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Pricing Insurance Derivatives: The Case of CAT Futures

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Abstract

Since their appearance on the market, catastrophe insurance futures have triggered a considerable interest from both practitioners as well as academics. As one example of a securitized (re)insurance risk, its pricing and hedging contains many of the key problems to be addressed in the analysis of more general insurance derivatives. In the present paper we review the main methodological questions underlying the theoretical pricing of such products. We discuss utility maximization pricing more in detail. A key methodological feature is the theory of incomplete markets. Our paper follows closely the exposition given in Meister (1995).

Catastrophe Insurance Futures

Introduction

In recent years the magnitude of catastrophic losses has been staggering. Catastrophic events put significant financial demands on society. In response to market conditions, the Chicago Board of Trade (CBOT) introduced catastrophe insurance futures and options in 1992. The catastrophe insurance (CAT) futures are based upon a loss index consisting of specified losses reported by insurance companies. If an insurance company's insurance portfolio is highly correlated to the loss index, then the company may achieve a reinsurance by the purchase of catastrophe insurance futures or options. Therefore, the catastrophe insurance futures and options can be considered as standardized reinsurance instruments. Similar products are also under consideration in Europe.

The exact specifications of the catastrophe insurance futures contracts are the subject of many recent papers. A complete description is given by the CBOT in its *Catastrophe Insurance Reference Guide* (1995) and *The Management of Catastrophe Losses using CBOT Insurance Options* (1994). Data information is to be found in the CBOT's *Catastrophe Insurance Background Report* (1995). A very good introduction on the use of CAT futures is due to Albrecht, König, and Schradin (1994), which also contains a useful list of references. In this introduction, we only recall some of the basics underlying CAT futures.

Four different catastrophe insurance futures are traded at the CBOT: eastern, midwestern, western, and national catastrophe insurance futures (from now on: insurance futures). Several options of the American type on each contract are also traded. At the moment, the trading volume of the insurance futures options is larger than the one of the insurance futures proper. A reason is certainly that the options tend to hedge nonproportional reinsurance contracts rather than proportional ones, and proportional reinsurance contracts are not very common in the context of catastrophe insurance. In the following, we will discuss the actual insurance futures, rather than the corresponding options. Since the exact specifications of the four insurance futures are the same with exception of the states concerned, we will not distinguish between them. An excellent overall introduction to futures in general is Duffie (1989).

Insurance futures trade in quarterly cycles with contract months March, June, September, and December. For example, the June contract covers losses from events occurring during the first quarter of the same year as reported by the end of June. Since the settlement value is based on losses incurred (paid plus estimated unpaid), a third quarter allows for loss settlement. Trading ends on the fifth day of the fourth month following the contract month. Therefore, the June contract settlement takes place on October 5. The settlement value of the contract is determined by a loss index.

The Loss Index

The loss index is the underlying instrument of the future's final value. It consists of losses reported to the Insurance Service Office (ISO). Losses are reported to the ISO by approximately 100 companies, but the ISO selects only some of the data on the basis of size, diversity of business and quality. As the selected losses should be representative for the different lines of insurance and states, they are replaced by weighted losses. The weights correspond to the percentages of (estimated) premiums received by the selected companies and total (estimated) premiums earned per line and state. Both the list of reporting companies included in the pool (selected companies) and the estimated premium volume are announced by the CBOT prior to the beginning of the trading period for that contract.

The different lines of insurance include homeowners, commercial multiple peril, earthquake, automobile physical damage, fire, allied lines, farmowners, and commercial inland marine. Reported losses can arise from the perils of windstorm, hail, earthquake, riot, and flood.

Now let L_T denote the sum of selected weighted losses incurred during the quarter corresponding to the contract and reported at the end of the following quarter. Let Π denote the announced premiums earned during the three months exposure period. Then, the insurance future's settlement value F_T is given by

$$F_T = \$25{,}000 \times \text{Min}\left(\frac{L_T}{\Pi}, 2\right). \tag{1}$$

A convenient way of rewriting (1) is as follows

$$F_T = \$25{,}000 \times \left(\frac{L_T}{\Pi} - \text{Max}\left(\frac{L_T}{\Pi} - 2, 0\right)\right),$$

so that in finance terminology, a catastrophe future (or more precisely its settlement value) is equivalent with a long position in the loss ratio and a short position in a European call option with maturity T, strike price 2, with an underlying loss ratio. Depending on the assumptions on the process (L_t), results from general mathematical finance may be used.

The Loss Process (L_t)

The goal of this section is to develop a plausible stochastic model for the process $(L_t)_{0\leq t\leq T}$ of losses reported to the selected insurance companies until time t (T being the end of the reporting period). Basically, for $0\leq t\leq T$,

$$L_t = \sum_{k=1}^{7} O_{k,t} + \sum_{\ell=1}^{4} C_{\ell,t} \equiv L_t^{(1)} + L_t^{(2)}$$

where losses are subdivided in seven classes of ordinary losses $(O_{k,t})$ $(k=1, \ldots, 7)$, arising from allied lines, automobile physical damage, commercial multiple peril, farmowners, fire, homeowners, and inland marine. There are four classes of catastrophic losses $(C_{\ell,t})$ $(\ell=1, \ldots, 4)$ arising from earthquakes, wind/hail/flood, hurricanes, and riot. The latter classes of losses are defined as losses exceeding a certain high threshold. For example, in their interesting analysis Huygues-Beaufond and Partrat (1992) classified losses due to hurricanes exceeding \$30 million and those due to wind/hail/flood above \$7.5 million as catastrophic. Both statistical analysis (as in Huygues-Beaufond and Partrat [1992]) and theoretical considerations (superposition and thinning of point processes as in Daley and Vere-Jones [1988], Propositions 9.2 VI and 9.3.I.) lead to Poisson-type of assumptions on both $L^{(1)}$ and $L^{(2)}$. Under the assumption that the processes $L^{(1)}$ and $L^{(2)}$ are only weakly correlated or indeed uncorrelated, again from a theoretical point of view one may safely assume that (L_t) obeys one of the following nested assumptions.

