IX

Is Your Standard of Living Sustainable during Retirement? Ruin Probabilities, Asian Options, and Life Annuities

by Moshe Arye Milevsky and Chris Robinson

Abstract

In this paper we compute the conditional and unconditional probability of ruin for an individual who wishes to consume a fixed periodic amount from an initial endowment invested in a portfolio earning a stochastic rate of return. The conditional probability of ruin is the probability that the net wealth becomes zero prior to the individual's stochastic date of death. Unconditional is the probability that the wealth ever becomes zero. We solve this problem using insights from option pricing theory. Specifically, we show that the probability of ruin corresponds to the probability that a suitably parameterized Asian call option (a type of derivative security) will expire with value in-the-money. Under standard assumptions for the investment process, the unconditional probability of ruin is obtained analytically using well-known results leading to the Gamma distribution. The conditional probability of ruin is then approximated with moment-matching techniques using the same Gamma distribution. Finally, using realistic market values for equity and fixed-income investments, we apply our approximation to demonstrate that the conditional probability of ruin is minimized with a relatively high allocation to equity (the high-risk asset) until quite late in life.

1. Motivation

Once an individual retires, lifetime consumption is funded by money saved and invested during the working part of the life cycle. The two classic finance problems for a retired individual are (1) What level of consumption can the individual enjoy from invested wealth, including investment earnings, without running out of money during his or her lifetime? and (2) How should the retirement fund be allocated to different investment assets?

If the date of death and the rate of return are known with certainty, this problem is easily solved, but, of course, these assumptions are not realistic. In this paper we compute the conditional and unconditional *probability of ruin* for an individual (retiree) with a stochastic life span who is consuming a fixed *real* amount from a diversified investment portfolio. By the term *conditional* we mean the probability that the net-wealth process will hit zero while the individual is still alive, otherwise referred to as bankruptcy. By *unconditional*[1] we mean the probability that the process will *ever* hit zero. The unconditional probability would be of interest to endowments or individuals with very strong bequest motives. In particular,

[1]Perhaps abusing conventions.

we tabulate the probability of ruin as an explicit function of the stochastic growth rate and volatility of the portfolio vis-à-vis the consumption rate. We view this research as an extension of the literature on ruin probabilities in insurance, such as the work by Pentikainen (1980) and Panjer (1986), as well as many others. The main distinction, of course, is that we focus on "ruin" from a personal perspective, and they do so on the company-firm level. In particular we assume that the (consumption) "claims" are deterministic and the (investment returns) "premiums" are stochastic.

Interestingly, we demonstrate that the probability of ruin is equivalent to the probability that a suitably parameterized Asian call option—a type of path-dependent derivative security—will expire in-the-money. The actual *price* of this Asian call option can be interpreted as the cost of ensuring the retiree's prespecified standard of living, which is also analogous to the cost of an appropriately defined life annuity. Finally, we use a Gamma distribution approximation for life annuities with realistic market parameters for equity and fixed-income investments to demonstrate that the conditional probability of ruin and the implicit cost of insurance is minimized with a relatively high allocation to equity until quite late in life. This analytical approximation can be used to confirm earlier *simulation-based* studies by Milevsky, Ho, and Robinson (1997), which documented the effect of asset allocation on ruin probabilities.

The essence of our approach is the actuarial intuition that the probability of ruin can be formulated as the probability that the stochastic present value—basically a life annuity or perpetuity—is greater than the initial wealth available to support the consumption. Thus, in our framework, an individual retires at age (x) with an initial wealth of $W_0 = w$ and a desired lifelong consumption stream of c *real* dollars per annum. In a deterministic world, with fixed time of death T and a fixed real interest rate r, the present value of the desired consumption stream is trivially calculated as

$$PV_T(c) = c\int_0^T e^{-rt}\,dt = \frac{c(1-e^{-rT})}{r}. \qquad (1)$$

If the expression in Equation (1) is greater than the initial wealth w, the retiree does not have enough to support the desired consumption stream, and ruin occurs with probability one. Likewise, when $T = \infty$, Equation (1) becomes $PV_\infty(c) = c/r$, which is the sum needed to fund a perpetuity of c dollars per annum.

On the other hand, in a stochastic world, both the time of death and the rate of return on investment are stochastic. The stochastic analogue to the deterministic present value of consumption is the stochastic present value of lifetime consumption (*SPV(c)*) denoted by

$$SPV_{\tilde{T}}(c) = c\int_0^{\tilde{T}} e^{-(\tilde{R}_t)t}\,dt, \qquad (2)$$

where the two sources of randomness, $\tilde{T}$ and $\tilde{R}_t$, are incorporated explicitly into the computation. The right-hand side (r.h.s.) of Equation (2) is the actuarial definition of a life annuity under stochastic discounting. In addition, the r.h.s. of Equation (2) can be identified as the scaled payoff from an Asian put option (see Section 3 for more on this result). The higher the *SPV, ceteris paribus,* relative to the initial wealth-to-consumption ratio, the higher the probability of ruin. Once we have the probability density function (pdf) of the stochastic present value of lifetime consumption we can compute the probability that this quantity is greater than the initial level of wealth w. We denote this by

$$P_{\text{ruin}}^{\text{alive}} := P(SPV_{\tilde{T}}(c) \geq w) = P(SPV_{\tilde{T}} \geq \frac{w}{c}), \qquad (3)$$

for the *conditional* case, and

$$P_{\text{ruin}} := P(SPV_\infty(c) \geq w) = P(SPV_\infty \geq \frac{w}{c}),$$

for the *unconditional* case.

