I

Bounds on the Price of Catastrophe Insurance Options on Futures Contracts

by Patrick L. Brockett, Samuel H. Cox, and James Smith

Abstract

Insurers can hedge the risk of assets and interest sensitive liabilities with readily available financial instruments. Similarly, certain types of loss exposures which are strongly correlated with the valuation of special commodities (for example, default risk in mortgage insurance in Houston, Texas is negatively related to the price of oil, etc.) can also be hedged by utilizing the commodity futures market. Until recently, however, the management of other underwriting risks has primarily been limited to reinsurance contracts. In 1992 the Chicago Board of Trade (CBOT) introduced insurance futures contracts based on the loss ratio of a portfolio of actual insurance policies over a given period of time. This is conceptually similar to more commonly traded stock index futures, with the exception that there is no intrinsic underlying tradable asset upon which to base no arbitrage pricing, and there is only partial information available about the dynamics of the underlying loss ratio series. This paper describes methods for determining bounds on the price of derivative instruments, such as options on insurance futures, given only market information.

Introduction

The first recorded use of derivative contracts to insurable risk was an option contract covering grain bound for Venice in the seventh century (Ferrick, Faber, and Dumas Limited [1994]). The CBOT began trading in grain futures in 1840, and more recently (since 1972), the Chicago Mercantile Exchange and many other exchanges throughout the world trade in currency futures and other financial derivatives. The rapid growth of the use of financial futures and options (and other financial derivative products such as interest swaps, look back options, compound options, etc.) has shown a development from the arena of *product* risk management (for example, bonds, swaps, currency futures and options, etc.) to an arena of *exposure* risk management (for example, interest rate, stock market, foreign exchange rate options and futures, etc.). The latest step in this process was the CBOT's introduction in December 1992 of insurance futures to manage exposure in the general area of the insurance line upon which the contract is based. Cox and Schwebach (1992) discuss the mechanics of these insurance futures, their purpose and use, and how they fit into the financial market in general. Insurance futures are based on the loss ratio for a portfolio of actual insurance policies over a given period. The loss ratio is analogous to the stock market index underlying a stock index futures. An important difference is that the insurance portfolio loss ratio is not readily available, and there is no purchasable set of insurance contracts by which to price the futures contract (in contrast to the stock market). The insurance futures contract is also the basis for put and call options, and option spreads.

The above-listed insurance futures contracts and options have payoffs which are similar to traditional reinsurance in the sense that if one is long in the futures contract, and the loss index goes up indicating increased losses in the insurance market, then these losses are partially offset by financial gains in the financial market. Thus, the CBOT complex offers an alternative to reinsurance that is easily reversible and low cost. The call spread is the most popular contract. It was designed to compete with reinsurance stop-loss contracts. Insurance futures contracts also provide an investor or speculator easy access to the insurance market since there is no licensing, only margin requirements. The idea behind the introduction of insurance futures is to allow more efficient use of capital. The beneficiaries ultimately are insurance consumers and insurance company owners. In fact, it has been speculated that more capital in the reinsurance market would possibly have helped to alleviate the insurance crisis of the late 1980s. [The market for reinsurance is relatively difficult to enter when compared to financial markets, a fact that some such as Berger, Cummins, and Tennyson (1992) assert may have contributed to the general insurance crisis of 1984–86.] Insurance futures provide an alternative way for capital to be used like reinsurance. The result should be more stable consumer prices.

Although the CBOT has, from time to time, proposed futures on other insurance lines such as health, as of May 1995 there was but a single insurance futures market in operation, and it involved what is known as catastrophe insurance, loss ratios being based upon specified lines of insurance susceptible to catastrophic losses from natural events such as hurricanes, earthquakes, floods, wind damage, etc.* The Insurance Services Office uses actual loss and premium data from participating insurance companies to aggregate these lines of insurance (homeowners property loss, automobile physical damage, etc.), and the CBOT creates a market based upon this loss ratio index. Cox and Schwebach (1992) suggest that prices of insurance futures and options can be priced using the Black-Scholes framework as a crude approximation. D'Arcy and France (1992) give a convincing argument for the viability of catastrophe futures. Boose and Graham (1993) assess the viability of an insurance futures market using an empirical model. A successful catastrophic insurance futures and option market could reduce the threat of insolvency due to lack of capacity to withstand catastrophic loss as occurred in Florida after Hurricane Andrew. Boose and Graham (1993), however, conclude that the positive factors effecting success of the insurance futures market (large size of the insurance market, low transactions cost, high liquidity) are offset by the infrequent release of loss information. The Standard & Poor's 500 stock index is distributed to the market every 15 seconds. In contrast, the loss ratio and related aggregate loss data underlying the insurance futures contract is released to the market only twice: once at the end of the loss period and once at the settlement date, although monthly reports of industry catastrophe losses are available to the public. This lack of information should be taken into account in models used to price insurance and futures options.