Basic Assumptions

1. (L_t) follows a compound Poisson process.
2. (L_t) follows a mixed compound Poisson process.
3. (L_t) follows a doubly stochastic (or Cox) compound Poisson process.

For a definition of these processes, see the section immediately following.

Any insurance derivative involving claim payments will fall within one of the above categories, or at least be an important modeling component. See, for instance, the already mentioned work of Huygues-Beaufond and Partrat (1992), Panjer and Willmont (1992), Grandell (1991), and the references therein. As noted by Sondermann (1991), over the decades, actuaries have collected bulks of statistical information which enable them to specify such processes one and two with considerable accuracy, so that one can safely say that a lot more is known about risk processes than about security price processes. For a model combining one of the above jump models together with a diffusion component, see Cummins and Geman (1994, 1995).

We assume the loss process $(L_t)_{0\le t\le T}$ to be defined on a basic probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$, where P is the so-called physical probability measure and $(\mathcal{F}_t)_{0\le t\le T}$ is an increasing family of sub-σ-algebras of $\mathcal{F}$ so that for all t, L_t is $\mathcal{F}_t$-measurable (i.e., (L_t) and is $(\mathcal{F}_t)$-adapted). We interpret $\mathcal{F}_t$ as containing all basic information up to and including time t, often (though not always) one assumes that $(\mathcal{F}_t)$ is the natural filtration belonging to (L_t), $\mathcal{F}_t=\sigma$ $(L_s, 0\le s\le t)$. This essentially exposes a weakness in the trading of CAT futures: though L_t is the main information needed to determine the future's price F_t at time t, (L_t) is only known on two days. An interim report publishes the value of L_t at the fourth day after the reporting period; the final index value is known at maturity. A further uncertainty is due to the actual quality or type of reporting. In this way, a CAT future is fundamentally different from a common (finance) future which is based on a regularly (in many cases very frequently or even tick-by-tick) reported spot price. This incomplete reporting adds an extra component to the overall volatility of insurance futures.

Some Basic Notations and Definitions

Consider a probability space $(\Omega, \mathcal{F}, P)$ and a second probability measure Q on $(\Omega, \mathcal{F})$. Let X and Y be random variables on $(\Omega, \mathcal{F}, (P, Q))$. $P_X(s)$: $=P[X\le s]$ denotes the distribution of the random variable X under the measure P; $E_P[X|Y]$ denotes the conditional expectation of X given the σ-algebra $\sigma(Y)$ with respect to the measure P. If $A\in\mathcal{F}$ and $s\in Y(\Omega)$, then $P_{Y=s}[A]$ is a second notation for $E_P\,[1_A|Y=s]$. We call a function f: $R \to R$ measurable if it is measurable with respect to the Borel σ-algebra $\mathcal{B}(R)$. $\mathcal{B}(A)$ always denotes the Borel sets of A (where A itself is a Borel set). A cadlag function on R is a right continuous function such that the limits from the left always exist. $Q|_{\mathcal{F}}\sim P|_{\mathcal{F}}$ is the notation for the equivalence of the measures P and Q on the σ-algebra $\mathcal{F}$. $Q|_{\mathcal{F}} \perp P|_{\mathcal{F}}$ means that P and Q are mutually singular on $\mathcal{F}$; $Q_X\sim P_X$ means that $Q|_{\sigma(X)}\sim P|_{\sigma(X)}$ and $Q_X \perp P_X$ means that $Q|_{\sigma(X)} \perp P|_{\sigma(X)}$.

A doubly stochastic compound Poisson or compound Cox process $(S_t)_{t\ge0}$ can be written as follows:

$$S_t = \sum_{k=1}^{N_t} X_k$$

where $(X_1, X_2, X_3, \ldots)$ are strictly positive, i.i.d. random variables on $(\Omega, \mathcal{F}, P)$, and (N_t) is an increasing point process, independent of (X_k), starting at zero, with

$$N_{s+t} - N_t \sim \text{Poisson } (\Lambda(s+t) - \Lambda(t))\ (s > 0)$$

where $(\Lambda(t))$ itself is a strictly increasing stochastic process on R^+, also defined on the probability space $(\Omega, \mathcal{F}, P)$. For a precise definition, see Grandell (1991).

A mixed compound Poisson process is a doubly stochastic compound Poisson process with

$$\Lambda(t) \equiv \Lambda \times t$$

and Λ being a strictly positive random variable on $(\Omega, \mathcal{F}, P)$. If the random variable Λ is constant, almost surely we call the resulting process a compound Poisson process. In this case, (N_t) is an homogeneous Poisson process with constant intensity $\lambda>0$. In this paper we always assume that $P[X_1>0]=1$, $X_1\in L^2(\Omega, \mathcal{F}, P)$, $P[\Lambda>0]=1$, $\Lambda\in L^2(\Omega, \mathcal{F}, P)$, and $\forall t\in R$: $\Lambda(t)\in L^2(\Omega, \mathcal{F}, P)$.