The remainder of this paper is organized as follows. Section 2 introduces the investment and mortality dynamics, using the techniques of continuous time financial economics, and then derives an expression for the probability of ruin. Section 3 describes the connection and analogy between our problem and Asian options. Section 4 develops some techniques for computing the relevant probabilities using the Gamma distribution. Section 5 provides some numerical examples of the conditional and unconditional probability of ruin using realistic capital market and mortality parameters. Section 6 concludes the paper.

2. Investment and Mortality

We start with the basic geometric Brownian motion (GBM) model of investment dynamics in which individual stocks (or asset classes) obey the stochastic differential equation (SDE) defined by

$$dS_t^i / S_t^i = \mu_i dt + \sigma_i dB_t^i, \qquad (4)$$

where B_t^i is a standard Brownian motion, μ_i and σ_i are the *real* (inflation-adjusted) mean and standard deviation of dS^i/S^i, and $d\langle B^i, B^j\rangle_t = \rho_{ij}$ is the correlation coefficient. An investor (retiree) allocates and rebalances wealth among the universe of investment assets, provided by Equation (4), and consumes a fixed *real* amount c, per unit of time. By construction, the *real* net-wealth process will obey the SDE

$$dW_t = (\mu_p W_t - c)\,dt + \sigma_p W_t dB_t, \quad W_0 = w, \quad (5)$$

where B_t is a one-dimensional Brownian motion, c is the real fixed consumption rate, w is the initial level of wealth, and (μ_p, σ_p) correspond to the portfolio mean and standard deviation as an implicit function of a static[2] asset allocation vector α. Specifically, the scalar-valued mean return is

$$\mu_p = \boldsymbol{\mu}\boldsymbol{\alpha}' = \left(\sum_i^n \alpha_i \mu_i\right). \quad (6)$$

and the scalar valued standard deviation (also known as volatility) of the portfolio is

$$\sigma_p = \sqrt{\boldsymbol{\alpha}\boldsymbol{\Sigma}\boldsymbol{\alpha}'} = \sqrt{\sum_{i=1}^n \sum_{j=1}^n \alpha_i \sigma_i \rho_{ij} \sigma_j \alpha_j}, \quad (7)$$

where $\boldsymbol{\mu}$ is the vector of expected returns and $\boldsymbol{\Sigma}$ is the variance-covariance matrix of the relevant assets in the market, all of which are lognormally distributed.

The net-wealth process defined by Equation (5) has a drift coefficient $\mu_p W_t - c$ that *may* become negative if c is large enough relative to $\mu_p W_t$. This, in turn, implies that the process W_t *may* eventually hit zero, in contrast to the classic geometric Brownian motion. Our intention is to compute the probability that W_t will ever hit zero and compute the probability that W_t will hit zero while the investor is still alive. Naturally, the former quantity will be an upper bound for the latter.

Lemma 1 *The stochastic process W_t, defined by Equation (5), can be written (solved) explicitly as*

$$W_t = H_t\left[w - c\int_0^t (H_s)^{-1}\,ds\right], \quad (8)$$

where the fundamental solution H_s is

$$H_s = \exp\left[\left(\mu_p - \frac{1}{2}\sigma_p^2\right)s + \sigma_p B_s\right]. \quad (9)$$

See Appendix for proof.

2.1 Mortality Function

Following the actuarial literature and recent work on annuity pricing by Frees, Carriere, and Valdez (1996) we assume a Gompertz law for mortality.[3] In this model the probability of survival to age $(x + t)$ conditional on survival at age (x) is denoted by ${}_tp_x$ and defined equal to

$$p(\tilde{T} \geq t \mid m, b, x) = {}_tp_x$$
$$= \exp\left\{\exp\left(\frac{x-m}{b}\right)\left[1 - \exp\left(\frac{t}{b}\right)\right]\right\}, \quad (10)$$

where m is the mode, b is the scale parameter, and $\tilde{T}$ denotes the time-until-death random variable. For example, when the "mode" of life is $m = 80$ and the "scale" of life is $b = 10$, Equation (10) stipulates that the probability a 65-year-old, lives to age 85 is $P(\tilde{T} \geq 20 \mid 80, 10, 65) = 0.2404$. The probability that a 75-year-old lives to age 85 is $P(\tilde{T} \geq 10 \mid 80, 10, 75) = 0.3527$. (The chances of reaching age 85 increase the older you are.) The Gompertz model, with two free parameters, can be "fitted" to any mortality table, which we will do in Section 5.

Substituting a value of $t \to \infty$ in Equation (10), with a finite value for m and b, results in $\exp\{-\infty\} \to 0$, which confirms the natural boundary condition of human life (you can't live for ever). Likewise, a value of $m \to \infty$ in Equation (10) results in $\exp\{0\} \to 1, \forall t$, which we call "the endowment" case. Therefore, the notation ${}_tp_x$ can be used, without loss of generality, to include the unconditional (perpetuity) case as well.

[2]A richer model would allow for dynamic portfolio strategies in which the investor can react to market conditions by optimizing asset allocation proportions to achieve greater utility over time. Indeed, a full theory of continuous time dynamic programming has been applied to investment-consumption problems by Samuelson (1969), Merton (1993), Richard (1975), and many others; see Karatzas and Shreve (1992), chapter 5.8, for further references. However, our intention is to simply (1) describe the analogy between the probability of ruin and Asian option pricing and (2) produce a reasonable, practical, and simple measure of *sustainability* as a function of consumption ratios and basic asset allocation proportions. Accordingly, we do not advocate that rational utility maximizing agents manage their portfolios (statically) so as to exclusively minimize the probability of bankruptcy. See Browne (1997) for a dynamic policy that does indeed minimize the unconditional probability of ruin in an infinite horizon framework.

[3]The probability of ruin, and the methodology we describe, can be applied using any analytic mortality law or mortality table, as will become evident in the next section.

