II

Crosshedging of Insurance Portfolios

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Motivation

New tools for managing insurance risk are being created everywhere. With a special emphasis on catastrophe insurance and reinsurance, we want to better understand what these tools are basically trying to achieve.

It is important to note that the concept of *crosshedging* gives us the key to the actuarial understanding of most new tools arising in the area of alternative risk transfer mechanisms. *Securitization* of insurance, which is the theme of this symposium can, as I see it, almost always be seen as an exercise in *crosshedging*.

The *practical* motivation for my presentation theme comes from catastrophe insurance and reinsurance. I am therefore first going to model this important branch of insurance. The model will then serve as a guide throughout the whole lecture.

The Catastrophe Risk Model

Assume that your insurance company has written a portfolio of homeowners policies in a clearly defined geographic area. The total claims originating from this portfolio for one specific time interval (year, quarter) is called S.

As standard in nonlife actuarial techniques, assume

1. $S = \Sigma_j^N = 1\ Y_j \sim$ doubly stochastic sum (1.1)
   number of claims : $N \sim$ Poisson distributed

   amount of claim j: $Y_j \sim$ identically distributed with cumulative distribution factor $F(x)$.

   Also assume $N, Y_1, Y_2, \ldots, Y_j, \ldots$ independent, which leads to the standard *compound Poisson model.*

The specific character of an *insurance portfolio exposed to catastrophe risk* is modeled in the parameter of the Poisson distribution $\sim$ *parameter of frequency of claims.*

Think of this parameter as follows

2. $\mu + \lambda$ (1.2)
   where μ is fixed and λ is a random variable of shocks (due to catastrophes).

It is instructive to look at the following numerical example:

3. $E[Y] = 1 \sim$ our unit is the average claim size
   $E[Y^2] = 3 \sim$ which is a rather high value for property claims (1.3)
   $\mu = 100$
   $\lambda = 0$ with probability 9/10 and
   $\lambda = 100$ with probability 1/10.

We have just constructed a model for a portfolio in which, on average, once in ten years the expected number of claims is doubled. Whenever this happens, common language will say that a catastrophe (windstorm, flood, earthquake) has happened.

Let us compute

$$E[S] = (\mu + E[\lambda]) * E[Y] = (100 + 10) * 1 = 110 \quad (1.4)$$

$$\mathrm{Var}[S] = \underbrace{(\mu + E[\lambda]) * E[Y^2]}_{A} + \underbrace{\mathrm{Var}[\lambda] * (E[Y])^2}_{B}$$

$$= (100 + 10) * 3 + 900 * 1 = 1230. \quad (1.5)$$

We observe two things:

1. The *variance* is big compared to the *mean*.
2. The *variance component B* gives us particular trouble.

What can we do to *reduce B*, which is the variation due to λ (in practical language: the variation due to the catastrophic risk)?

The Idea of Crosshedging

Assume that there is a second portfolio T covering risks in the same area (T could also be the aggregate portfolio written by all insurance companies other than yours in the same area). You are invited to participate in a quotashare of the insurance benefits of T.

Let us summarize:

You are given the opportunity to change the random variable

$$S \text{ into } S - \alpha * T \tag{2.1}$$

at the price of

$$\alpha * \Pi. \tag{2.2}$$

Clearly this leads to the following questions:

1. What is the optimal value of α? Call it α^*.
2. Which price Π is acceptable to you?

General Formulas for Crosshedging

The exercise:

$$\text{Find } \alpha^* \text{ such that } \mathrm{Var}[S - \alpha^* * T] = \min! \tag{3.1}$$

leads to

$$\alpha^* = \frac{\mathrm{Cov}(S,T)}{\mathrm{Var}[T]} \tag{3.2}$$

and with

$$S^* = S - \alpha^* * T \tag{3.3}$$

we find

$$\mathrm{Var}[S^*] = \mathrm{Var}[S] - \frac{(\mathrm{Cov}(S,T))^2}{\mathrm{Var}[T]}. \tag{3.4}$$

Remark: These formulas are derived in any introductory text on linear forecasting. It is actually the most elementary case which can be easily generalized to a higher dimension (using matrix notation).

As simple as these formulas are, the finance literature still fails to draw a lot of conclusions from them. This is due to the fact that, for example, in connection with catastrophe derivatives, nobody seems to have applied the crosshedging technique to the actuarial structure of the random variables S and T (assumed compound Poisson in this paper).

A Successful Area of Crosshedging in Insurance: Credibility Theory

Credibility can indeed be understood as *crosshedging the individual risk* against *its own past*.

Consider:

S~claims generated by the individual risk in *the year of study*

T~sum of the claims of the same risk in the past:

The credibility premium $P=M+\alpha^*T$ charged to the insured leads then to the net payment $S-\alpha^*T-M$ of the insurance company.