Recall that two measures P and Q on (Ω, P) are equivalent if they have the same nullsets. Finally, the integrable, adapted process (X_t) is an $(\mathcal{F}_t)$-Martingale with respect to Q if

$$\forall 0 \le s \le t \le T : E_Q\,(X_t - X_s \mid \mathcal{F}_s) = 0,$$
$$Q\text{-almost surely.}$$

Pricing by "No-Arbitrage"

Introduction

In order to highlight the fundamental differences between CAT futures (or indeed more general insurance derivatives) and traditional finance derivatives, let us look at the pricing problem for the latter in the context of no arbitrage. This introduction is based on the excellent paper by Föllmer (1990). Further basic references are Harrison and Kreps (1979), Harrison and Pliska (1981), and the more recent Delbaen and Schachermayer (1995). An excellent textbook treatment (in French) is Lamberton and Lapeyre (1991). For an English language edition, see Lamberton and Lapeyre (1996).

Consider a price process $(X_t)_{0\le t\le T}$ on $(\Omega, \mathcal{F}, P)$ and a random cash flow H before or at time T. H, as an $\mathcal{F}_T$-measurable random variable on $(\Omega, \mathcal{F}, P)$, is called a contingent claim. Typical examples are:

1. A European call option written on (X_t) with maturity T and strike price K,
$$H = (X_T - K)^+,$$
where $x^+=\max(x, 0)$;
2. If (X_t) describes a loss process and H is the final cash flow arising from a nonproportional reinsurance contract covering losses in a layer (K_1, K_2) at time T,
$$H = \min (X_T, K_2) - \min (X_T, K_1);$$
3. The settlement of a CAT future,
$$H = \$25{,}000 \times \min \left(\frac{X_T}{\Pi}, 2\right), \text{ and}$$
4. An Asian option with strike price K and maturity T,
$$H = \left(\frac{1}{T}\int_0^T X_u\, du - K\right)^+.$$

Recall that an arbitrage opportunity is the possibility of making a riskless sure profit. In a no-arbitrage market, such opportunities do not exist. The (fair) pricing of contingent claims in such a market standardly starts with the following assumption (construction): "Let Q be an equivalent Martingale measure to P, i.e. $Q \sim P$ and (X_t) is an $(\mathcal{F}_t)$-Q-Martingale." *It is precisely this assumption which crucially depends on the stochastic properties of the underlying process* (X_t). In standard finance markets like the Cox-Ross-Rubinstein binomial tree model, the Bachelier Brownian motion model or the Markovitz-Black-Scholes geometric Brownian motion model, one can show that not only such a Q exists but is moreover unique (as we shall discuss later, the latter is related to the notion of market completeness and is *crucially different in insurance markets*). However, let us for the moment assume that we have such a unique Q. Recall that H is $\mathcal{F}_T\,(=\sigma\,(X_s : 0\le s\le T)$, say)-measurable; then in the above standard finance cases one can write H as an *Itô representation*:

$$H = H_0 + \int_0^T \xi_s\, dX_s \tag{2}$$

for some H_0 and predictable (think left continuous) process (ξ_t). The representation (2) leads to a portfolio strategy replicating (riskless) the claim H if the premium H_0 is suitably chosen (and here Q will enter!). Indeed at time t we hold the amount ξ_t in the risky asset X_t, and the amount

$$\eta_t \equiv \left(H_0 + \int_0^t \xi_s\, dX_s\right) - \xi_t X_t$$

in the riskless asset ("money in the bank") given by the constant 1 (think of discounted amounts). The value V_t of this portfolio at time t is $V_t=\xi_t X_t+\eta_t$, and hence *by construction* $V_T=H$ (we neglect transaction costs!). This will all work if we can calculate the initial investment H_0 $(=V_0)$ and the process (ξ_t). Unfortunately, Itô's representation (2) is mostly only a nonconstructive existence theorem. Using the notion of self-financing strategies and Itô's lemma, one can derive a constructive solution (involving PDEs). Here is the final trick to calculate H_0:

1. (X_t) is a Q-Martingale (not necessarily a P-Martingale);
2. (ξ_s) is predictable and "nice," whence $(I_t\equiv\int_0^t\xi_s dX_s)_{0\le t\le T}$ is a Q-Martingale;
3. $E_Q(I_t)=E_Q(I_0)=0$; therefore $E_Q(H)=H_0$; since we know H and Q we have found the fair premium H_0.

Though admittedly we have left out various details, the above discussion clearly gives us a way in which to price and hedge insurance derivatives based on a risk process (X_t):

Step 1 Investigate the relationship between no-arbitrage conditions and the existence of equivalent Martingale measures Q.

Step 2 What about uniqueness of Q (related to completeness)?

Step 3 Find the Itô representation of (X_t) and investigate the explicit construction of hedging portfolios.

Step 4 What if any of the above fail?

In the following sections, we show that for risk processes in general many equivalent Martingale measures (and consequently fair premiums) exist so that a key discussion will be devoted to Step 4. By way of an important example, we restrict our discussion to mixed compound Poisson processes.

Mixed Compound Poisson Processes and Change of Measure

For a given process (X_t) on $(\Omega, \mathcal{F}, P)$, in order to investigate the existence of equivalent Martingale measures, one has to be able to characterize Radon-Nikodym derivatives dQ/dP. The following result is proven in Meister (1995). We use the notation discussed previously in Some Basic Notations and Definitions.

