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An Actuarial Bridge to Option Pricing

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1. Introduction

Actuaries measure, model, and manage risks. Risk associated with the investment function is a major uncertainty faced by many insurance companies. Actuaries should have knowledge of the asset side of the balance sheet of an insurance company and how it relates to the liability side. Such knowledge includes the operation of financial markets, the instruments available to the insurance companies, the options imbedded in these instruments, and the methods of pricing such options and derivative securities.

In this paper we study the pricing of financial options and derivative securities, and their synthetic replication by means of the primitive securities. We show that a time-honored concept in actuarial science, the Esscher transform, is an efficient tool for pricing many options and contingent claims if the logarithms of the prices of the primitive securities are certain stochastic processes with stationary and independent increments. The Swedish actuary F. Esscher (1932) suggested that the Edgeworth approximation (a refinement of the normal approximation) yields better results, if it is applied to a modification or transformation of the original distribution of aggregate claims. Here, this Esscher transform is defined more generally as a change of measure for certain stochastic processes. An Esscher transform of such a process induces an equivalent probability measure on the process. The Esscher parameter or parameter vector is determined so that the discounted price of each primitive security is a martingale under the new probability measure. A derivative security or contingent claim is valued as the expectation, with respect to this equivalent martingale measure, of the discounted payoffs.

Although there may be more than one equivalent martingale measure, in general, the risk-neutral Esscher measure provides a unique and transparent answer, which can be justified if there is a representative investor maximizing his or her expected utility. The option price is unique whenever a self-financing replicating portfolio can be constructed. This is the case in the multidimensional geometric Brownian motion model and also in the multidimensional geometric shifted compound Poisson process model. The latter is at the same time simpler (in view of its sample paths) and richer (the former can be retrieved as a limit). The Esscher method can be extended to pricing the derivative securities of (possibly) dividend-paying stocks.

We show that, in the case of a multidimensional geometric Brownian motion model, the price of a European option does not depend on the interest rate, provided that the payoff is a homogeneous function of degree one with respect to the stock prices. Moreover, with the aid of Esscher transforms, a change of numeraire can be discussed in a concise way.

2. The Esscher Transform of a Single Random Variable

Let Y be a given random variable and h a nonzero real number for which the expectation

$$E[e^{hY}]$$

exists. The positive random variable

$$\frac{e^{hY}}{E[e^{hY}]} \tag{2.1}$$

can be used (as the Radon-Nikodym derivative) to define a new probability measure, which is *equivalent* to the old measure in the sense that they both have the same null sets (sets of measure zero). In other words, the old and new measures are mutually absolutely continuous. For a measurable function ψ, the expectation of the random variable $\psi(Y)$ with respect to the new measure is

$$E[\psi(Y); h] = \frac{E[\psi(Y)e^{hY}]}{E[e^{hY}]}. \tag{2.2}$$

We call this new measure the *Esscher measure* of parameter h. The corresponding distribution is usually called the *Esscher transform* in the actuarial literature (Esscher, 1932; Philipson, 1963; Jensen, 1991). In some statistical literature, the term *exponential tilting* is used to describe this change of measure.

The method of Esscher transforms was developed to approximate the aggregate claim amount distribution around a point of interest, y_0, by applying an analytic approximation (the first few terms of the Edgeworth series) to the transformed distribution with the parameter $h=h_0$ chosen such that the new mean is equal to y_0. Let

$$c(h) = \ln(E[e^{hY}]) \tag{2.3}$$

be the cumulant-generating function. Its first- and second-order derivatives are

$$c'(h) = \frac{E[Ye^{hY}]}{E[e^{hY}]} = E[Y; h] \tag{2.4}$$

and

$$c''(h) = \frac{E[Y^2e^{hY}]}{E[e^{hY}]} - \left(\frac{E[Ye^{hY}]}{E[e^{hY}]}\right)^2 = \mathrm{Var}[Y; h]. \tag{2.5}$$

Since $\mathrm{Var}[Y; h]>0$ for a nondegenerate random variable Y, the function $c'(h)$ is strictly increasing; thus the number h_0 for which

$$y_0 = c'(h_0) = E[Y; h_0]$$

is unique. In using the Esscher transform to calculate a stop-loss premium, the parameter h_0 is usually chosen such that the mean of the transformed distribution is the retention limit.

3. Discrete-Time Stock-Price Models

A purpose of this paper is to show that the concept of Esscher measures is an effective tool for pricing stock options and other derivative securities. We need to extend the change of measure for a single random variable to that for a stochastic process. In this section we consider the simpler case of discrete-time stochastic processes.

For $j=0, 1, 2, \ldots$, let $S(j)$ denote the price of a stock a time j. Assume that there is a sequence of independent (but not necessarily identically distributed) random variables $\{Y_k\}$ such that

$$S(j) = S(0)\exp(Y_1 + Y_2 + \cdots + Y_j), \quad j = 1, 2, 3, \ldots \tag{3.1}$$

Assume that the moment generating function for each Y_i exists, and write

$$M_{Y_i}(h) = E[e^{hY_i}]. \tag{3.2}$$

For a sequence of real numbers $\{h_k\}$, define

$$Z_j = \exp\left(\sum_{k\le j} h_k Y_k\right)\Big/\prod_{k\le j} M_{Y_k}(h_k). \tag{3.3}$$

Then $\{Z_j\}$ is a positive martingale which can be used to define a change of measure for the stock-price process. For a positive integer m, let $\psi(m)$ be a random variable that is a function of $Y_1, \ldots, Y_m$,

$$\psi(m) = \psi(Y_1, \ldots, Y_m). \tag{3.4}$$

The expected value of $\psi(m)$, with respect to the new measure, is

$$E[\psi(m)\, Z_m]. \tag{3.5}$$

In (3.5) the random variable Z_m can be replaced by Z_j, $j\ge m$, because of the martingale property.

We assume that the risk-free interest rate is constant through time and the stock pays no dividends. Let δ denote the risk-free force of interest. The *risk-neutral Esscher measure* is the measure, defined by the sequence of numbers $\{h_k^*\}$, with respect to which

$$\{e^{-\delta j}S(j); j = 0, 1, 2, \ldots\} \quad (3.6)$$

is a martingale. This leads to

$$e^{\delta} = M_{Y_k}(1 + h_k^*)/M_{Y_k}(h_k^*), \quad k = 1, 2, 3, \ldots \quad (3.7)$$

As we pointed out at the end of the last section, the numbers $\{h_k^*\}$ are unique.

Suppose that each Y_k is a Bernoulli random variable, i.e., it takes on two distinct values, a_k and b_k, only. Then there is only one risk-neutral measure, given by

$$Pr^*(Y_k = a_k) = \frac{e^{\delta} - e^{b_k}}{e^{a_k} - e^{b_k}} \quad (3.8)$$

and

$$Pr^*(Y_k = b_k) = \frac{e^{\delta} - e^{a_k}}{e^{b_k} - e^{a_k}}. \quad (3.9)$$

(To rule out arbitrage opportunities we assume that the force of interest δ is between a_k and b_k for each k.)

If we assume that the random variables $\{Y_k\}$ are identically distributed in addition to being independent, then all h_k^* are the same number. This points to an approach to extend the change of measure to certain continuous-time models, as we shall see in Section 5. On the other hand, the risk-neutral Esscher measure can also be defined for dependent random variables $\{Y_k\}$. In this more general situation, each h_k^* is a function of $Y_1, Y_2, \ldots, Y_{k-1}$ and thus a random variable itself.

4. Fundamental Theorem of Asset Pricing

In this paper we assume that the market is frictionless and trading is continuous. There are no taxes, no transaction costs, and no restriction on borrowing or short sales. All securities are perfectly divisible. It is now understood that, in such a security market model, the absence of arbitrage is "essentially" equivalent to the existence of a *risk-neutral measure* or an *equivalent martingale measure*, with respect to which the *price* of a random payment is the expected discounted value. Dybvig and Ross (1987) call this result the *Fundamental Theorem of Asset Pricing*. In general, there may be more than one equivalent martingale measure. A merit of the risk-neutral Esscher measure is that it provides a general, transparent, and unambiguous solution.

That the condition of no arbitrage is intimately related to the existence of an equivalent martingale measure was first pointed out in Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983). Their results are rooted in the idea of the risk-neutral valuation of Cox and Ross (1976). In a finite discrete-time model, the absence of arbitrage opportunities is equivalent to the existence of an equivalent martingale measure (Dalang, Morton, and Willinger 1990; Schachermayer 1992b). In a more general setting the characterization is delicate, and we have to replace the term "equivalent to" by "essentially equivalent to." It is beyond the scope of the present paper to discuss the details. Some recent papers are Artzner and Heath (1995), Back (1991), Back and Pliska (1991), Christopeit and Musiela (1994), Cox and Huang (1989), Delbaen (1992), Delbaen and Schachermayer (1994a, 1994b), Frittelli and Lakner (1994), Müller (1989), Schachermayer (1992a, 1994), Schweizer (1992), and Stricker (1993).