2.2 Statement of Problem

We would like to compute the probability that the net-wealth process, defined by Equation (5), "hits" zero while the individual is still living and ever. Mathematically,

$$p_{\text{ruin}}^{\text{alive}} := p\left[\inf_{0 \le t \le \tilde{T}} W_t \le 0\right] \tag{11}$$

and

$$p_{\text{ruin}} := p\left[\inf_{0 \le t \le \infty} W_t \le 0\right] \tag{12}$$

is the probability that the smallest value of the process W_t, over the random time period $[0, \tilde{T}]$, or $[0, \infty]$, is less than or equal to zero, which is the definition of ruin.

Before we proceed to obtain an analytic expression for P_{ruin} and an approximation for $P_{\text{ruin}}^{\text{alive}}$, we state and prove the following useful lemma.

Lemma 2 *The stochastic process W_t, defined by Equation (5), obeys the following property:*

$$p\left[\inf_{0 \le t \le t^*} W_t \le 0\right] = p[W_{t^*} \le 0], \quad \forall t^* \ge 0.$$

Thus, W_t will not "cross zero" more than once. Once it enters the negative region, it stays there.
See Appendix for proof.

Relying on the lemma, which applies to any s, a stochastic $s = \tilde{T}$, as well as an infinite $s = \infty$, we can restate the probability of ruin in Equations (11) and (12), using Equation (8), as

$$p_{\text{ruin}}^{\text{alive}} := p\left[\inf_{0 \le t \le \tilde{T}} W_t \le 0\right] = p[W_{\tilde{T}} \le 0]$$

$$= \Pr\left[\frac{w}{c} \le \int_0^{\tilde{T}} (H_s)^{-1}\, ds\right] \tag{13}$$

and

$$p_{\text{ruin}} := p\left[\inf_{0 \le t \le \infty} W_t \le 0\right] = p[W_\infty \le 0]$$

$$= \Pr\left[\frac{w}{c} \le \int_0^{\infty} (H_s)^{-1} ds\right]. \tag{14}$$

The probability of ruin can be expressed as the probability that the stochastic present value of lifetime consumption *of one real dollar* is greater than the initial wealth to consumption ratio w/c.

The expression $\int_0^{\tilde{T}} (H_s)^{-1}\, ds$ has been studied extensively in the actuarial literature. It represents the stochastic present value of an immediate life annuity. See the research initiated by Boyle (1976) and continued by Panjer and Bellhouse (1980, 1981) for additional analysis of the interaction between the investment and mortality.

In Section 4 we compute the exact mean and variance of the (annuity) random variable $\int_0^{\tilde{T}} (H_s)^{-1}\, ds$ and the (perpetuity) random variable $\int_0^{\infty} (H_s)^{-1}\, ds$. In addition, we will illustrate the well-known result that $\int_0^{\infty} (H_s)^{-1}\, ds$ obeys a reciprocal Gamma distribution and therefore argue that $\int_0^{\tilde{T}} (H_s)^{-1}\, ds$ can be *approximated* by the reciprocal Gamma distribution using moment-matching techniques.

3. Asian Options and the Cost of Insurance

There is an interesting connection between the retirees probability of ruin and a financial derivative security known as an Asian option. An Asian option is a path-dependent contingent claim whose payoff at maturity is based on the *average price* observed over the life of the option. The payoff from a regular call option is max $[S_T - X, 0]$, where K is the exercise price and S_T is the price of the underlying security at maturity. The payoff from an Asian (fixed strike) call option is max $[\frac{1}{n} \Sigma_{i=1}^{n} S_i - K, 0]$, where n is the number of measurement-observation periods and S_i is the price of the underlying security on those discrete measurement-observation dates. By inspecting the two types of boundary conditions (payoff structures), one can see that the price of an arithmetic Asian option will always be less than the price of a regular call option as a result of the averaging. Of particular interest is the fact that when the number of measurement-observation periods is very large compared to the lifetime of the option, we can *approximate* the payoff from (and definition of) the Asian option using the integral instead of the summation sign. Thus, the payoff from a continuous arithmetic Asian option is max $[\frac{1}{T} \int S_t\, dt - K, 0]$. In other words, the Asian option will pay off at maturity the sum $\frac{1}{T} \int S_t\, dt - K$, provided that $\frac{1}{T} \int S_t\, dt > K$, the option expires in the money with intrinsic value. Otherwise, the payoff will be zero. Consequently, the probability that an Asian option will expire in-the-money can be stated mathematically as Pr $[\frac{1}{T} \int S_t\, dt > K]$. *Ceteris paribus*, the higher the exercise price K, the lower the probability of expiring in-the-money. Purchasing an Asian call option is akin to insuring (betting) that the weighted (nonlinear) average return from the underlying asset, *over the specified life of the option*, will exceed a predetermined threshold delineated by the exercise price.

In an analogous fashion this paper is concerned with the *discounted average* consumption from the portfolio over the lifetime of the retiree. If this quantity is greater than the (suitably scaled) initial wealth, the individual will eventually be ruined. If the quantity is less than the (suitably scaled) initial wealth, the individual will avoid ruin. In a stochastic environment, we focus on the probability of ruin.