Helyette Geman has presented informally joint work with David Cummins (1994) on valuation of insurance futures. Their approach is based on Asian options. This Asian option is based on an average of prices before the exercise date. Since the CBOT contract is based on a loss ratio, of which the numerator is the sum of prior losses, the Asian option feature fits the actual contract very well. However, their approach is based on the assumption that all traders can observe the loss ratio continuously. There is no published model which allows for this important feature; in the actual insurance futures market traders may have different sets of information regarding the underlying index.

The purpose of the current research is to establish a relation between information (or lack of information) about the loss ratio and insurance futures prices based solely on observable market prices. The potential for insurance futures to play an important role in hedging underwriting risk is enormous. However, the absence of models which take into account the lack of information presents a barrier to their success. The results should allow traders to better understand this unusual aspect of the catastrophe insurance futures market. With better understanding, speculators and hedgers may be more willing to enter the market.

Methodology

In this section we present one way to allow for lack of information. Let $S(t)$ denote the aggregate losses paid during the interval $[0,t]$. The loss ratio on the settlement date T is $S(T)/Q$ where Q is an estimate of the premiums written during the interval $[0,t]$. The value of $S(t)$ is known at only a few points in time (four points for a three-month contract, for example), but trading in the futures contract with a price denoted by $F(t)$ takes place continuously. In a financial market which is "frictionless" (in that traders can buy or sell as many securities as desired at posted market prices

*A new crop of insurance futures and options markets has been opened subsequently.

without incurring transactions costs) and allows no arbitrage, the fundamental theorem of asset pricing implies the existence of a "risk-neutral probability distribution" (or "equivalent martingale measure") such that the current price of any asset is equal to its expected future value discounted at a risk-free rate r. [See Harrison and Kreps (1979) for a detailed discussion of this theory.] For example, since a futures contract requires no cash outlay at time 0, the futures price must be equal to the expected value of the settlement value with the expectation calculated using the risk-neutral distribution:

$$F(t) = E\left[\frac{S(T)}{Q}\right].$$

As another example, a call option written on a futures contract gives the owner the right, but not the obligation, to buy the futures contract for some predetermined "strike price" K at expiration T. Assuming the owner of the option follows the optimal exercise policy (exercise at expiration if and only if the futures price exceeds the strike price), the value of the option at expiration is then given by $(F(T)-K)^{+}=\max[0, F(T)-K_i]$, and its value at any earlier time $t(t<T)$ is given by

$$E\left[\frac{(X-K)^{+}}{(1+r)^{T-t}}\right],$$

where E denotes expectations taken with respect to the risk-neutral distribution.

The values of more exotic options can be similarly represented as the expected value of some payoff function under the risk-neutral distribution. For example, a call option spread is a CBOT contract which provides the owner an exercise value of 100,000* $\{\text{Max}(F(T)-K_1, 0)-\text{Max}(F(T)-K_2, 0)\}$ where the strike prices K_1 and K_2 and exercise date T are specified in the contract. This is equivalent to buying a call option with an exercise price based on a loss ratio of K_1 and selling based on K_2. Let $u(s)$ denote the payoff function:

$$u(s) = \max(s-K_1, 0) - \max(s-K_2, 0)$$
$$= \begin{cases} 0 & \text{if } s \le K_1 \\ s-K_1 & \text{if } K_1 \le s \le K_2 \\ K_2-K_1 & \text{if } s \ge K_2. \end{cases}$$

Suppressing the contract face amount of 100,000, we can write the value at expiration as $u(F(T))$ and at any earlier time t, the price would be given by:

$$E\left[\frac{u(F(T))}{(1+r)^{T-t}}\right],$$

where, again, E denotes expectations taken with respect to the risk-neutral distribution.