We note that the idea of changing the probability measure to obtain a consistent positive linear pricing rule has appeared in the actuarial literature in the context of equilibrium reinsurance markets. See Borch (1960, 1990), Bühlmann (1980, 1984), Deprez and Gerber (1985), Lienhard (1986), Gerber (1987), Sonderman (1991), Aase (1993a, 1993b), and Chevallier and Müller (1994).

5. Continuous-Time Stock-Price Models

In the rest of the paper, we consider continuous-time stock-price models. For $t \geq 0$, let $S(t)$ denote the price at time t of a nondividend-paying stock. We assume that there is a stochastic process $\{X(t)\}$ with independent and stationary increments such that

$$S(t) = S(0)\, e^{X(t)}, \quad t \geq 0. \quad (5.1)$$

For a theoretical "justification" that stock prices should be modeled with such processes, see Samuelson (1965) or Parkinson (1977). (Some authors call $\{X(t)\}$ a *Lévy process*.) To rule out arbitrage opportunities, we need the condition that $X(t) - \delta t$ assumes positive and negative values. If this were not the case, for example, if $X(t) - \delta t \geq 0$ for all t, we would have

$$S(0)e^{\delta t} \leq S(0)e^{X(t)};$$

thus, by borrowing $S(0)$ and investing it in the stock, one could make a sure profit (unless $X(t) \equiv \delta t$). This condition is analogous to the condition in the Bernoulli example at the end of Section 3 that δ is between a_k and b_k.

We assume that the moment generating function of $X(t)$,

$$M(h, t) = E[e^{hX(t)}],$$

exists and that

$$M(h, t) = M(h, 1)^t. \tag{5.2}$$

The process

$$\{e^{hX(t)} M(h, 1)^{-t}\} \tag{5.3}$$

is a positive martingale and can be used to define a change of probability measure, i.e., it can be used to define the Radon-Nikodym derivative dQ/dP, where P is the original probability measure and Q is the Esscher measure of parameter h. The *risk-neutral Esscher measure* is the Esscher measure of parameter $h=h^*$ such that the process

$$\{e^{-\delta t} S(t)\} \tag{5.4}$$

is a martingale.

The condition

$$E[e^{-\delta t}S(t); h^*] = e^{-\delta 0}S(0) = S(0)$$

yields

$$\begin{aligned} e^{\delta t} &= E[e^{(1+h^*)X(t)} M(h^*, 1)^{-t}] \\ &= [M(1 + h^*, 1)/M(h^*, 1)]^t, \end{aligned}$$

or

$$e^{\delta} = M(1 + h^*, 1)/M(h^*, 1), \tag{5.5}$$

which is analogous to (3.7) with $\{Y_k\}$ being identically distributed. The parameter h^* is unique. There may be many other equivalent martingale measures.

Because, for $t \geq 0$,

$$e^{hX(t)} M(h, 1)^{-t} = \frac{e^{hX(t)}}{E[e^{hX(t)}]} = \frac{S(t)^h}{E[S(t)^h]}, \tag{5.6}$$

we have the following: Let g be a measurable function and h, k, and t be real numbers, $t \geq 0$; then

$$\begin{aligned} &E[S(t)^k g(S(t)); h] \\ &= E[S(t)^k g(S(t))\, e^{hX(t)} M(h, 1)^{-t}] \\ &= \frac{E[S(t)^{h+k} g(S(t))]}{E[S(t)^h]} \\ &= \frac{E[S(t)^{h+k}]}{E[S(t)^h]} \frac{E[S(t)^{h+k} g(S(t))]}{E[S(t)^{h+k}]} \\ &= E[S(t)^k; h]\, E[g(S(t)); h + k]. \end{aligned} \tag{5.7}$$

This *factorization formula* simplifies many calculations and is a main reason why the method of Esscher measures is an efficient device for valuing certain derivative securities. For example, applying (5.7) with $k=1$, $h=h^*$ and $g(x)=I(x>K)$ [where $I(A)$ denotes the indicator random variable of an event A], we obtain

$$\begin{aligned} &E[S(\tau)\, I(S(\tau) > K); h^*] \\ &\quad = E[S(\tau); h^*]\, E[I(S(\tau) > K); h^* + 1] \\ &\quad = E[S(\tau); h^*]\, Pr[S(\tau) > K; h^* + 1] \\ &\quad = S(0)e^{\delta\tau}\, Pr[S(\tau) > K; h^* + 1]. \end{aligned} \tag{5.8}$$

The last equality holds because (5.4) is a martingale with respect to the risk-neutral Esscher measure. Thus we have a pricing formula for a European call option on a nondividend-paying stock,

$$\begin{aligned} &E[e^{-\delta\tau} (S(\tau) - K)_+; h^*] \\ &= E[e^{-\delta\tau} (S(\tau) - K)\, I(S(\tau) > K); h^*] \\ &= e^{-\delta\tau}\{E[S(\tau)\, I(S(\tau) > K); h^*] \\ &\quad - KE[I(S(\tau) > K); h^*]\} \\ &= S(0)Pr[S(\tau) > K; h^* + 1] \\ &\quad - Ke^{-\delta\tau}Pr[S(\tau) > K; h^*]. \end{aligned} \tag{5.9}$$

For $\{X(t)\}$ being a Wiener process, (5.9) is the celebrated Black-Scholes formula; see also (10.20) below.

6. Representative Investor with Power Utility Function

When there is more than one equivalent martingale measure, why should the option price be the expectation, with respect to the risk-neutral Esscher measure, of the discounted payoff? This particular choice may be justified within a utility function framework. Consider a simple economy with only a stock and a risk-free bond and their derivative securities. There is a representative investor who owns m shares of the stock and bases his or her decisions on a risk-averse utility function $u(x)$. Consider a derivative security that provides a payment of $\pi(\tau)$ at time τ, $\tau>0$; $\pi(\tau)$ is a function of the stock price process until time τ. What is the investor's *price* for the derivative security, such that it is optimal for him or her not to buy or sell any fraction or multiple of it? Let $V(0)$ denote this price. Then, mathematically, this is the condition that the function

$$\phi(\eta) = E[u(mS(\tau) + \eta[\pi(\tau) - e^{\delta\tau}V(0)])] \tag{6.1}$$

is maximal for $\eta=0$. From

$$\phi'(0) = 0,$$

we obtain

$$V(0) = e^{-\delta\tau} \frac{E[\pi(\tau)u'(mS(\tau))]}{E[u'(mS(\tau))]} \tag{6.2}$$

(as a necessary and sufficient condition, since $\phi''(\eta)<0$ if $u''(x)<0$). In the particular case of a power utility function with parameter $c>0$,

$$u(x) = \begin{cases} \dfrac{x^{1-c}}{1-c} & \text{if } c \neq 1 \\ \ln x & \text{if } c = 1 \end{cases}, \tag{6.3}$$

we have $u'(x)=x^{-c}$, and

$$\begin{aligned} V(0) &= e^{-\delta\tau} \frac{E[\pi(\tau)[mS(\tau)]^{-c}]}{E[[mS(\tau)]^{-c}]} \\ &= e^{-\delta\tau} \frac{E[\pi(\tau)S(\tau)^{-c}]}{E[S(\tau)^{-c}]}. \end{aligned} \tag{6.4}$$

Formula (6.4) must hold for all derivative securities. For $\pi(\tau)=S(\tau)$ and therefore $V(0)=S(0)$, (6.4) becomes

$$\begin{aligned} S(0) &= e^{-\delta\tau} \frac{E[S(\tau)^{1-c}]}{E[S(\tau)^{-c}]} \\ &= e^{-\delta\tau} S(0) \frac{M(1-c,\tau)}{M(-c,\tau)}, \end{aligned}$$

or

$$e^{\delta} = \frac{M(1-c, 1)}{M(-c, 1)}. \tag{6.5}$$

On comparing (6.5) with (5.5), we see that the value of the parameter c is $-h^*$. Hence $V(0)$ is indeed the discounted expectation of the payoff $\pi(\tau)$, calculated with respect to the Esscher measure of parameter $h^*=-c$.

By considering different points in time τ, we get a consistency requirement. This is satisfied if the investor has a power utility function. We conjecture that it is violated for any other risk-averse utility function, which implies that the pricing of an option by the risk-neutral Esscher measure is a consequence of the consistency requirement. Some related papers are Rubinstein (1976), Bick (1987, 1990), Constantinides (1989), Naik and Lee (1990), Stapleton and Subrahmanyam (1990), He and Leland (1993), Heston (1993), and Wang (1993).

7. Logarithm of Stock Price as a Shifted Poisson Process

Here we consider the so-called pure jump model. The assumption is

$$X(t) = kN(t) - ct, \tag{7.1}$$

where $\{N(t)\}$ is a Poisson process with parameter λ, and k and c are constants with $k\neq 0$. Then the price of the nondividend-paying stock is modeled as

$$S(t) = S(0)e^{kN(t)-ct}. \tag{7.2}$$

The condition that $X(t)-\delta t$ assumes positive and negative values is that k and $c+\delta$ have the same sign. This model contains the classical Wiener process model as a limiting case. Note that

$$E[X(1)] = k\lambda - c \tag{7.3}$$

and

$$\mathrm{Var}[X(1)] = k^2\lambda. \tag{7.4}$$

Suppose that we vary k, λ, and c so that (7.3) and (7.4) remain the constant values μ and σ^2, respectively, i.e., we set

$$\lambda = (\sigma/k)^2 \tag{7.5}$$

and

$$c = (\sigma^2/k) - \mu. \tag{7.6}$$

In the limit as $k\to 0$, $\{X(t)\}$ has continuous sample paths, and hence it is a Wiener process with drift μ and infinitesimal variance σ^2. This is illustrated in the two graphs in Figure 1. We note that the discontinuities of the second sample path are not recognizable and that it appears to be a sample path of a Wiener process.