As

$$\mathrm{Var}[S - \alpha * T - M] = \mathrm{Var}[S - \alpha * T] \tag{4}$$

(you may omit the constant M),

we have reduced the credibility approach to our basic exercise in crosshedging!

The Fundamental Difference between the Basic Models in Credibility Theory and in Catastrophe Risk Crosshedging

The Credibility Model

The total claims of the individual risk in *one time period* are described by the random variable

$$S = \sum_{j=1}^{N} Y_j \sim \text{Compound Poisson} \quad \text{with } N \sim \text{Poisson}(\lambda). \tag{5.1}$$

The essential feature now relates to the fact that the *past variables* S_i, $i=1, 2, \ldots, n$, of the same risk are assumed compound Poisson *with the same* λ.

Hence, we can use the variable $T=\sum_{i=1}^{n} S_i$ to

1. *learn* about the value of λ~credibility language and
2. *reduce* the variance component due to the variability of λ~crosshedging language.

The Catastrophe Risk Model

S is defined as in (1.1)–(1.5):

1. $S=\sum_{j=1}^{N} Y_j$
2. N~*Poisson*$(\lambda+\mu)$

where

μ is fixed and
λ is a random variable of shocks.

Contrary to the credibility case, there is no use in looking at the past variables S_i, $i=1, 2, \ldots, n$. *Their* λ *parameter value has no relation whatsoever* to the λ value *of the recent random variable* S! Typically these values are drawn independently in each period.

But as catastrophes occur in whole areas, competing companies in the same area will also suffer from them. Hence, the increased value of λ *should also be effective in their portfolios*! This is the basic idea of crosshedging for catastrophe risks.

We, therefore, take T as total claims from a big competing company which is active in the same area (you can also think of T as representing the aggregate portfolio of all companies other than yours).

Let

$$T = \sum_{j=1}^{M} W_j \sim \text{Compound Poisson} \tag{5.2}$$

$$M \sim \text{Poisson}(\mu_T+\lambda_T). \tag{5.3}$$

Assume again μ_T is constant.

The *essential assumption* is, however, that

$$\lambda_T = A_T * \lambda \tag{5.4}$$

That is, we postulate that the expected frequency due to catastrophe events λ_T is a *multiple* of the corresponding λ in the distribution of S. In the following we explore two cases:

1. A_T is a deterministic factor, and
2. A_T is a random factor.

For simplicity we follow case 1 here, but will return to case 2 later.

Alternative Notation

From an intuitive point, one might prefer to write (5.4) as (5.4a):

$$\lambda = \frac{1}{A_T} * \lambda_T \quad \text{and then try to model } \frac{1}{A_T}. \tag{5.4a}$$

Of course for the case of deterministic A_T, this is just a change of notation, for the stochastic case one might however easily have $1/A_T=0$ with positive probability.

Remarks

1. We should think of T as substantially big relative to S; hence A_T is a big factor, and μ_T is big relative to μ.
2. For reasons of simplicity, we have assumed that S and T stem from different portfolios, which is contrary to some practical applications where T stands for the sum over all portfolios of all insurance companies in a given area. In such a case write

$$S - \alpha * T = S - \alpha * S - \alpha * (T - S).$$

S and $\tilde{T} = T - S$ stem now from disjoint portfolios.

Hence,

$$S - \alpha * T = (1 - \alpha) * S - \alpha * \tilde{T}$$

shows that crosshedging now implies a combination of crosshedging between distinct portfolios with a classical quotashare reinsurance.

Numerical Example

In order to do explicit calculations, let us also give numerical values for the parameters of the distribution of T.

$$E[W] = 1.5$$
$$E[W^2] = 6 \tag{5.5a}$$

$$\mu_T = 10000$$
$$A_T = 80 \tag{5.5b}$$

$$\lambda \text{ distributed as in (1.2) and (1.3).} \tag{5.5c}$$

Finally let us express the assumption that S and T stem from disjoint portfolios by the property:

$$S \text{ and } T \text{ are conditionally independent given } \lambda \quad (5.6)$$

Applying General Formulas for Crosshedging to the Microstructure of Variables S and T

For the covariance, we have

$$\begin{aligned}\mathrm{Cov}(S,T) &= E[\mathrm{Cov}(S,T)|\lambda] + \mathrm{Cov}[E[S|\lambda], E[T|\lambda]] \\ &= \mathrm{Cov}[\lambda * E[Y], A_T * \lambda * E[W]] \\ &= A_T * E[W] * E[Y] * \mathrm{Var}(\lambda).\end{aligned} \quad (6.1)$$

Similarly for the variances,

$$\mathrm{Var}[S] = E[\lambda + \mu] * E[Y^2] + \mathrm{Var}(\lambda) * (E[Y])^2 \quad (6.2)$$