In general these risk-neutral distributions will be unique if and only if the market is "complete" in that the underlying risk can be perfectly hedged by trading existing securities [see Harrison and Kreps (1979)]. This completeness assumption underlies the standard Black-Scholes valuation methods and may be appropriate when valuing put and call options on a stock or stock market index where the underlying asset (the stock or stocks making up the index) are continuously traded and prices are continually updated. But, in the case of insurance futures contracts and options on these futures contracts, the underlying asset (the loss ratio S) is not traded and its value is infrequently updated so that, consequently, the assumption of completeness seems inappropriate. In the incomplete markets case, we cannot identify a unique risk-neutral distribution, but we can use available information to restrict the set of possible equivalent risk-neutral distributions and compute upper and lower bounds on the price of any security.

The bounds on securities prices thus depend on what information we assume to be available about the underlying uncertainty. For example, rather than assuming that traders have precise information about the distribution for $S(T)$, we might assume, as in Brockett and Cox (1985) and Cox (1991), that only a range of values is known:

$$\mu_1 \le E[X(T)] \le \mu_2 \text{ and } \sigma_1^2 \le \text{Var}[X(T)] \le \sigma_2^2.$$

Alternatively, as in Smith (1995), we might assume that we know prices for certain securities and want to determine bounds on the prices for others that are consistent with the known prices. For example, we might be given prices for a futures contract and several call option contracts and seek bounds on some other call option contract. We will examine this second example in detail shortly.

The general framework of our analysis is concomitant with the mathematics of the general moment problem from statistics [cf., Kemperman (1987)]. Specifically, we assume that we are given $n+1$ real valued functions $f_i(x)$, $i=0, 1, \ldots, n$, whose expectations are assumed known and finite, that is, we know

$$\mu_i = E_p[f_i] = \int f_i(x)dP(x), \text{ for } i = 0, 1, \ldots, n.$$

Though we know these moment values, the specific underlying distribution P is unknown. For convenience we take $f_0(x)=1$ and $\mu_0=1$ so that the first moment constraints specify that we are dealing with a probability measure. We let $\mu=(\mu_0, \mu_1, \ldots, \mu_n)$ and $\mathbf{f}=(\mathbf{f}_0, \mathbf{f}_1, \ldots, \mathbf{f}_n)$ and assume the moment functions are linearly independent. Now, given a real valued *objective function* ϕ, our goal is to compute

$$\inf E_P[\phi] \text{ and } \sup E_P[\phi] \quad (1)$$

where the infimum and supremum are taken over the set of all distributions P such that $E_P[\mathbf{f}]=\boldsymbol{\mu}$. In our context, we allow P to vary over the set of all risk-neutral distributions consistent with the given information.

There are two different methods for solving these optimization problems, both of which lead to the same answer. In the "primal" approach, one seeks a distribution P that attains or approaches the bounds in (1): the theory of the moment problem [cf., Brockett and Cox (1985), Kemperman (1987), Smith (1995)] shows that the infimum and supremum are obtained by distributions which are discrete at most $n+1$ mass points; therefore, we can restrict our search to this subset of distributions. Alternatively, in the "dual" approach, we seek linear combinations of the given moment functions which lie above (for the upper bound) and below (for the lower bound) the function whose expectation is to be bounded and seek the polynomials with the least and greatest expectations. Under certain specific conditions regarding the moment functions f_i and objective function ϕ (for example, if they form a Chebycheff system of functions for example), we may find explicit formulas for the solutions to these optimization problems [cf., Brockett and Cox (1985)]. In the more general setting, while explicit formulae are not known, the problem can readily be solved using numerical methods. For example, Smith (1995) describes a procedure where one first solves a discretized version of (1) using standard linear programming methods and then obtains an exact solution by "polishing" this approximate solution by solving a low-dimensional, non-linear programming problem.

