher Transform. This is based on the reasoning given below, illustrating the power of the Esscher transform methodology.

Write $E(Y;\,h^*)$ for the expected value of the random variable $Y$ with respect to the Esscher transform measure. The martingale property implies that

$$E(e^{-\delta t}S(t);\,h^*) = e^{-\delta 0}S(0) = S(0). \tag{7.23}$$

This, in turn, gives

$$\begin{aligned} e^{\delta t} &= E(e^{X(t)};\,h^*) = \frac{E(e^{X(t)}e^{h^*X(t)})}{E(e^{h^*X(t)})} \\ &= \frac{M(h^*+1,\,t)}{M(h^*,\,t)} = \left(\frac{M(h^*+1,\,1)}{M(h^*,\,1)}\right)^t, \end{aligned} \tag{7.24}$$

or, more simply,

$$e^{\delta} = \frac{M(h^*+1,\,1)}{M(h^*,\,1)}. \tag{7.25}$$

This shows the parameter $h^*$ to be unique, although as indicated before, there may be other equivalent martingale measures. Note that

$$\frac{e^{hX(t)}}{M(h,\,1)^t} = \frac{e^{hX(t)}}{E(e^{hX(t)})} = \frac{S(t)^h}{E(S(t)^h)} \tag{7.26}$$

and, therefore, for any measurable function $g$

$$\begin{aligned} &E(S(t)^k g(S(t));\,h) \\ &\quad= \frac{E(S(t)^{h+k}g(S(t)))}{E(S(t)^h)} \\ &\quad= \frac{E(S(t)^{h+k})}{E(S(t)^h)}\,\frac{E(S(t)^{h+k}g(S(t)))}{E(S(t)^{h+k})} \\ &\quad= E(S(t)^k;\,h)E(g(S(t));\,h+k). \end{aligned} \tag{7.27}$$

Gerber and Shiu (1996) refer to Equation (7.27) as the *factorization formula*. This formula turns out to be an extremely useful tool in simplifying analysis of financial instruments with the use of the Esscher Transform. For example, let $k = 1$, $h = h^*$, and

$$g(x) = \begin{cases} 1 & \textit{if} \quad x > K \\ 0 & \textit{if} \quad x \leq K \end{cases}. \tag{7.28}$$

Note that $g(x)$ is the characteristic function of the set $\{x{:}x > K\}$. Then (7.27) implies:

$$\begin{aligned} &E(e^{-\delta\tau}\max(S(\tau) - K,0);\,h^*) \\ &\quad= E(e^{-\delta\tau}(S(\tau) - K)g(S(\tau));\,h^*) \\ &\quad= e^{-\delta\tau}(E(S(\tau)g(S(\tau));\,h^*) - KE(g(S(\tau));\,h^*)) \\ &\quad= S(0)\,\Pr(S(\tau) > K;\,h^*+1) \\ &\qquad - Ke^{-\delta\tau}\Pr(S(\tau) > K;\,h^*) \end{aligned} \tag{7.29}$$

For $X(t)$ being the classical Wiener process (stationary and independent increments with $X(t) - X(0) \sim N(0,\sigma^2 t))$, Equation (7.29) becomes the Black-Scholes (1973) formula for the price of a European call option on a nondividend paying stock

$$C = S(0)N(d_1) - Ke^{-\delta\tau}N(d_2) \tag{7.30}$$

with

$$d_1 = \frac{\ln\left(\dfrac{S(0)}{Ke^{-\delta\tau}}\right) + \dfrac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}} \text{ and}$$

$$d_2 = \frac{\ln\left(\dfrac{S(0)}{Ke^{-\delta\tau}}\right) - \dfrac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}$$

The Escher transform provides one of the ways to derive the Black-Scholes formula, this celebrated achievement of modern financial theory, via the the no-arbitrage pricing methodology.