We now determine the risk-neutral Esscher measure according to Section 5. Since

$$E[e^{zN(t)}] = \exp[\lambda t(e^z - 1)],$$

we have

$$\begin{aligned} M(z, t) &= E[e^{zX(t)}] \\ &= E(e^{z[kN(t)-ct]}) \\ &= \exp([\lambda(e^{zk} - 1) - zc]t). \end{aligned} \tag{7.7}$$

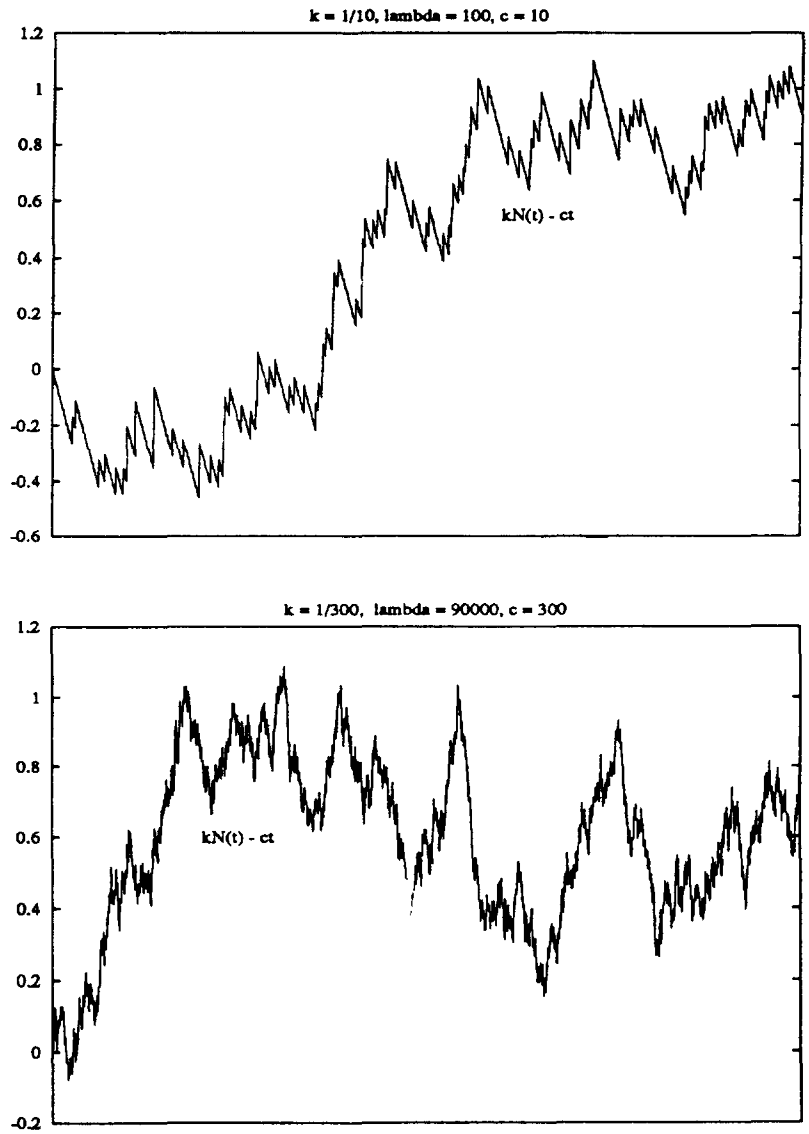
Because

$$\begin{aligned} E[e^{zX(t)}; h] &= \frac{M(z+h, t)}{M(h, t)} \\ &= \exp([\lambda e^{hk}(e^{zk} - 1) - zc]t), \end{aligned}$$

we see that, under the Esscher measure of parameter h, the process $\{X(t)\}$ remains a shifted Poisson process, but with modified Poisson parameter λe^{hk}. Formula (5.5) is the condition that

$$\delta = \lambda e^{h^*k}(e^k - 1) - c. \tag{7.8}$$

FIGURE 1
SAMPLES PATHS OF TWO SHIFTED POISSON PROCESSES WITH $\mu = 0$ AND $\sigma^2 = 1$



Hence, the risk-neutral Esscher measure is the measure with respect to which $\{N(t)\}$ becomes a Poisson process with parameter

$$\begin{aligned}\lambda^* &= \lambda e^{h^*k}.\\ &= (\delta + c)/(e^k - 1).\end{aligned} \tag{7.9}$$

The ratio in (7.9) is indeed a positive number because we have imposed the condition that k and $\delta+c$ have the same sign.

Consider a European option or contingent claim with exercise date τ and payoff function $\Pi(s)$. [At time τ the option owner receives $\Pi(S(\tau))$.] The method of the Esscher transforms prices the option as the expected discounted payoff, with the expectation taken with respect to the risk-neutral Esscher measure. That is, for $t \leq \tau$, the *Esscher option price* at time t, with $S(t)=s$, is the conditional expectation.

$$\begin{aligned}V(s, t) &= E[e^{-\delta(\tau-t)}\Pi(S(\tau))|S(t) = s; h^*]\\ &= e^{-\delta(\tau-t)} \sum_{n=0}^{\infty} Pr[N(\tau) - N(t) = n; h^*]\, \Pi(se^{nk-c(\tau-t)})\\ &= e^{-(\delta+\lambda^*)(\tau-t)} \sum_{n=0}^{\infty} \frac{[\lambda^*(\tau - t)]^n}{n!}\, \Pi(se^{nk-c(\tau-t)}).\end{aligned} \tag{7.10}$$

In this stock price model, the option price is unique and given by (7.10). To see this, we construct a self-financing portfolio of the stock and risk-free bond whose value at time t is $V(S(t), t)$. The amounts invested in the stock and bond are dynamically adjusted. The term *self-financing* means that, once started, no funds are added or withdrawn from the portfolio until the option exercise date τ. As $t \rightarrow \tau$, $V(S(t), t) \rightarrow \Pi(S(\tau))$ with certainty; hence, at time τ, the value of the self-financing portfolio is equal to the payoff of the option. Consequently, the option price at any previous point in time must be identical to the portfolio value at that time, i.e., the option price is indeed the Esscher option price (7.10).

For $t<\tau$, let $\eta(S(t), t)$ denote the amount in the portfolio invested in the stock at time t; therefore the difference $V(S(t), t) - \eta(S(t), t)$ is the amount invested in the risk-free bond at time t. The crucial question is whether we can define $\eta(S(t), t)$ so that the portfolio is self-financing, i.e., that the investment gain of the portfolio is identical to the change of the portfolio value, as defined by V, in any time interval and in any situation. We have to examine two scenarios.

First, we consider the case where the stock price has a discontinuity at time t, jumping from $S(t)$ to $S(t)e^k$. Then the condition that the instantaneous investment gain is equal to the instantaneous change of the portfolio value yields the equation

$$\begin{aligned}&\eta(S(t), t)e^k - \eta(S(t), t)\\ &\quad = V(S(t)e^k, t) - V(S(t), t),\end{aligned} \tag{7.11}$$

resulting in the condition

$$\eta(S(t), t) = \frac{V(S(t)e^k, t) - V(S(t), t)}{e^k - 1}. \tag{7.12}$$

Second, we consider the case where the stock price process does not have a jump in a time interval around a certain point t_0. Let $S(t_0)=s$. For t in the time interval, we have

$$S(t) = se^{-c(t-t_0)} \tag{7.13}$$

and

$$\begin{aligned}&V(S(t), t)\\ &= V(se^{-c(t-t_0)}, t)\\ &= e^{-(\delta+\lambda^*)(\tau-t)} \sum_{n=0}^{\infty} \frac{[\lambda^*(\tau - t)]^n}{n!}\, \Pi(se^{nk-c(\tau-t_0)}).\end{aligned} \tag{7.14}$$

Thus

$$\begin{aligned}&\frac{d}{dt} V(S(t), t)\\ &\quad = (\delta + \lambda^*)V(S(t), t) - \lambda^* V(S(t)e^k, t),\end{aligned} \tag{7.15}$$

and the instantaneous change of the portfolio value is

$$[(\delta + \lambda^*)V(S(t), t) - \lambda^* V(S(t)e^k, t)]dt. \tag{7.16}$$

On the other hand, the instantaneous investment gain of the portfolio is

$$\eta(S(t), t)(-cdt) + [V(S(t), t) - \eta(S(t), t)](\delta dt). \tag{7.17}$$

Here, the condition that the instantaneous investment gain is equal to the instantaneous change of the portfolio value yields the equation

$$\begin{aligned}&-c\eta(S(t), t) + \delta[V(S(t), t) - \eta(S(t), t)]\\ &\quad = (\delta + \lambda^*)V(S(t), t) - \lambda^* V(S(t)e^k, t),\end{aligned}$$

or

$$\eta(S(t), t) = \frac{\lambda^*}{c + \delta} [V(S(t\, e^k, t) - V(S(t), t)]. \tag{7.18}$$

Since λ^* is defined by (7.9), conditions (7.12) and (7.18) are equivalent, and with this choice of $\eta(S(t), t)$, the portfolio is indeed self-financing.