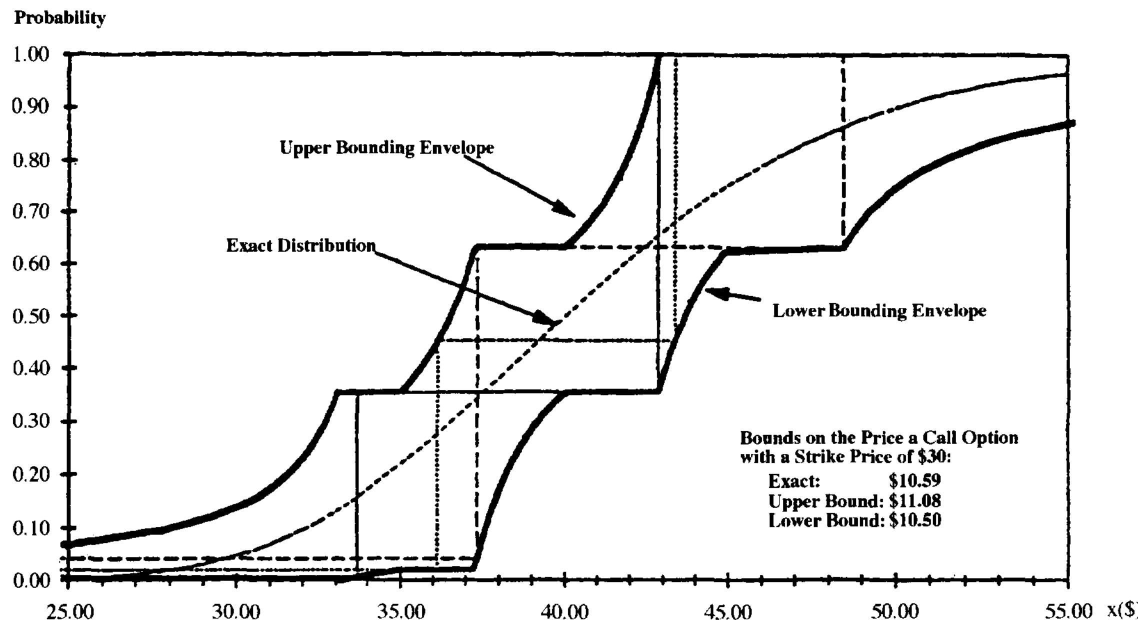
An Example

To illustrate this methodology, we consider a specific example where we are given the current (time 0) value of an asset (X_0) and the prices for a series of call options on this asset that expire at time T. To make the example concrete, suppose the current asset price is $40 and call options that expire in four months with strike prices $35, $40, and $45 have current prices $6.26, $3.08, and $1.26, respectively. We will then compute bounds on the underlying (cumulative) risk-neutral distribution as well as bounds on a call option with a $30 strike price. These prices are consistent with the Black-Scholes model with a risk-free discount rate of 5% per year and an annual volatility (σ) of 30%: in the Black-Scholes model, the risk-neutral distribution is log normal ($\ln(X/X_0)$ and is normally distributed with mean $(r-\sigma^2/2)t$ and variance $\sigma^2\, t$) so that the call option with a $30 strike price would have a price of $10.59.

To place this example in our general framework, we let $X=[0, \infty]$ represent the possible asset prices at time T. The moments correspond to the current prices for the traded securities, and the moment functions represent the discounted time T payoffs as a function of the underlying asset price at time T. For the asset itself, we take $f_1(x)=x/(1+r)^T$ and have a corresponding moment value $\mu_1=\$40$. For the call options, the moment function is given by $f_i(x)=(x-K_i)^+/(1+r)^T$ where K_i denotes the strike price and the moments μ_i correspond to the given prices for the call options. We can compute bounds on the underlying (cumulative) risk-neutral distribution at some point d by taking the objective function $\phi(x)$ to be a step function with a step at d (since $P(X \leq d)=E[\phi]$); by varying the point d we can trace out the entire distribution function. We can compute bounds on the call option with $30 strike price by taking the objective function $\phi(x)$ to be $(x-30)^+/(1+r)^T$.