Theorem 1 *Suppose $(S_t)_{t\geq 0}$ is a mixed compound Poisson process on $(\Omega, \mathcal{F})$ under P and Q. The following statements are equivalent*:

1. $\forall s\geq 0 : Q|_{\mathcal{F}_s} \sim P|_{\mathcal{F}_s}$;
2. $P_{X_1} \sim Q_{X_1}$, *and*
3. $\exists \gamma : R \to R$ *measurable with*

$E_{P_{X_1}}(\exp(\gamma(X_1)))=1$, $E_{P_{X_1}}(X_1^2 \exp(\gamma(X_1)))<\infty$, and

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_s} = \exp\left(\sum_{i=1}^{N_s} \gamma(X_i)\right) \frac{E_{Q_\Lambda}(\Lambda^{N_s} \exp(-\Lambda s))}{E_{P_\Lambda}(\Lambda^{N_s} \exp(-\Lambda s))}.$$

Theorem 1 allows us to explicitly calculate Radon-Nikodym derivatives for the most important insurance risk processes.

Example 1 (Compound Poisson case)

Let P and Q be probability measures such that $(S_t)_{t\geq 0}$ is a P- and Q-compound Poisson process, and let $P_{X_1} \sim Q_{X_1}$, $P_\Lambda = \delta_{\lambda_1}$, $Q_\Lambda = \delta_{\lambda_2}$, $\lambda_1, \lambda_2 > 0$. Then Theorem 1 immediately yields

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_s} = \exp\left(\sum_{i=1}^{N_s} \gamma(X_i)\right) \left(\frac{\lambda_2}{\lambda_1}\right)^{N_s} e^{-(\lambda_2-\lambda_1)s}$$

for some measurable function γ.

Example 2 (Gamma mixed compound Poisson case)

Let P and Q be probability measures such that $(S_t)_{t\geq 0}$ is a P- and Q-gamma mixed compound Poisson process and $P_{X_1} \sim Q_{X_1}$. Consider $\gamma_1, \gamma_2, c_1, c_2 > 0$ to be the parameters describing the distribution of Λ, i.e.

$$P_\Lambda \sim \Gamma(\gamma_1, c_1),\ Q_\Lambda \sim \Gamma(\gamma_2, c_2),$$

where $\Gamma(\gamma_i, c_i)$ denotes the gamma distribution with parameters γ_i, c_i.

Then again using Theorem 1, for some measurable function γ,

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_s} = \exp\left(\sum_{i=1}^{N_s} \gamma(X_i)\right) \frac{\Gamma(N_s+\gamma_2)}{\Gamma(N_s+\gamma_1)} \left(\frac{(c_1+s)}{(c_2+s)}\right)^{N_s} \theta$$

where θ is given by

$$\theta = \frac{c_2^{\gamma_2}\,(c_1+s)^{\gamma_1}\,\Gamma(\gamma_1)}{c_1^{\gamma_1}\,(c_2+s)^{\gamma_2}\,\Gamma(\gamma_2)}.$$

The above model is also called the compound negative binomial process.

Example 3 (General inverse Gaussian mixed compound Poisson case)

Let P and Q be probability measures such that $(S_t)_{t\geq 0}$ is a P- and Q-general inverse Gaussian mixed compound Poisson process and $P_{X_1} \sim Q_{X_1}$. Consider μ_1, μ_2, β_1, $\beta_2 > 0$ and $\lambda_1, \lambda_2 \in R$ to be the parameters determining the distribution of Λ under the two measures (λ_1, λ_2 are constants here and not realizations of the random variable Λ), i.e.,

$$P_\Lambda \sim GIG(\mu_1, \beta_1, \lambda_1),\ Q_\Lambda \sim GIG(\mu_2, \beta_2, \lambda_2).$$

For further details on the processes, see Panjer and Willmot (1992). In this case, we obtain

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_s} = \theta \exp\left(\sum_{i=1}^{N_s} \gamma(X_i)\right) \left(\frac{\mu_2}{\mu_1}\right)^{N_s} \left(\frac{(1+2\beta_1 s)}{(1+2\beta_2 s)}\right)^{\frac{N_s}{2}} \frac{k_{\lambda_2+N_s}(\mu_2\beta_2^{-1}(1+2\beta_2 s)^{1/2})}{k_{\lambda_1+N_s}(\mu_1\beta_1^{-1}(1+2\beta_1 s)^{1/2})}$$

with

$$\theta = \frac{(1+2\beta_1 s)^{\lambda_1/2}\,k_{\lambda_1}(\mu_1\beta_1^{-1})}{(1+2\beta_2 s)^{\lambda_2/2}\,k_{\lambda_2}(\mu_2\beta_2^{-1})}.$$

The above model is also called the compound Sichel process.

As explained in Meister (1995, Proposition 2.9), the above results can immediately be used to show that after a change of measure, the process $(S_t)_{t\geq 0}$ remains a mixed compound Poisson process if the Radon-Nikodym derivatives

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_s}$$

have the right structure.

Mixed Compound Poisson Processes and Martingales

Consider a mixed compound Poisson process $(S_t)_{t\geq 0}$ under P, and a premium density $p>0$. The following result shows how equivalent Martingale measures for the process $(S_t - pt)_{t\geq 0}$ can be obtained. A proof of the result can be found in Meister (1995, Proposition 2.11). For more details on premium densities and actuarial premium calculation principles, see Delbaen and Schachermayer (1995).

THEOREM 2 *Suppose $(S_t)_{t\geq 0}$ as above. $(S_t - pt)_{t\geq 0}$ is an $(\mathcal{F}_t)$-Martingale under Q and for all $s\geq 0$, $Q|_{\mathcal{F}_s} \sim P|_{\mathcal{F}_s}$ if and only if*

1. *$\exists \lambda > 0$, $\beta : R^+ \to R$ measurable with $E_P(e^{\beta(X_1)}) = \lambda$ and $E_P(X_1^2 e^{\beta(X_1)}) < \infty$ such that*

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_s} = \exp\left(\sum_{i=1}^{N_s} \beta(X_i) - \lambda s\right) \left(E_P(\Lambda^{N_s} e^{-\Lambda s})\right)^{-1},$$

and

2. $p = E_P(X_1 e^{\beta(X_1)})$.

