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GENERATING STOCHASTIC INTEREST RATE SCENARIOS

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Recorder: MICHAEL F. DAVLIN

What are the alternative approaches? Do you need a commercial scenario generator? What are the pitfalls associated with generating scenarios? How does one test for arbitrage freedom?

MR. MICHAEL F. DAVLIN: I was fortunate enough to be able to recruit four very capable, expert, and experienced individuals in this area. Accordingly, my role will be limited to introducing them. Our topic is trying to answer the question, "What constitutes a good interest rate scenario generator?"

Craig Merrill is an assistant professor of finance at Brigham Young University. Craig will start us off in the shallow end of the pool by giving us a very basic introduction to interest rate scenarios. Some of you may have seen or heard Craig's presentation in New Orleans at the spring meeting. It was very well received and covered what Craig called the arithmetic of option pricing. He is also co-authoring a monograph for the Society, which is titled " Valuation of Interest-Sensitive Financial Investments" [for ordering information, call 215-598-8926]. Craig's coauthor is David Babbel, and I understand that this work is nearly finished and should be available soon.

Gordon Klein will address the different ways that you can evaluate interest rate processes and talk about ways you might rank them from best to worst according to the purposes to which you intend to use resultant scenarios. Gordon caught my attention with a paper he wrote for the *Transactions* called "The Sensitivity of Cash-Flow Testing to the Choice of Statistical Model for Interest Rate Changes." [*TSA* XLV (1994): 79-186]. By using stable Paretian distributions, Gordon showed pretty dramatically that the tails on both ends of the distribution are very important for determining the capitalized value of your institution.

Mark Tenney is the president of Mathematical Finance Company in Arlington, VA. Mark is not the type of person who likes to follow what everyone else is doing. He has invented some new very novel and intriguing mathematical solution techniques for existing interest rate processes. Mark has also invented a new interest rate process that he calls a double mean reverting process (DMRP), which he'll explain during the course of his presentation. Mark will also address some of the qualitative aspects of interest rate behavior that should be reflected in any good scenario generator.

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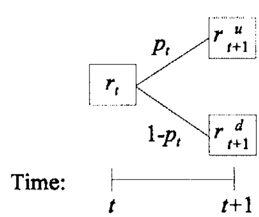
Dave Becker will wrap up our discussion. As I am sure most of you already know, Dave is chief actuarial officer at Lincoln National. Dave has done a great deal of work looking at the history of interest rate movements, and he has evaluated several interest rate scenario generators and interest rate processes. He will talk about some quantitative ways to evaluate a resultant set of interest rate scenarios to see if they comport well with what has been observed historically.

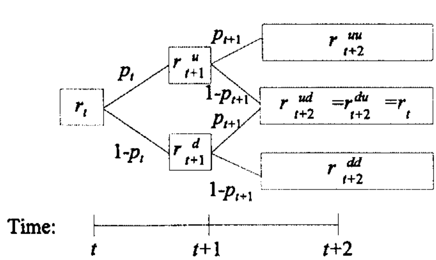
MR. CRAIG B. MERRILL: I've been asked to introduce the topic of stochastic interest rate modeling. To do this, I will start with the most basic building blocks of an interest rate model and hope that we gain some insight from that. We'll learn a little about what it means to have an arbitrage-free interest rate model, and we'll talk a little bit about what the differences are between equilibrium and arbitrage-free modeling techniques. I'll then introduce you to some extensions to the basic model—multifactors—which will set the stage for what Mark Tenney will talk about in his presentation.

The most basic model of interest rate uncertainty is a binomial lattice. In essence, we know that interest rates are going to change from one day to the next. We look out into the future, and we know that there's uncertainty. The question is, how do we model that? We could just randomly draw numbers from a hat, but that probably doesn't reflect reality too well. We could choose some distribution and make random drawings from that, but there's a question about what distribution to use. But to understand the basic ideas of interest rate modeling, the easiest setting is just a simple binomial model in which we assume that the interest rate can go up or down by some amount and with some probability between now and our next time period of interest. This simple time step from time t to time t+1 could be a minute, an hour, a day, a week, or a month. It depends on the period of time that we're interested in. But the uncertainty is being captured by this idea that over one time period the interest rate could move either up or down. Then we repeat this process over many time periods.

GENERAL BINOMIAL MODEL



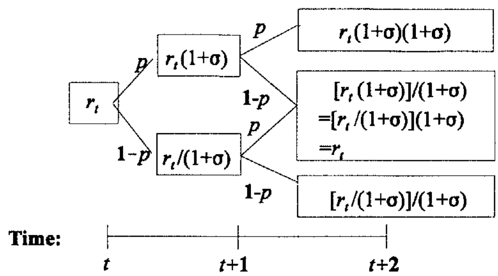
TWO-PERIOD BINOMIAL MODEL



For example, we have interest rate *r* at time *t*. At time *t*+1 it could be *r* up or *r* down. At *t*+2 it could be *r* after either an up-up, or an up-down, or a down-up, or a down-down movement. We fill this lattice, which goes out into the future with as much refinement as we'd like, by putting in many time steps. We can generate all kinds of distributions, in the limit, by carefully choosing how the interest rate moves up or down and the probability of that movement.