The results for this example are summarized in Figure 1. In all cases, the bounds are achieved by distributions with no more than three points of support; a few of these distributions are shown in the figure. Here we find that the bounds on the underlying risk-neutral distribution are quite loose. From the "dual" perspective described above, this is a reflection of the fact that step functions used in computing these bounds are poorly approximated by the given moment functions: a "step" security that pays $1 if and only if the stock price is less than, say, $35 cannot be approximated very well by a portfolio consisting of the stock and given call options. The bounds on the value of a call option with a strike price of $30, however, are somewhat tighter as we can better approximate its payoffs using the given securities.

FIGURE 1
BOUNDS ON THE RISK-NEUTRAL DISTRIBUTION IN THE OPTION PRICING EXAMPLE



Some Extensions

This general framework can be extended to incorporate additional information about the underlying distribution by including this information into the constraint set in the optimization outlined previously. For example, another useful constraint is to allow the external measure to be "not too unusual" in the sense that its deviation from some prespecified potential risk-neutral distribution is not too great. For example, the Black-Scholes model assumes the risk-neutral distribution to be the lognormal distribution, and we may want to allow limited deviations from this distribution. A measure of deviation which can be used in this context is the entropic or information theoretic distance measure

$$I(P_1,P_2) = \int p(x)\ln p(x)dP_2(x)$$

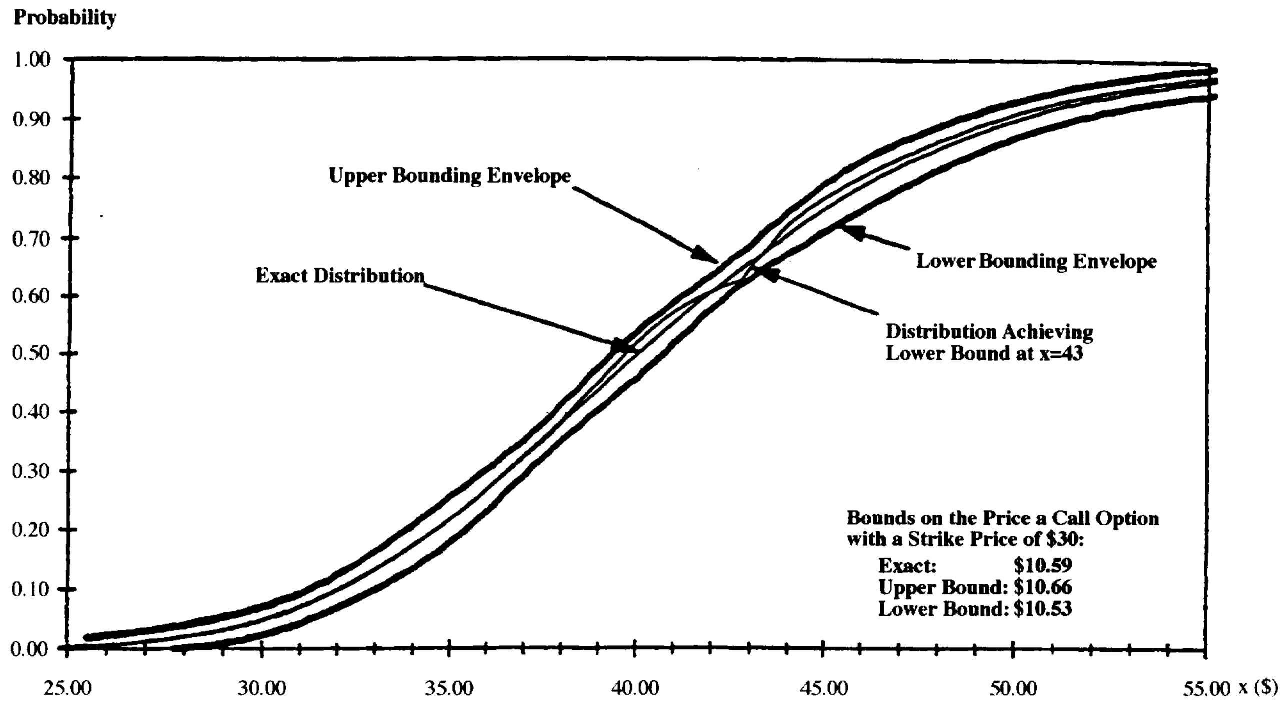
where $p(x)$ is the Radon-Nikodym derivative of P_1 with respect to P_2. If $P_1=P_2$, then $I(P_1, P_2)=0$, and $I(P_1, P_2)>0$ if $P_1 \neq P_2$. The minimum information distance estimate of P_2 subject to the constraint set is called the MDI estimate, and is discussed in detail in Brockett (1991).