Observe that, in constructing the self-financing portfolio which replicates the option payoff, we did not need $\{N(t)\}$ to be a Poisson process. The self-financing portfolio can be constructed because in each infinitesimal time interval exactly two scenarios are possible: a jump with known magnitude, or no jumps. Thus $N(t)$ in (7.1) and (7.2) can be assumed to come from a more general class of counting processes; the equivalent martingale measure is the measure with respect to which $\{N(t)\}$ becomes a Poisson process with parameter λ^* given by (7.9).

It is of interest to consider the limiting case where $k \to 0$, with λ and c varying according to (7.5) and (7.6). With respect to the risk-neutral Esscher measure, the drift and infinitesimal variance of the process (7.1) are $k\lambda^* - c$ and $k^2\lambda^*$, respectively. Because of (7.6),

$$\lim_{k \to 0} kc = \sigma^2. \tag{7.19}$$

Hence

$$\begin{aligned} k\lambda^* - c &= k\frac{\delta + c}{e^k - 1} - c \\ &= \delta\frac{k}{e^k - 1} - c\frac{e^k - 1 - k}{e^k - 1} \\ &\to \delta - \frac{\sigma^2}{2} \end{aligned} \tag{7.20}$$

and

$$\begin{aligned} k^2\lambda^* &= k^2\frac{\delta + c}{e^k - 1} \\ &\to \sigma^2 \end{aligned} \tag{7.21}$$

as $k \to 0$. Thus, in the limit the risk-neutral Esscher measure corresponds to the Wiener process with drift (7.20) and infinitesimal variance (7.21).

Furthermore, in the limit $(k \to 0)$, formula (7.12) becomes

$$\eta(S(t), t) = S(t)V_s(S(t), t), \tag{7.22}$$

showing that the number of shares in the replicating portfolio at time t, $\eta(S(t),t)/S(t)$, is given by the partial derivative $V_s(S(t),t)$, which is usually called *delta* in the option literature. Also, by means of the Taylor expansion, we have

$$\begin{aligned} \lambda^*[V(S(t)e^k, t) &- V(S(t), t)] \\ &= \lambda^*\{(e^k - 1)S(t)V_s(S(t), t) \\ &\quad + [(e^k - 1)S(t)]^2V_{ss}(S(t), t)/2 + \ldots\} \\ &= (\delta + c)S(t)V_s(S(t), t) \\ &\quad + \sigma^2S(t)^2V_{ss}(S(t), t)/2 + \ldots \end{aligned} \tag{7.23}$$

Substituting (7.23) in the right-hand side of (7.15) and

$$\frac{d}{dt}V(S(t), t) = V_s(S(t), t)[-cS(t)] + V_t(S(t), t) \tag{7.24}$$

in its left-hand side, canceling the cSV_s terms, and letting k tend to 0 yields the equation

$$\begin{aligned} V_t(S(t), t) &= \delta V(S(t), t) - \delta S(t)V_s(S(t), t) \\ &\quad - \frac{\sigma^2}{2}S(t)^2V_{ss}(S(t), t), \end{aligned} \tag{7.25}$$

which was first derived by Black and Scholes (1973) with a replicating portfolio argument.

8. Extension to Multiple Assets

In this section we extend the model in the last section to more than one nondividend-paying stock. For $j = 1, 2, \ldots, n$, let $S_j(t)$ denote the price of stock j at time t, $t \geq 0$, and write

$$X_j(t) = ln[S_j(t)/S_j(0)]. \tag{8.1}$$

Generalizing (7.1) we model the processes $\{X_1(t)\}$, $\{X_2(t)\}, \ldots, \{X_n(t)\}$ as shifted compound Poisson processes with

$$E[X_j(t)] = \mu_j t \tag{8.2}$$

and

$$\mathrm{Cov}(X_i(t), X_j(t)) = \sigma_{ij}t, \tag{8.3}$$

where μ_j and σ_{ij}, $1 \leq i, j \leq n$, are constants with the n-by-n matrix

$$(\sigma_{ij}) \tag{8.4}$$

being positive definite. Let $k \neq 0$, and let $\{N_1(t)\}$, $\{N_2(t)\}, \ldots, \{N_n(t)\}$ be n independent Poisson processes, each with the same parameter value

$$\lambda = k^{-2}. \tag{8.5}$$

Let $\{a_{ij};\ 1 \leq i, j \leq n\}$ be n^2 numbers such that

$$\sum_{j=1}^{n} a_{ij}\, a_{hj} = \sigma_{ih}, \quad 1 \leq i, h \leq n. \tag{8.6}$$

(The numbers a_{ij} are not unique. One way to obtain them is the Choleski factorization algorithm. As n-by-n matrices, they are related to each other by left multiplication with orthogonal matrices.) For $i=1, 2, \ldots, n$, define

$$c_i = \frac{1}{k}\sum_{j=1}^{n} a_{ij} - \mu_i \tag{8.7}$$

and

$$X_i(t) = k\sum_{j=1}^{n} a_{ij}N_j(t) - c_i t. \tag{8.8}$$

Then (8.2) and (8.3) are satisfied. (To reconcile with the notation in this section, one would replace the k in the last section by $k\sigma$.)

Let

$$\mathbf{X}(t) = (X_1(t), X_2(t), \ldots, X_n(t))'. \tag{8.9}$$

The process $\{\mathbf{X}(t)\}$ has independent and stationary increments with drift vector

$$(\mu_1, \mu_2, \ldots, \mu_n)' \tag{8.10}$$

and covariance matrix per unit time (8.4). With μ_i and a_{ij} held fixed, as $k \to 0$ [λ and c_i varying according to (8.5) and (8.7)], the limiting stochastic process $\{\mathbf{X}(t)\}$ has continuous sample paths and is thus an n-dimensional Wiener process with drift vector (8.10) and diffusion matrix (8.4).

To rule out arbitrage opportunities in the model of the last section, we imposed the condition that k and $\delta+c$ have the same sign, which, in turn, guarantees that the new Poisson parameter as defined by (7.9),

$$\lambda^* = (\delta + c)/(e^k - 1),$$

is positive. Here, we need to generalize the condition to one on the parameters a_{ij}, c_i, k, and δ. We make the following assumption: if $\eta_1, \eta_2, \ldots, \eta_n$ are any n real numbers such that

$$\sum_{i=1}^{n} \eta_i(e^{a_{ij}k} - 1) \geq 0, j = 1, 2, \ldots, n, \tag{8.11}$$

with strict inequality for at least one j, then this implies that

$$\sum_{i=1}^{n} \eta_i(c_i + \delta) > 0. \tag{8.12}$$

This assumption can be justified by an arbitrage argument. If it were violated, there would be n real numbers, $\eta_1, \eta_2, \ldots, \eta_n$, satisfying (8.11), with strict inequality for at least one j, and such that

$$\sum_{i=1}^{n} \eta_i(c_i + \delta) \leq 0. \tag{8.13}$$

Inequality (8.13) is equivalent to

$$\sum_{i=1}^{n} \eta_i + \left[\sum_{i=1}^{n} \eta_i(-c_i)\right] \bar{s}_{\overline{t}|} \geq \left(\sum_{i=1}^{n} \eta_i\right) e^{\delta t}, \quad t \geq 0, \tag{8.14}$$

both sides of which have economic interpretations. If the amount of

$$\sum_{i=1}^{n} \eta_i$$

is invested in the risk-free bond at time 0, the expression on the right-hand side is the (accumulated) value of the investment at time t. An alternative, more sophisticated investment strategy is to invest the amount of η_i in stock i at time 0 and keep this amount fixed at all subsequent times by investing all gains (or losses) in the risk-free bond, $i=1, 2, \ldots, n$. The expression on the left-hand side of (8.14) is the value at time t of this investment portfolio if no jumps have yet occurred. When the first jump occurs, say, due to Poisson process j, the instantaneous change of portfolio value is

$$\sum_{i=1}^{n} \eta_i(e^{a_{ij}k} - 1). \tag{8.15}$$

Because (8.15) is nonnegative for all j and positive for at least one, we see how a risk-free profit can be made: by selling a bond of the amount

$$\sum_{i=1}^{n} \eta_i$$

and investing this amount in the n stocks according to the strategy described above.