As per equations (13) or (14), the probability of ruin (both conditional and unconditional) can be rescaled and expressed, using Equation (9), as

$$P=\Pr\left[\frac{1}{t}\left(\frac{w}{c}\right)\leq\frac{1}{t}\int_0^t \exp\left\{-\left(\mu_p-\frac{1}{2}\sigma_p^2\right)s+\sigma_p B_s\right\}ds\right], \quad (15)$$

where $t=\infty$, for the unconditional case, and $t=\tilde{T}$, for the conditional case. Now, define a "new" variable $K=w/tc$, referred to as an exercise price, and a new variable $\eta_p=-\mu_p+\frac{1}{2}\sigma_p^2$, referred to as an expected return. By symmetry of the Brownian motion, the term $-\sigma_p B_t$ is equivalent to $\sigma_p B_t^*$, where $B_t^*=-B_t$. We can therefore rewrite Equation (15) as

$$P=\Pr\left[\frac{1}{t}\int_0^t \exp(\eta_p s+\sigma_p B_s^*)ds\geq K\right]$$

$$=\Pr\left(\frac{1}{t}\int_0^t Z_s\,ds\geq K\right), \quad (16)$$

where we define a new (pseudo) stock $Z_s=\exp\{\eta_p s+\sigma_p B_s^*\}$, $Z_0=1$.

Remarkably, Equation (16) corresponds to the probability that an (arithmetic) Asian option, with exercise K, will expire in-the-money.[4] As per the definition of K, the exercise price is the wealth-to-consumption ratio scaled by the time horizon. The longer the time horizon of the option (the longer one lives), the lower is the value of K. A lower exercise price on an Asian (or any) call option results in a higher premium and higher probability of exercise. An individual can, in theory, insure against outliving his/her money by purchasing an Asian call option with a stochastic exercise price and maturity date. Therefore, the actuarial cost of insuring against retirement ruin can be obtained by using any of the algorithms for pricing Asian options. See the work by Turnbull and Wakeman (1991), Keman and Vorst (1990), and Geman and Yor (1993) as well Milevsky and Posner (1998) for some well-known option pricing algorithms. In this paper we adopt the Milevsky and Posner approximation.

It is important to note that we are not suggesting that individuals insure against ruin by purchasing Asian options. This would depend on risk preferences embodied by a utility function of consumption. Rather, the value of the Asian option would provide a good indication of the implicit cost of any particular (fixed) investment/consumption strategy. In fact, buying such an insurance policy would reduce the initial wealth available for investment and would thus require even more insurance to support the same level of consumption. This iterative process would only converge once the individual selected a consumption level equivalent to purchasing a risk-free life annuity.[5]

[4]The probability is under the real-world and not the risk-neutral measure.

4. Moments and Densities

In this section we briefly sketch how to compute moments of the stochastic present value of lifetime consumption (of one dollar). We conclude the section with an easy-to-implement expression for the conditional and unconditional probability of ruin.

4.1 Moments of the SPV

To simplify notation somewhat we denote the stochastic present value of lifetime consumption (of one dollar) by

$$I_t=\int_0^t (H_s)^{-1}ds, \quad (17)$$

where, without loss of generality, I_∞ is the stochastic present value of a perpetual consumption and $I_{\tilde{T}}$, is the stochastic present value of lifetime consumption. Recalling the definition, from Equation (9),

$$H_s=\exp\left[\left(\mu_p-\frac{1}{2}\sigma_p^2\right)s+\sigma_p B_s\right],$$

and using the rules for conditional expectations, we obtain that

$$\mathrm{E}[I_{\tilde{T}}]=\mathrm{E}[\mathrm{E}[I_{\tilde{T}}|\mathcal{F}_\infty^B]],$$

where $\mathcal{F}_\infty^B$ is the sigma field generated by the entire path of the Brownian motion. We are conditioning on the realization of the investment return. Using Fubini's theorem,

[5]See the work by Yagi and Nishigaki (1993), Williams (1986), Sinha (1986), and Warshawsky (1988) for details on the optimal demand for life annuities which would reduce the probability of ruin to zero, provided that w/c is exactly equal to a_x, the price of a \$1 life annuity.

and the moment generating function for the normal random variable, one gets[6]

$$\mathrm{E}[I_T] = \int_0^\infty \exp[-(\mu_p - \sigma_p^2)\, s]\, {}_s p_x\, ds. \tag{18}$$

For convenience, we define the function

$$\Upsilon(\xi|m,b,x) := \int_0^\infty \exp\{-\xi s\}\, {}_s p_x\, ds, \tag{19}$$

which, after substituting ${}_s p_x$ and changing variables, is equivalent to

$$\Upsilon(\xi|m,b,x) := \exp\left[\exp\left(\frac{x-m}{b}\right) + (x-m)\xi\right] b\Gamma\left[-b\xi, \exp\left(\frac{x-m}{b}\right)\right], \tag{20}$$

where $\Gamma(u, v) = \int_v^\infty e^{-t} t^{(u-1)}\, dt$ denotes the incomplete Gamma function. By construction of Equation (19), the term $\Upsilon(\xi \mid m,b,x)$ coincides with the Gompertz price of a life annuity under a continuously compounded force of interest ξ. Without loss of generality, we use

$$\lim_{m\to\infty} \Upsilon(\xi|m,b,x) = \frac{1}{\xi}, \tag{21}$$

which makes Equation (20) applicable to the perpetuity case as well. Going back to Equation (18), the expectation of the stochastic present value of lifetime consumption (of one dollar) is

$$\mathrm{E}[I_T] = \Upsilon(\mu_p - \sigma_p^2|m,b,x). \tag{22}$$

The same technique can be employed to obtain the second (central) moment:

$$\mathrm{E}[I_T^2] = \frac{\Upsilon(\mu_p - \sigma_p^2|m,b,x) - \Upsilon(2\mu_p - 3\sigma_p^2|m,b,x)}{\frac{1}{2}\mu_p - \sigma_p^2}. \tag{23}$$

The variance of the stochastic present value is

$$V[I_T] = \mathrm{E}[I_T^2] - \mathrm{E}^2[I_T]. \tag{24}$$

Higher moments can be obtained with the same method. For the sake of completeness, we explicitly provide the first and second moments for the stochastic present value of perpetual consumption, as

$$\mathrm{E}[I_\infty] = \frac{1}{\mu_p - \sigma_p^2}, \qquad \mathrm{E}[I_\infty^2] = \frac{2}{(\mu_p - \sigma_p^2)(2\mu_p - 3\sigma_p^2)}. \tag{25}$$