(see 1.5) and

$$\begin{aligned}\mathrm{Var}(T) &= \underbrace{E[A_T * \lambda + \mu_T]} * E[W^2] \\ &\quad + \mathrm{Var}[A_T * \lambda] * (E[W])^2 \\ &= (\mu_T + A_T * E[\lambda]) * E[W^2] \\ &\quad + A_T^2 * \mathrm{Var}(\lambda) * (E[W])^2.\end{aligned} \quad (6.3)$$

From (3.2) we then get

$$\alpha^* = \frac{A_T * E[W] * E[Y] * \mathrm{Var}[\lambda]}{A_T^2 * (E[W])^2 * \mathrm{Var}[\lambda] + (\mu_T + A_T * E[\lambda]) * E[W^2]}$$

$$\alpha^* = \frac{E[Y]}{E[W]} * \frac{1}{A_T + \dfrac{E[W^2]}{(E[W])^2} * \dfrac{\mu_T + A_T * E[\lambda]}{A_T * \mathrm{Var}[\lambda]}} \quad (6.4)$$

Using the numerical values of 1 and 5, we obtain

$$\alpha^* = \frac{1}{1.5} * \frac{1}{80 + \dfrac{6}{2.25} * \dfrac{10000 + 80 * 10}{80 * 90}} = \frac{1}{1.5} * \frac{1}{84}$$

For later purposes, we are introducing here the *reduction factor* η

$$\eta = \frac{A_T}{A_T + \dfrac{E[W^2]}{(E[W])^2} * \dfrac{\mu_T + A_T E[\lambda]}{A_T * \mathrm{Var}[\lambda]}}. \quad (6.5)$$

We must comment on formula (6.4).

1. There are two critical values: μ_T and A_T.

2. Crosshedging is the more useful (i.e. α^* becomes bigger) *if* μ_T *becomes smaller.*

The conclusion is that the portfolio T used for crosshedging should (as much as possible) contain no other claims than those originating from catastrophes. Ideally we should have $\mu_T=0$. We call such a portfolio a *pure catastrophe portfolio.*

3. More critical is A_T, which we have assumed to be constant. If A_T is stochastic (and independent of λ), then the first A_T in the denominator of (6.4) has to be replaced by

$$\frac{E[A_T^2] * \mathrm{Var}[\lambda] + (E[\lambda])^2 * \mathrm{Var}[A_T]}{E[A_T] * \mathrm{Var}[\lambda]},$$

all others by $E[A_T]$.

The changes in formula (6.5) follow from those in (6.4).

For illustrative purposes, assume

$$\begin{aligned}A_T &= 40 \text{ with probability } 1/2 \\ &= 120 \text{ with probability } 1/2\end{aligned}$$

which leads to

$$\begin{aligned}E[A_T] &= 80 \qquad \text{as in the deterministic case} \\ E[A_T^2] &= 8000 \\ \mathrm{Var}[A_T] &= 1600.\end{aligned}$$

The figure 80 in our numerical example then jumps up to 102. The reduction factor η reduces from 0.95 to 0.75.

4. Nevertheless we see from the numerical exercise that the formula (6.4) derived from the deterministic case for A_T gives the right order of magnitude for α^*. Observe however that the pragmatic argument for

$$\alpha^* = \frac{E[Y]}{E[W]} * \frac{1}{A_T}$$

which is advocated in practice typically leads to values of α^* which are too high!

Safety and Cost

So far we have not addressed the question of whether our crosshedging exercise increases the safety of the insurance carrier. This question is obviously also related to the cost of the envisaged operation. Hence, we should ask *at which cost level* crosshedging can actually be used such that it contributes toward improving the safety of the insurance carrier. These are the questions that we are going to address now.

Assume the insurance carrier is collecting the premium P as its total of income for covering its obligations S. We propose to measure the safety of the insurance carrier by the *adjustment coefficient* κ.

Definition

$\kappa>0$ is called adjustment coefficient of a portfolio at premium income P if

$$E[e^{-\kappa(P-S)}]=1. \tag{7.1a}$$

If $P>E[S]$, such a $\kappa>0$ exists (under suitable mathematical assumptions, i.e., if the moment generating function of S is finite).

Remarks

1. Any text on risk theory will explain how to relate the adjustment coefficient κ to the probability of ruin criterion.
2. Our interpretation is going to be:
The *higher* κ, the higher the *safety* of S at premium P.
You can rewrite (7.1a) in the form

$$P = \frac{1}{\kappa} * \ln E[e^{\kappa S}], \tag{7.1b}$$

and by Taylor expansion of the right side we obtain the approximation

$$P = E[S] + \frac{\kappa}{2} * \mathrm{Var}[S]. \tag{7.2}$$

Hence, measuring safety means studying

$$\kappa = 2 * \frac{P - E[S]}{\mathrm{Var}[S]} = 2 * \frac{\text{Loading}}{\text{Variance}} \tag{7.3}$$

The factor 2 is, of course, irrelevant if we want just to *compare* different levels of safety.