An immediate consequence from the above result is that if the jumpsizes of $(S_t)_{t\geq 0}$ are not constant under the measure P (i.e., nonconstant claimsizes), then an equivalent Martingale measure for $(S_t - pt)_{t\geq 0}$ *cannot be unique*! In the less interesting case of constant claims, we have uniqueness.

A Word about Completeness

Completeness essentially means that the underlying process $(X_t)_{t\geq 0}$ is so that every contingent claim can be replicated by a (self-financing) strategy. Not surprisingly is this notion linked to Itô representations.

DEFINITION 1 *The market $((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, Q), (X_t)_{0\leq t\leq T})$ is complete if every contingent claim $H \in L^2(\Omega, \mathcal{F}_T, Q)$ admits an Itô representation* (2) *with respect to the process (X_t).*

We basically know that the no-arbitrage condition is "equivalent" with the existence of an equivalent Martingale measure. For a mathematically precise statement see Delbaen and Schachermayer (1995) or Stricker (1990). Sondermann (1991) showed that a similar result holds in reinsurance markets. Besides proving the existence of replicating strategies in various no-arbitrage models, completeness implies (or is indeed equivalent with) uniqueness of equivalent Martingales. The following models are known to be complete:

1. One-dimensional Brownian motion (Itô, 1951),
2. Multidimensional Brownian motion and some special types of diffusions (Jacod, 1979),
3. $(N_t - \lambda t)_{t\geq 0}$ with (N_t) a homogeneous Poisson process (Kunita and Watanabe, 1967), and
4. Square integrable point process Martingales $(N_t - \int_0^t \lambda_s ds)_{t\geq 0}$ (Brémaud, 1981).

As soon as we move to compound processes based on homogeneous, mixed, or doubly stochastic Poisson processes, even when completeness is present, *we may lose the uniqueness of the equivalent Martingale property and therefore the unique pricing property.* For further details see Meister (1995, Section 3.3). We introduced the notion of completeness via the existence of hedge portfolios (Itô representation). In various markets completeness turns out to imply the uniqueness of an equivalent Martingale measure. These cases are finite probability spaces or continuous price processes (Jacod, 1979), of a price process for which the natural filtration ($\mathcal{F}_t = \sigma(X_s : s\leq t)$) is strictly left-continuous (Pratelli, 1994). Unfortunately in an insurance market framework in general, none of these cases apply. *Clearly further research on this topic is called for.*

Before embarking on the pricing problem for catastrophe insurance futures, we quote some words of warning concerning incompleteness and insurance pricing.

1. There is no "right" price of insurance; there is simply *the transacted market price* which is high enough to bring forth sellers, and low enough to induce buyers (Lane, 1995).
2. In incomplete markets, exact replication is impossible and holding an option is a genuinely risky business, meaning that *no preference independent pricing formula is possible.* If, however, option pricing is imbedded in a utility maximization framework, i.e., the potential option purchaser's attitude to risk is specified, then a unique measure emerges in a very natural way (Davis and Robeau, 1994).
3. Arbitrage-based pricing theories are theories about *relative prices* and do *not* attempt to explain *why* the prices of a particular stock reached their observed level. Only the *interrelationship between prices* is explained (Jensen and Nielsen, 1994).

Back to CAT Futures

The key questions concerning catastrophe insurance futures markets are:

1. Do strategies of taking long and short positions in insurance futures contracts exist which yield a sure profit?

2. What are necessary and sufficient conditions to exclude the opportunities of a sure profit?

Recall that we denote the process of total losses flowing into the insurance futures index by (L_t). Also we assume (L_t) to follow either a compound Poisson process, a mixed compound Poisson process or a doubly stochastic compound Poisson process, defined on some probability space $(\Omega, \mathcal{F}, P)$. We denote the insurance future's price process by $(F_t)_{0\leq t\leq T}$. A short position at time t means the commitment to pay the random amount F_T-F_t at time T. A holder of a long position will receive F_T-F_t at time T. There are no cashflows before T. This is a theoretical assumption as the CBOT clearing system requires certain payments from agents before T. Furthermore, the insurance future contract has the starting value zero, and settlement takes place at time T. Consider the market to be liquid and the contracts to be divisible in the sense that any agent can buy or sell any fraction of a contract at any time. We consider deterministic interest rates, though our results also hold for stochastic interest rate models. For expository purposes, we assume interest rates to be zero; because there are no cashflows between 0 and T, we can think of all cashflows to be discounted to their value at time zero. We, furthermore, recall that F_T is determined by \$25,000 min

$$\left(\frac{L_T}{\Pi}, 2\right),$$

hence the process (F_t) is bounded. Both (L_t) and (F_t) will be adapted with respect to the filtration $(\mathcal{F}_t)$, denoting the increasing information σ-algebras available through time. We have to give a new definition for the meaning of a strategy: a strategy is used by an agent who agrees at certain random times to several long or short positions in an insurance future contract, where the number of the positions only depends on the information which is available at those (random) times. To be more precise,

DEFINITION 2 *The set* $(\xi)=\{n, \tau_1, \ldots, \tau_n, \xi_1, \ldots, \xi_n\}$, *where* $n\in N$, $\tau_i : \Omega \to [0, T]$, $i=1 \ldots n$ *are* $(\mathcal{F}_t)$*-stopping times*; $\xi_i : \Omega \to R$ *are* $\mathcal{F}_{\tau_i}$*-measurable, and square integrable is called a strategy. The final gain of trade* $G_T(\xi)$ *of a strategy* (ξ) *is given by*

$$G_T(\xi) = \sum_{k=1}^{n} \xi_k (F_T - F_{\tau_k}) \ (\in L^2 (\Omega, \mathcal{F}_T, P)).$$

It may not seem to be obvious that we do not allow for continuous strategies. Note that usually a continuous strategy describes the amount held in some risky asset. The price fluctuation then determines the final gain of trade. The above context is slightly different: we do not hold a certain amount of an underlying asset, but at any time we can agree to an insurance future contract, which is settled at time T. As it stands, definition 2 allows for quite realistic strategies.