4.2 Gamma Distribution

Parker (1993) uses approximation techniques to derive a cumulative density function for the present value of a portfolio of annuities. Vanneste, Goovaerts, and Labie (1994) use Laplace transforms. We use a slightly different method. In particular, we refer to Milevsky, (1997) and Dufresne (1990) for a proof that $\int_0^\infty (H_s)^{-1}\, ds$ is reciprocal Gamma distributed. The Milevsky (1997) proof uses the scale function of the net-wealth process W_t in conjunction with (our) Lemma 1 to show that the probability of W_t ever crossing zero is equivalent to the probability that a suitably defined Gamma variate is less than c/w. This result serves as the impetus for *approximating*[7] the distribution of the stochastic present value, $\int_0^{\tilde{T}} (H_s)^{-1}\, ds$, by the reciprocal Gamma distribution. The reciprocal Gamma distribution, as its name implies, is a random variable whose reciprocal obeys a Gamma distribution. The probability density function (pdf) of the Gamma distribution is parameterized by two variables, g_1 and g_2, and is mathematically represented as

$$p[Y \le d] = G(d|g_1,g_2) = \int_0^d g(y|g_1,g_2)\, dy = \int_0^d \frac{\exp\left(-\frac{1}{g_2} y\right) y^{(g_1-1)}}{\Gamma(g_1) g_2^{g_1}}\, dy. \tag{26}$$

The probability density function (pdf) of the reciprocal Gamma distribution is defined by $X = 1/Y$ and is mathematically represented as

$$p[1/Y \le d] = p[X \le d] = G_R(d|g_1,g_2), \tag{27}$$

which, by a simple change of variables, equals

$$\int_0^d g_r(x|g_1,g_2) := \int_0^d \frac{\exp\left(-\frac{1}{x g_2}\right) x^{-(g_1+1)}}{\Gamma(g_1) g_2^{g_1}}\, dx. \tag{28}$$

[6] These results are confirmed by Boyle (1976) in discrete time and by Beekman and Fuelling (1990, 1991, 1992), who derived the first two moments of the annuity present value in continuous time using function space integral techniques under a variety of interest rate dynamics.

[7] In the context of Asian option pricing Milevsky and Posner (1998) use Monte Carlo simulations to demonstrate that moment matching the finite integral to the reciprocal Gamma distribution provides accurate values.

The expected value of the RG distribution is

$$\mathrm{E}[RG] = \frac{1}{g_2(g_1 - 1)}, \tag{29}$$

and the variance is

$$V[RG] = \frac{1}{(g_2)^2(g_1 - 1)^2(g_1 - 1)}. \tag{30}$$

Conveniently, we can express the parameters of the reciprocal Gamma distribution as a function of the mean and variance. Specifically,

$$g_1 = \frac{\mathrm{E}[RG]^2 + 2V[RG]}{V[RG]} \tag{31}$$

and

$$g_2 = \frac{V[RG]}{\mathrm{E}[RG]\,(\mathrm{E}[RG]^2 + V[RG])}. \tag{32}$$

Using the mean and variance from the previous subsection, we can thus compute the ruin probability as

$$P^{\text{alive}}_{\text{ruin}} := p\left[I_T \geq \frac{w}{c}\right] = \left[(I_T)^{-1} \leq \frac{c}{w}\right] \approx G\left(\frac{c}{w}|\hat{g}_1, \hat{g}_2\right) \tag{33}$$

and

$$P_{\text{ruin}} := p\left[I_\infty \geq \frac{w}{c}\right] = \left[(I_\infty)^{-1} \leq \frac{c}{w}\right] = G\left(\frac{c}{w}|\bar{g}_1, \bar{g}_2\right), \tag{34}$$

where $\hat{g}_1$ and $\hat{g}_2$ are the Gamma parameters for the conditional case

$$\hat{g}_1 = \frac{\mathrm{E}[I_T]^2 + 2V[I_T]}{V[I_T]}, \qquad \hat{g}_2 = \frac{V[I_T]}{\mathrm{E}[I_T](\mathrm{E}[I_T]^2 + V[I_T])}, \tag{35}$$

and $\bar{g}_1$ and $\bar{g}_2$ are the Gamma parameters for the unconditional case

$$\bar{g}_1 = \frac{\mathrm{E}[I_\infty]^2 + 2V[I_\infty]}{V[I_\infty]}, \qquad \bar{g}_2 = \frac{V[I_\infty]}{\mathrm{E}[I_\infty](\mathrm{E}[I_\infty]^2 + V[I_\infty])}. \tag{36}$$

4.3 Discrete Mortality Tables

For those who prefer to work with discrete mortality tables, we present expressions for $\mathrm{E}[I_{\tilde{T}}]$ and $\mathrm{E}[I_{\tilde{T}}^2]$, which can be used in Equation (35) to obtain the values of $\hat{g}_1$ and $\hat{g}_2$ needed for Equation (33). They are discrete-time versions of Equations (22) and (23), using a summation instead of an integral in Equation (19). Specifically,

$$\mathrm{E}[I_{\tilde{T}}] = \sum_{i=1}^{N} \exp\left\{-(\mu_p - \sigma_p^2)\frac{i}{12}\right\} {}_i p_x \left(\frac{1}{12}\right) \tag{37}$$

and

$$\mathrm{E}[I_{\tilde{T}}^2] = \frac{\sum_{i=1}^{N} \exp\left\{-(\mu_p - \sigma_p^2)\frac{i}{12}\right\} {}_i p_x - \sum_{i=1}^{N} \exp\left\{-(2\mu_p - 3\sigma_p^2)\frac{i}{12}\right\} {}_i p_x}{12\left(\frac{1}{2}\mu_p - \sigma_p^2\right)}, \tag{38}$$

where ${}_i p_x$ is the conditional probability of survival from *month* x to *month* $(x + i)$, and N is the number of months in the mortality table.