Applying the Safety Concept to the Microstructure of S and S*

We first need for $S^*=S-\alpha^* * T$

$$\mathrm{Var}[S^*] = \mathrm{Var}[S] - \frac{(\mathrm{Cov}(S,T))^2}{\mathrm{Var}[T]}.$$

Using the formulas (6.1), (6.2), and (6.3) we obtain

$$\mathrm{Var}[S^*] = E[\lambda + \mu] * E[Y^2] + (E[Y])^2 * \mathrm{Var}[\lambda] * \{1 - \eta\} \tag{8}$$

with the *reduction factor* η as defined in (6.5).

Remarks

1. We have indeed achieved our goal, namely a reduction of the variance component B as introduced in the section titled Catastrophe Risk Model.
2. Observe that in our numerical example, we improve from

$\mathrm{Var}[S] = 1230$ Variance component B: 900 to
$\mathrm{Var}[S^*] = 373$ Variance component B: 43.

3. The reduction in the variance component B is excellent. But recall our discussion of formula (6.4). If A_T is no longer assumed to be deterministic (the assumption indeed is not realistic), then the reduction factor η is reduced. In our explicit calculations it dropped from 0.95 to 0.75 which would lead to $\mathrm{Var}[S^*]=555$, variance component B: 225.

We conclude from this exercise that also under more realistic assumptions the crosshedging exercise gives us a considerable reduction of our variance. Again, the deterministic A_T reflects the general case but it overstates the effect.

Can We Afford the Crosshedging Operation?

All operations of securitization need to be seen in relation to their cost. As in all forms of reinsurance (securitization is one of them) cost is identical to the *amount of loading* which is transferred.

We define
$L=P-E[S]$~Amount of loading in the original portfolio at premium P; $L_T=\Pi-E[T]$~Amount of loading in the portfolio used for crosshedging, Π being the premium (e.g. futures price) charged for receiving the benefits of T.

Now compare the situation before and after crosshedging:
Before crosshedging

$$\frac{\kappa}{2} = \frac{P - E[S]}{\mathrm{Var}[S]} = \frac{L}{\mathrm{Var}[S]}. \tag{9.1}$$

After crosshedging

$$\frac{\kappa^*}{2} = \frac{P - E[S] - \alpha^*[\Pi - E[T]]}{\text{Var}[S^*]}$$
$$= \frac{L - \alpha^* * L_T}{\text{Var}[S^*]}. \qquad (9.2)$$

The crosshedging operation improves our safety exactly if $\kappa^* > \kappa$.

For the case $\kappa^* = \kappa$, we obtain the upper limit for all the premiums Π that we can afford. Call this limit Π^* (the corresponding loading L_T^*). From (9.1) and (9.2), we conclude

$$\frac{L}{\text{Var}[S]} = \frac{L - \alpha^* * L_T^*}{\text{Var}[S^*]} \Rightarrow \alpha^* * L_T^*$$
$$= L * \frac{\text{Var}[S] - Var[S^*]}{\text{Var}[S]}.$$

From (6.2) and (8) we have

$$\frac{\text{Var}[S] - \text{Var}[S^*]}{\text{Var}[S]} = \frac{(E[Y])^2 * \text{Var}[\lambda] * \eta}{(E[\lambda] + \mu) * E[Y^2] + (E[Y])^2 * \text{Var}[\lambda]}.$$

Hence,

$$\alpha^* * L_T^* = L * \frac{1}{1 + \dfrac{E[Y^2] * E[\lambda + \mu]}{(E[Y])^2 * \text{Var}[\lambda]}} * \eta. \qquad (9.3)$$

With our numerical values as used before, the middle factor turns out to be 0.7317; hence we conclude that in order to increase the safety of the portfolio S by crosshedging, we must have: Transferred loading < Collected loading $*0.7317*\eta$ (in money units).

The critical quantity is again the reduction factor η.

Epilogue

The approach chosen for this presentation is truly an actuarial one:

1. the construction of a sufficiently structured model,
2. the insistence that data must be available to estimate the relevant parameters in the model, and
3. the comparison of results in a practical situation with expected results from the model.

Of course, we have only been able to give some ideas for the first point. The ideas for points two and three are obvious and follow.

The main point, however, is that we have taken a truly scientific route and hope that this scientific attitude also will be accepted ultimately by the financial community. Let me take this opportunity to remind you that the methods used by the life actuary were in existence for more than a century before they were recognized as the fundamental basis for the life insurance industry. The nonlife actuary is still struggling for this recognition. It is certainly high time to take up the struggle for the Actuary of the Third Kind as well.