A strategy (ξ) allowing for a sure profit is called an *arbitrage strategy*, i.e., a strategy (ξ) satisfying:

$$G_T(\xi) \geq 0 \quad P \text{ - almost surely, and}$$
$$E_P [G_T(\xi)] > 0.$$

Under the natural assumption that (F_t) is a right-continuous process, the following theorem gives a necessary and sufficient condition for the absence of arbitrage strategies. The formulation turns out to be exactly the same as in the context of standard financial markets.

THEOREM 3 *Consider the insurance futures market* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$, *with right-continuous price process* $(F_t)_{0\leq t\leq T}$. *The following are then equivalent*:

1. $(F_t)_{0\leq t\leq T}$ *does not allow arbitrage strategies, and*
2. *there exists an equivalent measure* Q *(i.e.* $P\sim Q$*) such that* (F_t) *is an* $(\mathcal{F}_t)$*-Martingale under* Q.

For a proof, see Meister (1995, Proposition 3.6, Theorem 3.7).

Example 4

A common premium principle in insurance mathematics is the Esscher principle, see for instance Gerber and Shiu (1995). Applied to the pricing of insurance futures, the Esscher principle leads to

$$F_t \equiv E_Q (F_T \mid \mathcal{F}_T), \qquad 0\leq t\leq T,$$

where for some $\alpha>0$

$$\frac{dQ}{dP} = \frac{e^{\alpha F_T}}{E_P (e^{\alpha F_T})}.$$

Obviously $Q\sim P$ and (F_t) is a Q-Martingale. If (L_t) follows a doubly stochastic compound Poisson process, then (F_t) is certainly right-continuous; hence, the Esscher model is arbitrage free. For a further discussion on the relevance of the Esscher principle in general finance, see Bühlmann, Embrechts, and Shiryaev (1996).

Pricing and Replication in the Insurance Futures Market

Introduction

An insurance market is often not complete, and even complete insurance markets do in general allow for many equivalent Martingale measures. Hence, there exists several possibilities to price contingent claims excluding arbitrage opportunities. In this context the common approach to pricing is a preference dependent model. Preferences are usually described by von Neumann-Morgenstern utility functions. We shall distinguish between prices calculated by individual agents and equilibrium prices over a whole market.

An individual agent's objective is to maximize expected utility of wealth at a certain fixed time. The agent, therefore, only agrees to a position in an insurance future contract if it is an attractive investment compared with other possible investments. Hence, the insurance future's price should only depend on the agent's preferences and investment opportunities. A market equilibrium is the situation where by exchanging risks all agents can maximize their expected utility at the same time. Equilibrium prices are derived by changing the measure and taking the corresponding expectations under the new measure. In this case the insurance future's price should only depend on all the agents' utilities and investment opportunities.

In some situations, an agent wants to replicate a contingent claim by an engagement in the corresponding market. In an incomplete market, there exists claims which cannot be replicated in the sense that they do not admit an Itô representation. Thus, there remains some uncertainty about the replication cost.

Pricing of Insurance Futures in a Utility Maximization Framework

A recent study on the subject of option pricing in a utility maximization framework is due to Davis and Robeau (1994). Although one cannot apply the results to an insurance market context in a straightforward manner, we find it important to discuss these basic ideas. An interesting paper giving an easy introduction to utility functions in an insurance context is Gerber (1987).

An investor with utility u and a certain initial endowment x forms a dynamic portfolio. To determine the portfolio, he or she can make the choice of a strategy π out of the set $\mathcal{S}$ of possible strategies. The cash value of the portfolio at time t is $X_x^\pi(t)$. The objective is to maximize expected utility of wealth $X_x^\pi(T)$ at a fixed final time T. The investor asks the question whether the maximum utility can be increased by the purchase or short-selling of a European option whose cash value at time T is some nonnegative random variable, the purchase price at time $t=0$ being p. Thus, from the investor's point of view, p is a fair price for the option if diverting a little of the funds into it at time $t=0$ has a neutral effect on the investor's achievable utility.

This "marginal rate of substitution" argument leads then (under additional assumptions) to a general option pricing formula basically dependent on the set of strategies $\mathcal{S}$, the initial endowment x, and the utility function u.

Let the investor now be an insurance company. The company holds a portfolio of insurance policies for which it receives premiums, but also has to pay for occurred losses. Let $(P_t)_{0\le t\le T}$ denote the total value of premiums received up to time t, and $(Y_t)_{0\le t\le T}$ be the total value of claims occurring up to time t, both processes defined on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ where $\mathcal{F}_t=\sigma(P_s, Y_s, s\le t)$. We assume the existence of a liquid reinsurance market, i.e. at any time $t\le T$ the insurance company can decide to sell any fraction of the remaining risk $(Y_s)_{t\le s\le T}$, based on the information available at time t. To be more precise,

Definition 3 *If $t\in[0, T]$, a reinsurance strategy $(\xi_s)_{t\le s\le T}$ is a predictable stochastic process on $(\Omega, \mathcal{F}, (\mathcal{F}_s), P)$ with*

$$0 \le \xi_s \le 1 \text{ for all } s \in [t, T].$$

$\mathcal{H}_t$ denotes the set of all reinsurance strategies which "start" at time t.