In the above option pricing example, we are given prices for an asset and three call options on the asset, and we seek bounds on the underlying risk-neutral distribution and the price of call option with a $30 strike price. The results for this example are summarized in Figure 2. Here we have taken the prior P_2 to be a log-normal distribution (as assumed in the standard Black-Scholes model) with parameters consistent with the given asset and call prices. In this case, the minimum possible information distance is 0, and we take our entropy or information cutoff to be 0.02 (so that the unknown risk-neutral distribution is somewhat "close" to lognormal. Here, we see that the bounds on the risk-neutral distribution are much tighter than they were in the unrestricted case (compare Figures 1 and 2), and the bounds on the price of a call option with a $30 strike price are tighter as well: [10.53, 10.66] versus [10.50, 11.08]. (See Smith [1995] for a discussion of how to compute these values.)

Conclusions and Implications for Future Research

This paper has shown how to use the information available from market prices to determine bounds on the value of derivative instruments like options on insurance futures without completely specifying all

FIGURE 2
RESULTS FOR THE OPTION PRICING EXAMPLE WITH ENTROPY CONSTRAINTS



information about the distributions involved. That is, unlike the Black-Scholes option pricing formulae which require a lognormal probability distribution to obtain exact values for all options, this method assumes no knowledge of the probability distribution other than the values of observed market prices which are then viewed as "moments" relative to this unknown probability distribution. For a specific option under investigation, bounds on the value of the option are constructed using solutions to the moment problem. For catastrophic insurance futures, the assumption of lognormality is questionable, and the extent to which the Black-Scholes formulas give reasonable answers is a subject for further research. Also to be investigated are other insurance futures and options markets such as the crop insurance futures and options markets that have recently been created.

REFERENCES

Berger, L., J.D. Cummins, and S. Tennyson. 1992. "Reinsurance and the liability insurance crisis." *Journal of Risk and Uncertainty* 5, no. 3:253–72.

Boose, M.A. and A.S. Graham. 1993. "Predicting trading activity for catastrophe insurance futures." *Journal of Reinsurance* 1, no. 2:41–54.

Brockett, Patrick L. 1991. "Information theoretic approach to actuarial science: A unification and extension of relevant theory and applications." *Transactions of the Society of Actuaries* 43:73–135.

Brockett, Patrick L. and Robert C. Witt. 1982. "The underwriting risk and return paradox revisited." *Journal of Risk and Insurance* 49, no. 4:621–27.

Brockett, P.L. and S.H. Cox. 1985. "Insurance calculations using incomplete information." *Scandinavian Actuarial Journal*, 94–108.

Cox, S.H. 1991. "Bounds on expected values of insurance payments and option prices." *Transactions of the Society of Actuaries* 43, 231–60.

Cox, S.H. and R.G. Schwebach. 1992. "Insurance futures and hedging insurance price risk." *Journal of Risk and Insurance* 59, no. 4:628–44.

Cummins, J.D. and H. Geman. 1994. Presentation at the risk Theory Conference, Oberwolfach, Germany, September.

D'Arcy, S.P. and V.G. France. 1992. "Catastrophe futures. A better hedge for insurers." *Journal of Risk and Insurance* 59, no. 4:575–600.

Ferrick, Gary, Willis Faber, and Dumas Limited. 1994. "Introduction to derivatives." In *Derivatives in Insurance: a Research Report.* London: Co-published with Willis Corroon Group PLC, DJ Freeman, and Alexander Howden Reinsurance Brokers Ltd., Reactions Publishing Group Ltd.

Harrison, J.M. and D. Kreps (1979). "Martingales and arbitrage in multiperiod securities markets." *Journal of Economic Theory* 20:381–408.

Kemperman, J.H.B. (1987). "Geometry of moment problems." In the *Proceedings of Symposia in Applied Mathematics* 37:16–53.

Smith, James (1995). "Generalized Chebychev inequalities: Theory and applications in decision analysis." *Operations Research* 43, no. 5:807–25.