We assume that the n-by-n matrix $(e^{a_{ij}k}-1)$ is nonsingular. [This is a relatively weak assumption: since the matrix (a_{ij}) is nonsingular and $k\neq 0$, it is satisfied if $|k|$ is sufficiently small or if (a_{ij}) is triangular.] Let $\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*$ be the solution of the system of equations

$$\sum_{j=1}^{n} \lambda_j^* (e^{a_{ij}k} - 1) = c_i + \delta, \quad i = 1, 2, \ldots, n. \tag{8.16}$$

Analogous to λ^* defined by (7.9), each λ_j^* is positive. To see this, let $\eta_1, \eta_2, \ldots, \eta_n$ be the solution of the system of equations

$$\sum_{i=1}^{n} \eta_i(e^{a_{ij}k} - 1) = 1$$

$$\sum_{i=1}^{n} \eta_i(e^{a_{im}k} - 1) = 0,\ m \neq j,\ m = 1, 2, \ldots, n. \quad (8.17)$$

Hence, (8.11) is satisfied with one strict inequality, and according to (8.12) we have

$$\begin{aligned} 0 &< \sum_{i=1}^{n} \eta_i(c_i + \delta) \\ &= \sum_{i=1}^{n} \eta_i \sum_{m=1}^{n} \lambda_m^* (e^{a_{im}k} - 1) \\ &= \sum_{m=1}^{n} \lambda_m^* \sum_{i=1}^{n} \eta_i(e^{a_{im}k} - 1) \\ &= \lambda_j^*. \end{aligned} \quad (8.18)$$

Extending the self-financing portfolio argument in the last section, we show that the price of a European option or contingent claim of the n stocks is the expectation of its discounted payoff, with the expectation taken with respect to a certain modified probability measure, which is unique. We introduce the function $V(s_1, s_2, \ldots, s_n, t)$, $t \leq \tau$, defined as the discounted conditional expectation

$$\begin{aligned} &V(s_1, s_2, \ldots, s_n, t) \\ &\quad = e^{-\delta(\tau - t)} E^*[\Pi(S_1(\tau), S_2(\tau), \ldots, S_n(\tau)) \\ &\qquad |S_1(t) = s_1, S_2(t) = s_2, \ldots, S_n(t) = s_n], \end{aligned} \quad (8.19)$$

where the expectation is to be taken with respect to the new Poisson parameters $\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*$ defined by (8.16). [For $j=1, 2, \ldots, n$, the Poisson process $\{N_j(t)\}$ is to have the new Poisson parameter λ_j^*.] The option price at time t, $t \leq \tau$, is necessarily

$$V(S_1(t), S_2(t), \ldots, S_n(t), t). \quad (8.20)$$

To prove this, we note that (8.20) converges to $\Pi(S_1(\tau), S_2(\tau), \ldots, S_n(\tau))$ with certainty for $t \to \tau$, and we construct a self-financing portfolio of the stocks and risk-free bond whose value at time t is precisely given by (8.20). In this portfolio, let

$$\eta_j(S_1(t), S_2(t), \ldots, S_n(t), t)$$

be the amount invested in stock j at time t; therefore the difference

$$V(S_1(t), \ldots, S_n(t), t) - \sum_{j=1}^{n} \eta_j(S_1(t), \ldots, S_n(t), t) \quad (8.21)$$

is the amount invested in the risk-free bond at time t. We have to show that it is possible to choose the quantities $\{\eta_j(S_1(t), S_2(t), \ldots, S_n(t), t), j=1, 2, \ldots, n\}$ such that the portfolio is self-financing, i.e., that the change of the portfolio value is equal to the investment gain in any time interval under each scenario.

We have to examine $n+1$ scenarios. (In an infinitesimally small time interval, exactly one of $n+1$ events will take place: either none of the n independent Poisson processes has a jump, or else exactly one of them has a jump.) If Poisson process j has a jump at time t, the price of stock i jumps from $S_i(t)$ to $S_i(t)e^{a_{ij}k}$, $i=1, 2, \ldots, n$, and the portfolio value changes from $V(S_1(t), S_2(t), \ldots, S_n(t), t)$ to

$$V(S_1(t)e^{a_{1j}k}, S_2(t)e^{a_{2j}k}, \ldots, S_n(t)e^{a_{nj}k}, t).$$

For the portfolio to be self-financing, the change must be identical to the investment gain, yielding the equation

$$\begin{aligned} &V(S_1(t)e^{a_{1j}k}, \ldots, S_n(t)e^{a_{nj}k}, t) - V(S_1(t), \ldots, S_n(t), t) \\ &\qquad = \sum_{i=1}^{n} \eta_i(S_1(t), \ldots, S_n(t), t)\,(e^{a_{ij}k} - 1). \end{aligned} \quad (8.22)$$

There are n such equations, one for each of the Poisson processes ($j=1, 2, \ldots, n$). The solution values $\{\eta_i(S_1(t), \ldots, S_n(t), t), i=1, 2, \ldots, n\}$ of these n simultaneous equations are the amounts of stocks in the self-financing portfolio at time t.

Next, we examine the scenario of a time interval, say around t_0, in which none of the Poisson processes has a jump. For t in this interval,

$$S_i(t) = S_i(t_0)e^{-c_i(t - t_0)}, \quad i = 1, 2, \ldots, n. \quad (8.23)$$

Then, generalizing (7.15), we have

$$\begin{aligned} &\frac{d}{dt} V(S_1(t), \ldots, S_n(t), t) \\ &\qquad = (\delta + \sum_{j=1}^{n} \lambda_j^*) V(S_1(t), \ldots, S_n(t), t) \\ &\qquad\quad - \sum_{j=1}^{n} \lambda_j^* V(S_1(t)e^{a_{1j}k}, \ldots, S_n(t)e^{a_{nj}k}, t), \end{aligned} \quad (8.24)$$

and the instantaneous change of the portfolio value is

$$\begin{aligned} &[(\delta + \sum_{j=1}^{n} \lambda_j^*) V(S_1(t), \ldots, S_n(t), t) \\ &\qquad - \sum_{j=1}^{n} \lambda_j^* V(S_1(t)e^{a_{1j}k}, \ldots, S_n(t)e^{a_{nj}k}, t)]dt. \end{aligned}$$

On the other hand, the instantaneous investment gain of the portfolio is

$$\sum_{i=1}^{n} \eta_i (-c_i dt) + [V - \sum_{i=1}^{n} \eta_i](\delta dt).$$

[For simplicity we write η_i for $\eta_i(S_1(t), \ldots, S_n(t), t)$, and V for $V(S_1(t), \ldots, S_n(t), t)$.] Hence, the condition that the instantaneous change of the portfolio value is equal to the instantaneous investment gain is that

$$(\delta + \sum_{j=1}^{n} \lambda_j^*)V - \sum_{j=1}^{n} \lambda_j^* V(S_1(t)e^{a_{1j}k}, \ldots, S_n(t)e^{a_{nj}k}, t)$$
$$= -\sum_{i=1}^{n} c_i\eta_i + \delta\left[V - \sum_{i=1}^{n} \eta_i\right]$$
$$= \delta V - \sum_{i=1}^{n} \eta_i(\delta + c_i), \tag{8.25}$$

or

$$\sum_{i=1}^{n} \eta_i(\delta + c_i) = \sum_{j=1}^{n} \lambda_j^*[V(S_1 e^{a_{1j}k}, \ldots, S_n e^{a_{nj}k}, t) - V(S_1, \ldots, S_n, t)]. \tag{8.26}$$

It follows from (8.22) and (8.16) that the right-hand side of (8.26) is

$$\sum_{j=1}^{n} \lambda_j^* \left[\sum_{i=1}^{n} \eta_i(e^{a_{ij}k} - 1)\right]$$
$$= \sum_{i=1}^{n} \eta_i \left[\sum_{j=1}^{n} \lambda_j^* (e^{a_{ij}k} - 1)\right]$$
$$= \sum_{i=1}^{n} \eta_i (\delta + c_i), \tag{8.27}$$

which is the left-hand side. Hence, the portfolio so constructed is indeed self-financing, which completes the proof that the option price at time t is given by (8.20).

Let us now consider the limiting case where $k \to 0$ and λ and c_i vary according to (8.5) and (8.7). It follows from (8.7) that, for $i = 1, 2, \ldots, n$,

$$\lim_{k\to 0} kc_i = \sum_{j=1}^{n} a_{ij}. \tag{8.28}$$

Expanding the exponential functions in (8.16) as a Maclaurin series yields

$$\delta + c_i = k\sum_{j=1}^{n} \lambda_j^* a_{ij} + \ldots, \tag{8.29}$$

from which and (8.28) we obtain that, for $i = 1, 2, \ldots, n$,

$$\lim_{k\to 0} k^2 \sum_{j=1}^{n} \lambda_j^* a_{ij} = \lim_{k\to 0} k(\delta + c_i)$$
$$= \sum_{j=1}^{n} a_{ij}.$$

Since the matrix (a_{ij}) is nonsingular, we have, for $j = 1, 2, \ldots, n$,

$$\lim_{k\to 0} k^2 \lambda_j^* = 1. \tag{8.30}$$

It now follows from (8.6) that

$$\lim_{k\to 0} k^2 \sum_{j=1}^{n} \lambda_j^* a_{ij} a_{hj} = \sum_{j=1}^{n} a_{ij} a_{hj}$$
$$= \sigma_{ih}. \tag{8.31}$$