Again, for expository purposes, we assume interest rates to be deterministic, described by the function $(r(t))_{0\le t\le T}$, where $r(t)$ denotes the value at time T of a cashflow arising from an investment of 1 at time t. We introduce, furthermore, the process $(X_t)_{0\le t\le T}$ given by

$$X_t = r(t)\,(P_t - Y_t),\ 0 \le t \le T,$$

denoting the inflated net earnings from the insurance business up to time t.

If the insurance company at time t chooses some reinsurance strategy $(\xi_s)\in\mathcal{H}_t$, then the company's final gain at time T (positive or negative) is given by

$$G_T(\xi) = \int_t^T \xi_s dX_s,$$

where we assume the reinsurance companies to receive primary insurer premiums for their engagement.

Assume the insurance company to have a utility function u, denoting the company's preferences. We assume u to be a C^2-function on R with $u'>0$, and $u''\leq 0$. The insurer's objective is to maximize expected utility of the final gain at time T by using the information $\mathcal{F}_t$. Let

$$V = \sup_{\xi \in \mathcal{A}_t} E_P\,[u(G_T(\xi))|\mathcal{F}_t]$$

be the maximum expected utility of the (inflated) final gain, and $\mathcal{A}_t$ be some subset of $\mathcal{H}_t$. The insurance company now asks the question whether its maximum utility V could be increased by the agreement to a long or a short position in an insurance future contract. The "marginal rate of substitution" argument is used as follows: F_t is a fair price for the insurance future at time t if "agreeing a little" into a contract has a neutral effect on the company's achievable utility. For the purpose of giving a mathematical formulation, we define for any δ and F_t:

$$W(\delta, F_t) = \sup_{\xi \in \mathcal{A}_t} E_P\,[u(G_T(\xi) + \delta(F_T - F_t))|\mathcal{F}_t]$$

and give the following:

Definition 4 *Suppose that for each F_t the function*

$$\delta \rightarrow W(\delta, F_t)$$

is differentiable at $\delta=0$, and there is a unique solution $\hat{F}_t$ of the equation

$$\frac{\partial W}{\partial \delta}(0, F_t) = 0.$$

Then $\hat{F}_t$ is the fair price (in the above sense) for the insurance future at time t.

In the following theorem, we can give *a* pricing formula for insurance futures:

Theorem 4 *Suppose that there exists $\xi \in \mathcal{A}_t$ such that*

$$V = E_P\,[u(G_T(\hat{\xi})) \mid \mathcal{F}_t]$$

and the function

$$\delta \rightarrow W(\delta, F_t)$$

is differentiable at $\delta=0$ for each F_t. Then the fair insurance future price at time t is given by

$$\hat{F}_t = \frac{E_P\,[u'(G_T(\hat{\xi}))\,F_T \mid \mathcal{F}_t]}{E_P\,[u'(G_T(\hat{\xi})) \mid \mathcal{F}_t]}.$$

The proof of this result is given in Meister (1995, Theorem 4.3). This result should also be compared with formula (17) in Gerber (1987).

Example 5

Consider the insurance company to have an exponential utility $u(x)=(1-e^{-\alpha x})$ with risk averseness $\alpha>0$ when it decides not to reinsure its claims, i.e., $\mathcal{A}_t=\{(1)_{t\leq s\leq T}\}$. Then we have

$$\hat{F}_t = \frac{E_P\,[e^{-\alpha(X_T - X_t)}\,F_T \mid \mathcal{F}_t]}{E_P\,[e^{-\alpha(X_T - X_t)} \mid \mathcal{F}_t]}.$$

Assume, furthermore, the premium process (P_t) to be deterministic. Then, using that Y_t is $\mathcal{F}_t$-measurable and $r(T)=1$, it follows that

$$\hat{F}_t = \frac{E_P\,[e^{\alpha(r(T)Y_T - r(t)Y_t)}\,F_T \mid \mathcal{F}_t]}{E_P\,[e^{\alpha(r(T)Y_T - r(t)Y_t)} \mid \mathcal{F}_t]} = \frac{E_P\,[e^{\alpha Y_T}\,F_T \mid \mathcal{F}_t]}{E_P\,[e^{\alpha Y_T} \mid \mathcal{F}_t]}.$$

Now replace the insurance company by the insurance market holding those policies which lead to losses flowing into the insurance future's index, i.e., for all $0\leq t\leq T$, $Y_t=L_t$. The fair insurance future price (from the insurance market's point of view), therefore, is (c=\$25,000/Π):

$$\hat{F}_t = \frac{E_P\,[e^{\alpha L_T}\,c\,(L_T \wedge 2\Pi) \mid \mathcal{F}_t]}{E_P\,[e^{\alpha L_T} \mid \mathcal{F}_t]}. \quad (3)$$

Showing that in this case the utility maximization approach essentially leads to the Esscher principle.