5. Numerical Demonstration

In this section we provide some numerical examples of the conditional ruin probabilities, using Canadian mortality and capital market estimates. Specifically, we focus on the situation of *2+1 assets* in which there is one risk free asset (r) and two risky assets (μ_1, σ_1)—the equity market—and (μ_2, σ_2)—the bond market—with correlation coefficient ρ. The appropriate risk-return vectors and matrices are

$$\boldsymbol{\mu} = [r, \mu_1, \mu_2], \tag{39}$$

and

$$\Sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_1 & \rho \\ 0 & \rho & \sigma_2 \end{pmatrix}, \tag{40}$$

respectively, where

$$\boldsymbol{\alpha} = [(1 - \alpha_1 - \alpha_2), \alpha_1, \alpha_2] \tag{41}$$

is parameterized by two independent variables for convenience. Thus, Equation (6) becomes

$$\mu_p = \boldsymbol{\mu\alpha}' = (1 - \alpha_1 - \alpha_2)r + \alpha_1\mu_1 + \alpha_2\mu_2 \tag{42}$$

TABLE 1
PROBABILITY OF RUIN FOR FEMALE AGED 65, w/c = 14, EQUITY VS. BONDS VS. T. BILLS

E\B	0%	20%	40%	60%	80%	100%
0%	[1.00].548	[1.00].518	[1.00].495	[1.00].479	[.999].472	[.993].470
20%	[.999].426	[1.00].399	[.999].380	[.996].371	[.979].370	
40%	[.991].342	[.981].319	[.957].306	[.921].300		
60%	[.884].299	[.849].281	[.811].269			
80%	[.755].284	[.719].267				
100%	[.673].285					

while Equation (7) becomes

$$\sigma_p = \sqrt{\boldsymbol{\alpha}\sum\boldsymbol{\alpha}'} = \sqrt{\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 + 2\alpha_1\alpha_2\rho\sigma_1\sigma_2}. \quad (43)$$

We will revisit the *2+1 asset* case in the section with numerical examples.

5.1 Mortality Data

We fit a Gompertz distribution using a nonlinear optimization routine in S-plus to the *Life Tables, Canada and Provinces 1990–1992* (Statistics Canada) and obtain the parameter estimates of $m = 81.95$, $b = 10.6$ for males and $m = 87.8$, $b = 9.5$ for females. For example, the probability that a 65-year-old male lives to age 85, using Equation (10), is $p(\tilde{T} \geq 20 \mid 81.95, 10.6, 65) = 0.3226$; likewise, the probability that a 65-year-old female lives to age 85, using Equation (10), is $p(\tilde{T} \geq 20 \mid 87.8, 9.5, 65) = 0.5199$.

Although analytic mortality laws are currently not in vogue in the actuarial community (see Bowers et al. 1986), we prefer to use an analytic Gompertz formulation because of its analytic tractability and our heuristic agenda.[8] Indeed, the concept underlying this paper can be applied using any mortality table.

5.2 Capital Markets Data

Ibbotson Associates, a consulting firm located in Chicago, provides forecast data for long-term Canadian investment returns. All the rates of return are continuously compounded. In this example we focus on three asset classes: the classic cash, bonds, and equities division:

- A deterministic *real* risk-free rate of $r = 2\%$,
- A government bond index with *real* parameters $\mu_1 = 3.5\%$ and $\sigma_1 = 11\%$, and
- An equity index with *real* parameters $\mu_2 = 8\%$ and $\sigma_2 = 19\%$.

Our numerical example assumes that the correlation coefficient between the *real* rate of return on the bond fund bond fund and the *real* rate of return on the equity fund is zero.

5.3 Numerical Results

We can substitute the data into Equations (38) and (39) to obtain estimates of the unconditional and conditional probability of ruin for a male or a female of a given age with an initial wealth and desired fixed level of consumption. In other words, we can estimate the sustainability of a given standard of living (conditional) or an endowment in perpetuity (unconditional, unrelated to gender or age).

Table 1 demonstrates a single application: a female, aged 65, with a *real* wealth to consumption ratio, $w/c = 14$. The absolute values of initial wealth and desired real consumption are irrelevant; only the ratio matters. The investment allocations are varied by increments of 20%. In the headers we show the proportion of the funds in bonds (B, or α_1). Down the stub column we show the proportion in the equity fund (E, or α_2). It is implicitly assumed that $1 - \alpha_1 - \alpha_2$ is allocated to the risk-free rate, short sales are not allowed, and hence we only display the upper-triangular region of the table. (There is nothing in our general methodology that precludes short sales or leverage.) The number in the square brackets is the unconditional probability of ruin P_{ruin}, and the number next to it is the conditional probability of ruin $P_{\text{ruin}}^{\text{alive}}$. As intuition dictates, P_{ruin} is always greater than $P_{\text{ruin}}^{\text{alive}}$. However, we remind the reader that while the former is precise, the latter is only an approximation.

[8]In fact, real-world use of this technique should involve a dynamic adjustment to a group annuity mortality table such as the Johansen (1995) update of the 1983 I.A.M. table.