Considering one more term in the Maclaurin series expansion of (8.29), we have

$$\delta + c_i = k\sum_{j=1}^{n} \lambda_j^* a_{ij} + \frac{k^2}{2}\sum_{j=1}^{n} \lambda_j^* a_{ij}^2 + O(k)$$
$$= k\sum_{j=1}^{n} \lambda_j^* a_{ij} + \frac{\sigma_{ii}}{2} + O(k), \tag{8.32}$$

by (8.31). [We write $f(k) = O(g(k))$ if $f(k)/g(k)$ is bounded as $k \to 0$.] Let E^* and Cov* denote the expectation and covariance operators with respect to the equivalent martingale measure [the probability measure such that the (independent) Poisson processes $\{N_1(t)\}$, $\{N_2(t)\}, \ldots, \{N_n(t)\}$ have parameters $\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*$, respectively]. Then

$$E^*[X_i(1)] = k\sum_{j=1}^{n} a_{ij}\, E^*[N_j(1)] - c_i$$
$$= k\sum_{j=1}^{n} a_{ij}\lambda_j^* - c_i$$
$$\to \delta - \frac{\sigma_{ii}}{2}, \tag{8.33}$$

as $k \to 0$, by (8.32), and

$$\text{Cov}^*(X_i(1), X_h(1))$$
$$= \text{Cov}^*(k\sum_{j=1}^{n} a_{ij}\, N_j(1), k\sum_{m=1}^{n} a_{hm}\, N_m(1))$$
$$= k^2 \sum_{j=1}^{n} a_{ij}\, a_{hj}\, \lambda_j^*$$
$$\to \sigma_{ih}, \tag{8.34}$$

as $k \to 0$, by (8.31). In the limit the process (8.9) is an n-dimensional Wiener process, which, with respect to the equivalent martingale measure, has a drift vector given by (8.33) instead of (8.10), and unchanged diffusion matrix (8.4). See also (10.15) below.

Furthermore, in the limit as $k \to 0$,

$$\eta_i(s_1, \ldots, s_n, t) = s_i V_{s_i}(s_1, \ldots, s_n, t), \quad (8.35)$$

showing that the number of shares of stock i in the replicating portfolio at time t is simply the partial derivative

$$V_{s_i}(S_1(t), S_2(t), \ldots, S_n(t), t). \quad (8.36)$$

To derive (8.35), divide (8.22) [with $S_1(t)=s_1, \ldots, S_n(t)=s_n$] by k and let k tend to 0 to obtain

$$\sum_{i=1}^{n} V_{s_i} s_i a_{ij} = \sum_{i=1}^{n} \eta_i a_{ij}, \quad j = 1, 2, \ldots, n. \quad (8.37)$$

Since the matrix (a_{ij}) is nonsingular, we have (8.35). For another proof that, in a multidimensional Brownian motion model, the number of shares of stock i in the replicating portfolio is V_{s_i}, see Theorem 1 of Pedersen (1995).

If the payoff function Π is homogeneous of degree one in the stock-price variables, then the option-price function V is also homogeneous of degree one in the stock-price variables. By Euler's theorem for homogeneous functions,

$$V(s_1, \ldots, s_n, t) = \sum_{i=1}^{n} s_i V_{s_i}(s_1, \ldots, s_n, t). \quad (8.38)$$

It follows from (8.35) that, in the multidimensional geometric Brownian motion model,

$$V(s_1, \ldots, s_n, t) = \sum_{i=1}^{n} \eta_i(s_1, \ldots, s_n, t); \quad (8.39)$$

hence there is no bond component in the replicating portfolio and the option price V does not depend on the interest rate δ. In Section 10 we derive this independence of interest rate result in the context of change of numeraire. For a proof by means of differential equations, see Pedersen (1995).

To conclude this section, we derive the generalization of (7.25). It follows from (8.23) and an application of the chain rule that (8.24) can be rewritten as

$$-\sum_{i=1}^{n} c_i s_i V_{s_i} + V_t$$

$$= \delta V - \sum_{j=1}^{n} \lambda_j^* [V(s_1 e^{a_{1j}k}, \ldots, s_n e^{a_{nj}k}, t) - V]. \quad (8.40)$$

By the multivariate Taylor expansion formula,

$$V(s_1 e^{a_{1j}k}, \ldots, s_n e^{a_{nj}k}, t) - V(s_1, \ldots, s_n, t)$$

$$= \sum_{i=1}^{n} (e^{a_{ij}k} - 1) s_i V_{s_i}$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{h=1}^{n} (e^{a_{ij}k} - 1)(e^{a_{hj}k} - 1) s_i s_h V_{s_i s_h} + \ldots \quad (8.41)$$

Multiplying (8.41) with λ_j^* and summing over j gives the sum in the right-hand side of (8.40). It follows from (8.16) that

$$\sum_{j=1}^{n} \lambda_j^* \sum_{i=1}^{n} (e^{a_{ij}k} - 1) s_i V_{s_i}$$

$$= \sum_{i=1}^{n} \delta s_i V_{s_i} + \sum_{i=1}^{n} c_i s_i V_{s_i}, \quad (8.42)$$

the last sum of which cancels with the one on the left-hand side of (8.40). Because of (8.31),

$$\sum_{j=1}^{n} \lambda_j^* (e^{a_{ij}k} - 1)(e^{a_{hj}k} - 1)$$

$$= \sum_{j=1}^{n} \lambda_j^* k^2 a_{ij} a_{hj} + \ldots \to \sigma_{ih}, \quad (8.43)$$

as $k \to 0$. It follows from (8.42) and (8.43) that, in the limit as $k \to 0$, (8.40) becomes the parabolic differential equation

$$V_t = \delta V - \delta \sum_{i=1}^{n} s_i V_{s_i} - \frac{1}{2} \sum_{i=1}^{n} \sum_{h=1}^{n} \sigma_{ih} s_i s_h V_{s_i s_h}. \quad (8.44)$$

9. Extension to Dividend-Paying Stocks

The results in Section 5 can be extended to the case where the stock pays dividends continuously, at a rate proportional to its price. In other words, we assume that there is a nonnegative number ϕ such that the dividend paid between time t and $t+dt$ is

$$\phi\, S(t)\, dt. \quad (9.1)$$

(The number ϕ may be called the dividend-yield rate.) If all dividends are reinvested in the stock, each share of the stock at time 0 grows to $e^{\phi t}$ shares at time t. The risk-neutral Esscher measure is the Esscher measure of parameter $h=h^*$ such that the process

$$\{e^{-(\delta-\phi)t} S(t)\} \quad (9.2)$$

is a martingale. Condition (5.5) now becomes

$$e^{\delta-\phi} = M(1 + h^*, 1)/M(h^*, 1). \tag{9.3}$$

Since

$$E[S(\tau); h^*] = S(0)\, e^{(\delta-\phi)\tau}, \tag{9.4}$$

the European call option pricing formula (5.9) is generalized as

$$E[e^{-\delta\tau}\,(S(\tau) - K)_+; h^*] \\ = S(0)e^{-\phi\tau}\, Pr[S(\tau) > K; h^* + 1] \\ - Ke^{-\delta\tau}\, Pr[S(\tau) > K; h^*]. \tag{9.5}$$

Formula (9.5) may also be used to price currency exchange options, with $S(\tau)$ denoting the spot exchange rate at time τ, δ the domestic force of interest, and ϕ the foreign force of interest. For $\{S(t)\}$ being a geometric Brownian motion, (9.5) is known as the *Garman-Kohlhagen formula*; see also (10.20) below.

We can extend the model to more than one dividend-paying stock. As in the last section, we let $S_j(t)$ denote the price of stock j at time t, $j=1, 2, \ldots, n$. For each j, we assume that there exists a nonnegative constant ϕ_j such that stock j pays dividends of amount

$$\phi_j\, S_j(t)\, dt$$

between time t and $t+dt$. Same as (8.1) and (8.9), we write

$$X_j(t) = ln[S_j(t)/S_j(0)], \quad j = 1, 2, \ldots, n, \tag{9.6}$$

and

$$\mathbf{X}(t) = (X_1(t), X_2(t), \ldots, X_n(t))'. \tag{9.7}$$

Let R^n denote the linear space of column vectors with n real entries, and

$$M(\mathbf{z}, t) = E[e^{\mathbf{z}'\mathbf{X}(t)}], \qquad \mathbf{z} \in R^n, \tag{9.8}$$

be the moment generating function of $\mathbf{X}(t)$. We assume that $\{\mathbf{X}(t)\}_{t\geq0}$ is a stochastic process with independent and stationary increments and that

$$M(\mathbf{z}, t) = [M(\mathbf{z}, 1)]^t, \qquad t \geq 0. \tag{9.9}$$

Let $\mathbf{h}=(h_1, h_2, \ldots, h_n)' \in R^n$ for which $M(\mathbf{h}, 1)$ exists. The positive martingale

$$\{e^{\mathbf{h}'\mathbf{X}(t)}\, M(\mathbf{h}, 1)^{-t}\}_{t\geq0} \tag{9.10}$$

can be used to define a new measure, the Esscher measure of parameter vector $\mathbf{h}$. The risk-neutral Esscher measure is the Esscher measure of parameter vector $\mathbf{h}=\mathbf{h}^*$ such that, for each j, $j=1, 2, \ldots, n$,

$$\{e^{-(\delta-\phi_j)t}\, S_j(t)\} \tag{9.11}$$

is a martingale. Condition (9.3) is generalized as n simultaneous conditions:

$$e^{\delta-\phi_j} = M(\mathbf{1}_j + \mathbf{h}^*, 1)/M(\mathbf{h}^*, 1), j = 1, \ldots, n. \tag{9.12}$$

Here,

$$\mathbf{1}_j = (0, \ldots, 0, 1, 0, \ldots, 0)', \tag{9.13}$$

where the 1 in the column vector $\mathbf{1}_j$ is in the j-th position.