In order to work out formula (3), one needs to impose specific conditions on $(L_t)_{0\leq t\leq T}$, like the three basic assumptions in the loss process (L_t) section. For instance, under assumption one (compound Poisson case) one obtains

$$\hat{F}_t = c\,e^{\lambda(t-T)E_P(e^{\alpha X_1})} \sum_{k=0}^{\infty} \frac{(\lambda(T-t))^k}{k!}\,\eta_k$$

where

$$\eta_k = \int_0^{(2\Pi - L_t)^+} e^{\alpha s}(s + L_t)\,dF^{*k}(s) + 2\Pi(1 - F^{*k}(2\Pi - L_t)^+).$$

Here $F(x)=P(X_1 \leq x)$ is the jump (or claim) distribution, and F^{*k} denotes the kth convolution of F.

In the uncapped case,

$$F_T = \$25{,}000 \frac{L_T}{\Pi} \equiv c L_T$$

and gamma distributed claim amounts $X_1 \sim \Gamma(n, \mu)$, the above formula reduces to the very easy

$$\hat{F}_t = c \left\{ L_t + \lambda(T - t) \frac{n\mu^n}{(\mu - \alpha)^{n+1}} \right\}$$

which explicitly shows that $\hat{F}_t$ increases with L_t, λ, and α; $\hat{F}_t$ decreases as time to maturity T becomes shorter. For generalizations and further analysis see Aase (1994) and Meister (1995).

Remarks

1. As we have remarked in the section titled, Pricing By No-Arbitrage, insurance markets are often not complete; hence, there exist contingent claims which cannot be perfectly hedged in the sense that they do not admit an Itô representation. However, one can show that under rather weak assumptions, there exist "best" risk minimizing reinsurance strategies in the sense of Föllmer and Sondermann (1986), Föllmer and Schweizer (1989), and Schweizer (1990, 1994). Details on this approach concerning CAT futures are to be found in Meister (1995, Section 4.4). An excellent paper using equilibrium pricing is Aase (1994).
2. Whenever pricing formulas concerning a financial instrument are to be worked out in an incomplete market (allowing for various equivalent Martingale measures), it is useful to estimate the distribution of the contingent claim under the physical measure P and apply standard loading techniques as is often done in an insurance context. The latter is mostly better than a blindfold application of some (nonunique) Martingale pricing formula. Such an analysis has been given by Klüppelberg and Mikosch (1995). In the latter paper it is assumed that $(N_t)_{t\geq 0}$ is either a renewal process or a family of Poisson rv's with intensities $(\lambda(t))_{t\geq 0}$, such that $E(N_t) \equiv \lambda(t) \to \infty$, $t \to \infty$. The claim process equals $S_t = \Sigma_{i=1}^{N_t} X_i$, where $(X_n)_{n\in N}$ is a sequence of i.i.d. rv's having distribution function F and mean μ. In the presence of large (catastrophic) claims, a natural condition on F is of the type $\bar{F}(x) \equiv 1 - F(x) \sim x^{-\alpha} L(x)$ for $x \to \infty$, where $\alpha > 1$ and L is slowly varying, i.e., for all $t>0$,

$$\lim_{x\to\infty} \frac{L(xt)}{L(x)} = 1.$$

We denote this condition by $\bar{F} \in \mathcal{R}(-\alpha)$. See Bingham, Goldie, and Teugels (1987) for a detailed discussion on the latter conditions. For a discussion on the modelling of catastrophic claims, see Embrechts, Klüppelberg, and Mikosch (1997). Motivated by CAT futures, Klüppelberg and Mikosch (1997) estimate for large t, the distributional behavior of

$$V(t) \equiv \$25{,}000 \times \min \left(\frac{S_t}{c\lambda(t)\mu}, 2 \right)$$

for some safety loading $c>0$.

Theorem 5 *Under the above assumptions, the following asymptotic estimates hold.*

1. *If* $\alpha>1$, *then as* $t \to \infty$,

$$E_P(V(t)) = \$25{,}000 \frac{1}{c} \Big(1 + (1 + o(1)) \frac{(2c - 1)\lambda(t)}{\alpha - 1} \bar{F}((2c - 1)\mu\lambda(t)) \Big).$$

2. *If* $\alpha>2$, *then as* $t \to \infty$,

$$\mathrm{Var}_P(V(t)) = (\$25{,}000)^2 \frac{1}{c^2} \left(\frac{E(X_1^2)}{\mu^2\lambda(t)} + (1 + o(1)) \frac{2(2c - 1)^2\lambda(t)}{(\alpha - 2)} \bar{F}((2c - 1)\mu\lambda(t)) \right).$$

The key problem in obtaining results of the above type can be seen as follows:

$$E_P \left(\frac{S_t}{c\mu(t)} - K \right)^+ = \frac{1}{c\mu(t)} \int_{\gamma\mu(t)}^{\infty} P(S_t - \mu(t) > x)\, dx, \quad (4)$$

where we assume that $\gamma \equiv Kc - 1 > 0$. Letting $t \to \infty$ in (4), we need estimates on $P(S_t - \mu(t) > x)$ for $x = x(t) \to \infty$. This leads to the well-known area of large deviation results, however, under the nonstandard (heavy-tail) condition $\bar{F} \in \mathcal{R}(-\alpha)$. This is exactly the theory worked out in Klüppelberg and Mikosch (1995).

3. Throughout this paper, we have concentrated mostly on the pricing of the CAT futures themselves; clearly the same theory may be used to price derivatives on the CAT futures like options and

call-option spreads. See, for instance, Aase (1994) and Meister (1995).

4. The final word on the pricing of insurance derivatives has not yet been said. The present paper provides some insight into the underlying mathematical methodology. More and more, insurance products are coming onto the market containing a financial component of some sort. Both finance experts as well as actuaries will have to get to know the other expert's field better. We hope that our paper contributes toward closing the existing gap so that with the right methodology at hand, we can seriously start tackling risk securitization.

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