For example, a 60% allocation to bonds with a 20% allocation to equity and (the remainder) 20% allocation to cash will result in a $\mu_p = 0.041$ and $\sigma_p = 0.07615$ as per Equations (42) and (43). This, in turn, will result in a value of $E[I_{\tilde{T}}] = 13.596$ and $\sqrt{V[I_{\tilde{T}}]} = 5.5308$, as per Equations (22) and (23). In other words, the expected value of the discounted stochastic lifetime consumption of one dollar—the PV of the annuity—is equal to 13.596, which is slightly lower than the initial wealth-to-consumption ratio of 14. The volatility of the present value is 5.5308. Intuitively, we see that there is a strong possibility of conditional (and obviously unconditional) ruin. Finally, we compute the parameters $\hat{g}_1 = 8.0428$ and $\hat{g}_2 = 0.010443$ using Equation (35). These numbers are plugged into the cumulative density function of the Gamma distribution evaluated at the reciprocal of the wealth-to-consumption ratio, to arrive at $G(\frac{1}{14} \mid 8.0428, 0.010443) = 0.3712$, which is a conditional probability of ruin of 37.12%. Thus we conclude that the $w/c = 14$ *standard of living is sustainable,* under the above mentioned allocation, with 62.88% probability. The lowest conditional probability of ruin occurs (very roughly) with an allocation of 80% equity, 20% bonds, and 0% cash and is equal to 26.7%

Table 2 displays the results for a 65-year-old *male* with $w/c = 14$. As one would expect intuitively, the conditional probabilities of shortfall are uniformly lower for all asset allocations as a direct result of the shorter life span. The unconditional probabilities remain the same since we have not modified the capital market parameters.

Once again, the lowest probability of ruin occurs with a high allocation to equity and very little in bonds.

Other values can easily be generated using a spreadsheet and the "optimal allocation"—in the ruin probability minimizing sense—can be located by visual inspection or by differentiating Equations (33) and (34) and finding a vector α^* that satisfies first and second-order conditions.

5.4 Alternative Perspective

Appealing to the notion of value-at-risk, an alternative use of this framework is to fix a certain ruin tolerance level ε, and then locate the maximum lifetime consumption c^* that can be achieved as a function of asset allocation.

Mathematically,

$$\max_{\{\vec{\alpha}\}} \mathbf{c}$$

subject to

$$\Pr\left[\inf_{0 \le s < \tilde{T}} \{W_s \le 0\} | W_0\right] \le \varepsilon,$$

where $\vec{\alpha}$ is the asset allocation vector. For example, a 65-year-old female with an initial wealth of $w = \$100{,}000$, can consume up to $c = \$5{,}000$, per annum and still have a probability of ruin that is less than or equal to $\varepsilon = 5\%$, provided she maintains a 50% allocation to equity and 50% allocation to bonds.

6. Conclusion

In this paper we introduce a method to estimate the conditional and unconditional probability of ruin for an individual (retiree) with a stochastic life span who is consuming a fixed real amount from a diversified investment portfolio. Conceptually, we show that the probability of ruin is equivalent to the probability that a suitably parametrized Asian call option will expire in-the-money, thus allowing the use of derivative pricing technology to compute the relevant probabilities.

Finally, we apply our formula—using Canadian data—with realistic market values for equity and fixed income investments to show that for an individual male or female the *conditional* probability of ruin is minimized with a relatively high allocation to equity until quite late in life.

TABLE 2
PROBABILITY OF RUIN FOR MALE AGED 65, $W/C = 14$, EQUITY VS. BONDS VS. T. BILLS

E\B	0%	20%	40%	60%	80%	100%
0%	[1.00].325	[1.00].307	[1.00].295	[1.00].291	[.999].292	[.993].230
20%	[.999].250	[1.00].234	[.999].225	[.996].223	[.979].228	
40%	[.991].206	[.981].193	[.957].186	[.921].185		
60%	[.884].188	[.849].177	[.811].170			
80%	[.755].186	[.719].176				
100%	[.673].195					

Appendix: Proof of Lemmas 1 and 2

The stochastic differential equation (5), which defines the dynamics of W_t, can be solved to yield

$$W_t = H_t\left[w - c\int_0^t (H_s)^{-1} ds\right], \qquad (44)$$

where the fundamental solution H_s is

$$H_s = \exp\left\{\left(\mu_p - \frac{1}{2}\sigma_p^2\right)s + \sigma_p B_s\right\}. \qquad (45)$$

This can be confirmed by applying Ito's lemma,

$$dW_t = \frac{\partial f(t,B_t)}{\partial t}dt + \frac{\partial f(t,B_t)}{\partial x}dB_t + \frac{1}{2}\frac{\partial^2 f(t,B_t)}{\partial x^2}dt, \quad (46)$$

to the function $f(t,x)$ defined by Equation (8) and demonstrating that it leads to the SDE in Equation (5).

From a qualitative point of view, Equation (8) contains two parts, an exponential function, H_s which is always greater than zero, multiplied by the term in square bracket, whose sign is indeterminate. Therefore, the process W_t, will be less than or equal to zero (ruin) at some future time t^*, *if and only if* the term in square brackets is less than or equal to zero. In other words,

$$W_{t^*} \le 0 \quad \Leftrightarrow \quad w \le c\int_0^{t^*} (H_s)^{-1} ds. \qquad (47)$$

On the other hand, the integral $\int_0^{t^*} (H_s)^{-1}\, ds$ is monotonically nondecreasing with respect to the upper bound of integration t^*. This means that once $c\, \int_0^{t^*} (H_s)^{-1}\, ds$ *becomes* greater than w, it *stays* greater than w. Consequently, we arrive at our result that the probability W_t crosses zero, prior to some time t^*, is equivalent to the probability that $W_{t^*} \le 0$. More precisely,

$$p\left[\inf_{0\le t\le t^*}\{W_t \le 0\}\right] = \Pr\left[\frac{w}{c} \le \int_0^{t^*} (H_s)^{-1} ds\right]$$

$$= \Pr[W_{t^*} \le 0]. \qquad (48)$$

This completes the proof.