As an illustration, let us consider the model in Section 8, where $\{\mathbf{X}(t)\}$ is defined by n independent Poisson processes $\{N_1(t)\}, \ldots, \{N_n(t)\}$, each with the same parameter value

$$\lambda = k^{-2}; \tag{9.14}$$

see (8.8). Because

$$E\left(\exp\left[\sum_{j=1}^n b_j N_j(t)\right]\right) = \prod_{j=1}^n E(\exp[b_j N_j(t)]) \\ = \prod_{j=1}^n \exp(t\lambda[e^{b_j} - 1]) \\ = \exp\left(t\lambda\sum_{j=1}^n [e^{b_j} - 1]\right),$$

we have

$$M(\mathbf{z}, 1) = E[e^{\mathbf{z}'\mathbf{X}(t)}] \\ = \exp\left(-\sum_{i=1}^n z_i c_i\right) E\left(\exp\left[k \sum_{j=1}^n \sum_{i=1}^n z_i a_{ij} N_j(t)\right]\right) \\ = \exp\left(-\sum_{i=1}^n z_i c_i + \sum_{j=1}^n \lambda\left[\exp\left(k \sum_{i=1}^n z_i a_{ij}\right) - 1\right]\right). \tag{9.15}$$

Hence,

$$M(\mathbf{z} + \mathbf{h}, 1)/M(\mathbf{h}, 1) \\ = \exp\left\{-\sum_{i=1}^n z_i c_i + \sum_{j=1}^n \lambda\exp\left[k \sum_{i=1}^n (z_i + h_i)a_{ij}\right] - \sum_{j=1}^n \lambda\exp\left(k \sum_{i=1}^n h_i a_{ij}\right)\right\} \\ = \exp\left\{-\sum_{i=1}^n z_i c_i + \sum_{j=1}^n \lambda\exp(k \sum_{i=1}^n h_i a_{ij}) \times \left[\exp(k \sum_{i=1}^n z_i a_{ij}) - 1\right]\right\}. \tag{9.16}$$

On comparing (9.16) with (9.15), we see that the Esscher measure of parameter vector $\mathbf{h}$ is the probability measure such that, for $j=1, 2, \ldots, n$, the Poisson process $\{N_j(t)\}$ has the parameter value

$$\lambda_j(\mathbf{h}) = \lambda \exp(k \sum_{i=1}^{n} h_i a_{ij}) = \frac{1}{k^2} \exp(k \sum_{i=1}^{n} h_i a_{ij}). \quad (9.17)$$

It follows from (9.12) and (9.16) that the risk-neutral Esscher measure is determined by the equations

$$\delta - \phi_i = -c_i + \sum_{j=1}^{n} \lambda_j(\mathbf{h}^*)(e^{ka_{ij}} - 1),$$
$$i = 1, 2, \ldots, n. \quad (9.18)$$

Not surprisingly, with $\phi_i=0$ and $\lambda_j(\mathbf{h}^*)=\lambda_j^*$, (9.18) is identical to (8.16).

It follows from (9.17) that, for $1 \le j, m \le n$,

$$\lambda_j(\mathbf{h}) e^{ka_{mj}} = \lambda_j(\mathbf{h} + \mathbf{1}_m). \quad (9.19)$$

Hence, an interesting way to express (9.18) is

$$\phi_i - c_i + \sum_{j=1}^{n} \lambda_j(\mathbf{h}^* + \mathbf{1}_i) = \delta + \sum_{j=1}^{n} \lambda_j(\mathbf{h}^*),$$
$$i = 1, 2, \ldots, n, \quad (9.20)$$

where the right-hand side is constant for all i.

For $\mathbf{k}=(k_1, \ldots, k_n)'$, write

$$\mathbf{S}(t)^{\mathbf{k}} = S_1(t)^{k_1} \ldots S_n(t)^{k_n}. \quad (9.21)$$

Then,

$$\begin{aligned} E[\mathbf{S}(t)^{\mathbf{k}} g(\mathbf{S}(t)); \mathbf{h}] &= \frac{E[\mathbf{S}(t)^{\mathbf{k}} g(\mathbf{S}(t))\, e^{\mathbf{h}'\mathbf{X}(t)}]}{E[e^{\mathbf{h}'\mathbf{X}(t)}]} \\ &= \frac{E[\mathbf{S}(t)^{\mathbf{k}} g(\mathbf{S}(t))\, \mathbf{S}(t)^{\mathbf{h}}]}{E[\mathbf{S}(t)^{\mathbf{h}}]} \\ &= \frac{E[\mathbf{S}(t)^{\mathbf{k}+\mathbf{h}}]}{E[\mathbf{S}(t)^{\mathbf{h}}]} \frac{E[g(\mathbf{S}(t))\, \mathbf{S}(t)^{\mathbf{k}+\mathbf{h}}]}{E[\mathbf{S}(t)^{\mathbf{k}+\mathbf{h}}]} \\ &= E[\mathbf{S}(t)^{\mathbf{k}}; \mathbf{h}]\, E[g(\mathbf{S}(t)); \mathbf{k} + \mathbf{h}], \end{aligned} \quad (9.22)$$

which generalizes the factorization formula (5.7). An immediate consequence of formula (9.22) and that (9.11) is a martingale under the risk-neutral Esscher measure is the formula:

$$\begin{aligned} E[e^{-\delta t} S_j(t) g(\mathbf{S}(t)); \mathbf{h}^*] &= E[e^{-\delta t} S_j(t); \mathbf{h}^*]\, E[g(\mathbf{S}(t)); \mathbf{h}^* + \mathbf{1}_j] \\ &= S_j(0)\, e^{-\phi_j t} E[g(\mathbf{S}(t)); \mathbf{h}^* + \mathbf{1}_j]. \end{aligned} \quad (9.23)$$

The *Margrabe option* (Margrabe 1978) is the option to exchange one stock for another at the end of a stated period, say time τ, $\tau>0$. The payoff of this European option is

$$[S_1(\tau) - S_2(\tau)]_+. \quad (9.24)$$

Its value at time 0, calculated with respect to the risk-neutral Esscher measure, is

$$E(e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+ ; h^*). \quad (9.25)$$

Since

$$(s_1 - s_2)_+ = s_1 I(s_1 > s_2) - s_2 I(s_1 > s_2),$$

it follows from (9.23) that

$$\begin{aligned} &E(e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+ ; \mathbf{h}^*) \\ &\quad = S_1(0)e^{-\phi_1\tau} E(I[S_1(\tau) > S_2(\tau)]; \mathbf{h}^* + \mathbf{1}_1) \\ &\qquad - S_2(0)e^{-\phi_2\tau} E(I[S_1(\tau) > S_2(\tau)]; \mathbf{h}^* + \mathbf{1}_2) \\ &\quad = S_1(0)e^{-\phi_1\tau} Pr[S_1(\tau) > S_2(\tau); \mathbf{h}^* + \mathbf{1}_1] \\ &\qquad - S_2(0)e^{-\phi_2\tau} Pr[S_1(\tau) > S_2(\tau); \mathbf{h}^* + \mathbf{1}_2]. \end{aligned} \quad (9.26)$$

A special case of (9.26) is (9.5).

10. Change of Numeraire and Homogeneous Payoff Function

Consider a European option or derivative security with exercise date τ and payoff

$$\Pi(S_1(\tau), \ldots, S_n(\tau)). \quad (10.1)$$

Let $E_t[\cdot]$ denote the expectation conditional on all information up to time t. For $0 \le t \le \tau$, let $V(t)=V(S_1(t), S_2(t), \ldots, S_n(t), t)$ denote the option price at time t, calculated with respect to the risk-neutral Esscher measure,

$$\begin{aligned} V(t) &= E_t[e^{-\delta(\tau-t)}\, \Pi(S_1(\tau), \ldots, S_n(\tau)); \mathbf{h}^*] \\ &= E_t[e^{-\delta(\tau-t)}\, S_j(\tau)\, \Pi(S_1(\tau), \ldots, S_n(\tau))/S_j(\tau); \mathbf{h}^*] \\ &= E_t[e^{-\delta(\tau-t)}\, S_j(\tau); \mathbf{h}^*] \\ &\quad \times E_t[\Pi(S_1(\tau), \ldots, S_n(\tau))/S_j(\tau); \mathbf{h}^* + \mathbf{1}_j] \\ &= e^{-\phi_j(\tau-t)}\, S_j(t)\, E_t[\Pi(S_1(\tau), \ldots, \\ &\qquad S_n(\tau))/S_j(\tau); \mathbf{h}^* + \mathbf{1}_j]. \end{aligned} \quad (10.2)$$

Thus,

$$\frac{V(t)}{e^{\phi_j t}\, S_j(t)} = E_t[\frac{1}{e^{\phi_j \tau}\, S_j(\tau)}\, \Pi(S_1(\tau), \ldots, S_n(\tau)); \mathbf{h}^* + 1_j], \quad (10.3)$$

from which it follows that, with respect to the Esscher measure of parameter vector $\mathbf{h}^*+\mathbf{1}_j$, the process

$$\left\{\frac{V(t)}{e^{\phi_j t}\, S_j(t)};\ 0 \le t \le \tau\right\} \tag{10.4}$$

is a martingale. In particular, with respect to this measure, the processes

$$\left\{\frac{e^{\delta t}}{e^{\phi_j t}\, S_j(t)}\right\} \tag{10.5}$$

and

$$\left\{\frac{e^{\phi_k t}\, S_k(t)}{e^{\phi_j t}\, S_j(t)}\right\} \tag{10.6}$$

are martingales (and conversely these conditions determine the parameter vector $\mathbf{h}^*$). To explain the denominator $e^{\phi_j t}S_j(t)$, we consider stock j as a standard of value or a *numeraire*. We imagine that there is a mutual fund consisting of stock j only and all dividends are reinvested; all other securities are measured in terms of the value of this mutual fund. See also Geman, El Karoui, and Rochet (1995).

Now, we assume that the payoff function Π is homogeneous of degree one. It follows from

$$\Pi(s_1, \ldots, s_n) = s_j\, \Pi(s_1/s_j, \ldots, s_{j-1}/s_j, 1, s_{j+1}/s_j, \ldots, s_n/s_j) \tag{10.7}$$

that (10.3) becomes

$$\frac{V(t)}{e^{\phi_j t}\, S_j(t)} = E_t\left[\Pi\left(\frac{S_1(\tau)}{e^{\phi_j \tau}\, S_j(\tau)}, \ldots, \frac{S_n(\tau)}{e^{\phi_j \tau}\, S_j(\tau)}\right);\ \mathbf{h}^* + \mathbf{1}_j\right]. \tag{10.8}$$

The right-hand side is a conditional expectation, with respect to the Esscher measure of parameter vector $\mathbf{h}^*+\mathbf{1}_j$, of a function of the $(n-1)$-dimensional random vector

$$(X_1(\tau) - X_j(\tau), \ldots, X_{j-1}(\tau) - X_j(\tau),\ X_{j+1}(\tau) - X_j(\tau), \ldots, X_n(\tau) - X_j(\tau))'. \tag{10.9}$$

Consider the special case that $\{\mathbf{X}(t)\}$ is an n-dimensional Wiener process, with $\boldsymbol{\mu}=(\mu_1, \mu_2, \ldots, \mu_n)'$ and $\mathbf{V}=(\sigma_{ij})$ denoting the mean vector and the covariance matrix of $\mathbf{X}(1)$, respectively. It is assumed that $\mathbf{V}$ is nonsingular. Because

$$M(\mathbf{z}, t) = \exp\left[t\left(\mathbf{z}'\boldsymbol{\mu} + \frac{1}{2}\mathbf{z}'\mathbf{V}\mathbf{z}\right)\right],\ \mathbf{z} \in R^n, \tag{10.10}$$

we have, for $\mathbf{h} \in R^n$,

$$\begin{aligned} &E[e^{\mathbf{z}'\mathbf{X}(t)}; \mathbf{h}] \\ &= M(\mathbf{z} + \mathbf{h}, t)/M(\mathbf{h}, t) \\ &= \exp\left\{t\left[\mathbf{z}'(\boldsymbol{\mu} + \mathbf{V}\mathbf{h}) + \frac{1}{2}\mathbf{z}'\mathbf{V}\mathbf{z}\right]\right\},\ \mathbf{z} \in R^n, \end{aligned} \tag{10.11}$$

showing that, under the Esscher measure of parameter vector $\mathbf{h}$, $\{\mathbf{X}(\mathbf{t})\}$ remains an n-dimensional Wiener process with modified drift vector

$$\boldsymbol{\mu} + \mathbf{V}\mathbf{h}$$

and unchanged diffusion matrix $\mathbf{V}$. It follows from (9.12) that, for $k=1, 2, \ldots, n$,

$$\delta - \phi_k = \mathbf{1}_k'(\boldsymbol{\mu} + \mathbf{V}\mathbf{h}^*) + \frac{1}{2}\mathbf{1}_k'\mathbf{V}\mathbf{1}_k. \tag{10.12}$$

Thus,

$$\begin{aligned} \boldsymbol{\mu}^* &= E[\mathbf{X}(1); \mathbf{h}^*] && (10.13)\\ &= \boldsymbol{\mu} + \mathbf{V}\mathbf{h}^* && (10.14)\\ &= \delta\mathbf{1} - \left(\phi_1 + \frac{1}{2}\sigma_{11}, \phi_2 + \frac{1}{2}\sigma_{22}, \ldots, \phi_n + \frac{1}{2}\sigma_{nn}\right)', && (10.15) \end{aligned}$$

where

$$\mathbf{1} = (1, 1, 1, \ldots, 1)'. \tag{10.16}$$

[Recall (8.33).] Also,

$$\begin{aligned} &E[\mathbf{X}(1); \mathbf{h}^* + \mathbf{1}_k] \\ &= \boldsymbol{\mu} + \mathbf{V}(\mathbf{h}^* + \mathbf{1}_k) \\ &= \boldsymbol{\mu}^* + \mathbf{V}\mathbf{1}_k \\ &= \delta\mathbf{1} - \left(\phi_1 - \sigma_{1k} + \frac{1}{2}\sigma_{11}, \phi_2 - \sigma_{2k} + \frac{1}{2}\sigma_{22}, \ldots, \phi_n - \sigma_{nk} + \frac{1}{2}\sigma_{nn}\right)'. \end{aligned} \tag{10.17}$$

For an n-dimensional Wiener process $\{\mathbf{X}(t)\}$, (10.9) is a normal random vector under the Esscher measure of parameter vector $\mathbf{h}^*+\mathbf{1}_j$, and it follows from (10.17) that its mean does not involve the force of interest δ, and, of course, its $(n-1)$-dimensional covariance matrix, which is the same for all $\mathbf{h}$, does not depend on

δ. Thus $V(t)$, the price of a derivative security with a payoff function which is homogeneous of degree one, does not depend on δ. For example, consider the European Margrabe option, which has the payoff function

$$\Pi(s_1, s_2) = (s_1 - s_2)_+.$$

Let

$$\nu^2 = \mathrm{Var}[X_1(1) - X_2(1)] = \sigma_{11} - 2\sigma_{12} + \sigma_{22}, \qquad (10.18)$$

$$\zeta(\tau) = \frac{1}{\nu\sqrt{\tau}} \ln\left(\frac{e^{-\phi_1\tau}S_1(0)}{e^{-\phi_2\tau}S_2(0)}\right), \qquad (10.19)$$

and Φ denote the standardized normal distribution function. Then (9.26) becomes

$$E(e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+; \mathbf{h}^*) = e^{-\phi_1\tau}S_1(0)\Phi\left(\zeta(\tau) + \frac{1}{2}\nu\sqrt{\tau}\right) - e^{-\phi_2\tau}S_2(0)\Phi\left(\zeta(\tau) - \frac{1}{2}\nu\sqrt{\tau}\right), \qquad (10.20)$$

which does not depend on δ. For nondividend-paying stocks ($\phi_1=\phi_2=0$), formula (10.20) has been given by Margrabe (1978). Fischer (1978) has also derived (10.20) with $\phi_1=0$ as a European call option formula; for him, $S_2(\tau)$ is the stochastic exercise price at time τ.

This independence of the interest rate is not valid in general. Consider the shifted compound Poisson model discussed earlier, where

$$X_i(t) - X_m(t) = k\sum_{j=1}^{n} (a_{ij} - a_{mj})N_j(t) - (c_i - c_m)t. \qquad (10.21)$$

It follows from (9.19) that, under the Esscher measure of parameter vector $\mathbf{h}^*+\mathbf{1}_m$, the process N_j has parameter value $\lambda_j^* e^{ka_{mj}}$. The risk-neutral parameter values $\{\lambda_j^*\}$, which are the solution of (9.18) [or (8.16)], depend on δ. To see how the interest rate δ gets canceled away as $k\to 0$ [with λ and c_i varying according to (8.5) and (8.7)], we apply (8.33) with the interest rate, replacing its difference with the dividend-yield rate and (8.34) to obtain

$$\begin{aligned} &E[X_i(1) - X_m(1); \mathbf{h}^* + \mathbf{1}_m] \\ &= k\sum_{j=1}^{n} (a_{ij} - a_{mj})\lambda_j^* e^{ka_{mj}} - (c_i - c_m) \\ &= k\sum_{j=1}^{n} (a_{ij} - a_{mj})\lambda_j^*(1 + ka_{mj} + \ldots) - (c_i - c_m) \\ &\to \left(\delta - \phi_i - \frac{\sigma_{ii}}{2} + \sigma_{im}\right) - \left(\delta - \phi_m - \frac{\sigma_{mm}}{2} + \sigma_{mm}\right). \end{aligned} \qquad (10.22)$$

The last expression is identical to the one obtained from (10.17).

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