Q.E.D.

Acknowledgments

This research was supported by a grant from the York University Research Authority and the Social Sciences and Humanities Research Council of Canada. The authors would like to acknowledge Kwok Ho for conversations that stimulated the development of this paper. In addition, the authors would like to thank Sharon Kim for excellent research assistance. Any errors are the exclusive fault of the authors.

REFERENCES

Beekman, J. A., and Fuelling, C. P. 1990. "Interest and Mortality Randomness in Some Annuities," *Insurance: Mathematics and Economics* 9:185–96.

Beekman, J. A., and Fuelling, C. P. 1991. "Extra Randomness in Certain Annuity Models," *Insurance: Mathematics and Economics* 10:275–87.

Beekman, J. A., and Fuelling, C. P. 1993. "One Approach to Dual Randomness in Life Insurance," *Scandinavian Actuarial Journal* 12:173–82.

Bowers, N., Gerber, H., Hickman, J., Jones, D., and Nesbitt, C. 1986. *Actuarial Mathematics*. Schaumburg, Ill.: Society of Actuaries.

Boyle, P. P. 1976. "Rates of Return as Random Variables," *Journal of Risk and Insurance* 43: 693–713.

Browne, S. 1997. "Survival and Growth with a Liability: Optimal Portfolio Strategies in Continuous Time," *Mathematics of Operations Research* 22:468–93.

Dufresne, D. 1990. "The Distribution of a Perpetuity with Applications to Risk Theory and Pension Funding," *Scandinavian Actuarial Journal* 9:39–79.

Frees, E. W., Carriere, J., and Valdez, E. 1996. "Annuity Valuation with Dependent Mortality," *Journal of Risk and Insurance* 63, no. 2:229–61.

Geman, H., and Yor, M. 1993. "Bessel Processes, Asian Options and Perpetuities," *Mathematical Finance* 3, no. 4:349–75. October.

Johansen, R. J. 1995. "Review of Adequacy of 1983 Individual Annuity Mortality Table," *Transactions of the Society of Actuaries* XLVII:101–25.

Karatzas, I., and Shreve, S. E. 1992. *Brownian Motion and Stochastic Calculus*. 2nd ed. New York: Springer.

Kemna, A., and Vorst, A., 1990. "A Pricing Method for Options Based on Average Values," *Journal of Banking and Finance* 14:113–29.

Merton, R. C. 1993. *Continuous-Time Finance*. Rev. ed. Cambridge, Mass.: Basil Blackwell.

Milevsky, M. A. 1997. "The Present Value of a Stochastic Perpetuity and the Gamma Distribution," *Insurance: Mathematics and Economics* 20:243–50.

Milevsky, M. A., Ho, K., and Robinson, C. 1997. "Asset Allocation Via the Conditional First Exit Time or How to Avoid Outliving Your Money," *Review of Quantitative Finance and Accounting* 9, no. 1: 53–70. July.

Milevsky, M. A., and Posner, S. E. 1998. "Asian Options, The Sum of Lognormals and the Reciprocal Gamma Distribution," *Journal of Financial and Quantitative Analysis* 33, no. 3:409–22.

Panjer, H. 1986. "Direct Calculation of Ruin Probabilities," *Journal of Risk and Insurance* 53, no. 3: 521–29.

Panjer, H., and Bellhouse, D. 1980. "Stochastic Modeling of Interest Rates with Applications to Life Contingencies," *Journal of Risk and Insurance* 47: 91–110.

Panjer, H., and Bellhouse, D. 1981. "Stochastic Modeling of Interest Rates with Applications to Life Contingencies, Part II," *Journal of Risk and Insurance* 48:628–37.

Parker, G. 1993. "Distribution of the Present Value of Future Cash Flows," in *Proceedings of the 3rd Annual AFIR Colloqium,* pp. 832–43. Schaumburg, Ill.: Society of Actuaries.

Parker, G. 1994. "Stochastic Analysis of a Portfolio of Endowment Insurance Policies," *Scandinavian Actuarial Journal* 2:119–30.

Pentikainen, T. 1980. "The Theory of Risk and Some Applications," *Journal of Risk and Insurance* 47:16–43.

Richard, S. F. 1975. "Optimal Consumption, Portfolio and Life Insurance Rules for an Uncertain Lived Individual in a Continuous Time Model," *Journal of Financial Economics* 2:187–203.

Samuelson, P. A. 1969. "Lifetime Portfolio Selection by Dynamic Stochastic Programming," *Review of Economics and Statistics* 51, no. 3:139–246. August.

Sinha, T. "The Effects of Survival Probabilities, Transactions Costs and the Attitude towards Risk on the Demand for Annuities," *Journal of Risk and Insurance* 53, no. 2:301–7.

Turnbull, S., and Wakeman, L. 1991. "A Quick Algorithm for Pricing European Average Options," *Journal of Financial and Quantitative Analysis* 26:377–89.

Vanneste, M., Goovaerts, M., and Labie, E. 1994. "The Distribution of Annuities," *Insurance: Mathematics and Economics* 15, no. 1:37–48.

Warshawsky, M. J. 1988. "Private Annuity Markets in the United States: 1919–1984," *Journal of Risk and Insurance* 55, no. 4:518–28.

Williams, A. C. 1986. "Higher Interest Rates, Longer Lifetimes, and the Demand for Life Annuities," *Journal of Risk and Insurance* 53, no. 1:164–71.

Yagi, T., and Nishigaki, Y. 1993. "The Inefficiency of Private Constant Annuities," *Journal of Risk and Insurance* 60, no. 3:385–412.