A very simple example would be something I call the multiplicative binomial model.

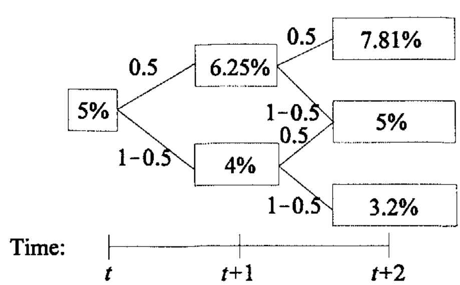
MULTIPLICATIVE BINOMIAL MODEL



An interest rate at the beginning period can either go up by multiplying times a factor of $(1+\sigma)$, or go down by dividing by $(1+\sigma)$. If you do this enough times, in the limit you will generate a lognormal distribution of interest rate. This is a very simple model that we might use to capture the uncertainty surrounding interest rate movements. So, for example, if we had a 50/50 probability of moving up or down, and we had a sigma of 25%, then the interest rate can go from 5% up to 6.25%, or down to 4% and so on through the lattice.

With two periods in our lattice, we can illustrate the basic ideas of an arbitrage-free interest rate model.

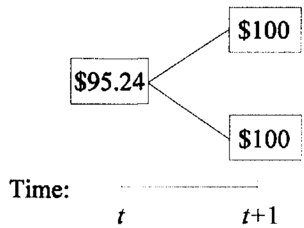
EXAMPLE OF MULTIPLICATIVE MODEL



I will use these numbers: 0.50%, 6.25%, and 4.00%. We won't need the last period's interest rate(s). Consider a one-year Treasury strip. To value it, I take the one-year rate of interest and discount \$100, if that is the face value. We have no need for an interest rate model. We have today's one-year spot rate; we're done. But, if we were to put it in the lattice framework, we would say that if the interest rate goes up or down, we get $100 at time t+1; so we discount that at 5.00%, and we get a value of $95.24. So far, we have introduced nothing of great interest, and it has all been easy.

ONE-YEAR STRIP

- $95.24=$100/(1.05)



However, with a two-year strip, we have a question about where the one-year spot rate will be in one year. Now there is some uncertainty to start to model and to try to capture. There are two ways we can do that. The first would be what I call discounting by paths. For example, the interest path could be 5.00% for the first year and then 6.25% for the second, or it could be 5.00% for the first year and 4.00% for the second; we face an equal 50% probability of either.

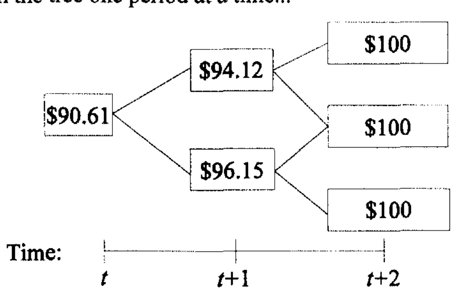
I would just discount the value of $100 down one path and multiply times a probability of 0.5, and then do likewise for the other path which, too, has a probability of 0.5, and by adding them together, I would arrive at a value of $90.61. That approach works great for two periods, but what if we have 30 periods? We would have 2^{30} as the number of paths we must evaluate—well over one billion—and we would have to calculate each and every one of them individually. That gets to be a little difficult. This is where our lattice model becomes very useful, because we can do this same valuation more efficiently by using iterations of single-period valuations.

TWO-YEAR STRIP

- Discounting by paths:

0.5[100/(1+0.0625)(1+0.05)]+0.5[100/(1+0.04)(1+0.05)]=$90.61

- Or, work through the tree one period at a time...



For example, we set this up on the lattice. At the end of the second period, we're going to receive \$100. So, at each end node of the lattice, I just write down \$100. That much is known, so we start with that. Then we consider this simple branch up here. Remember that the interest rate here is 6.25%. So we can discount the \$100 back to here, and we get this value of \$94.12. We can do the same thing on the lower-end node, after which we have a single period valuation problem right here: going back from time $t+1$ to t. So we take the \$100, and we bring one back at 4%, and one back at 6.25%.

Now we have a valuation problem that involves some uncertainty. We have a 50% chance of having our two-year strip move to a value of \$94.12 in one year if interest rates go up, or to a value of \$96.15 in one year if interest rates go down. So we take these two values, multiply them by their probability, and discount them back at 5.00% to get a value of \$90.61, which is exactly what we got by discounting by paths. So now we have this lattice structure that we can move back through iteratively, and we can do as many periods as we would like with as much refinement as we would like, but with a greatly reduced number of calculations in comparison to the technique of discounting by paths.

Within this two-period framework, we can say a little bit about spot rates and forward rates and what it means to be arbitrage-free. The spot rate is 5%; that was given, we know that at time t. We can deduce the two-year spot rate by knowing that the value of the two-year strip is \$90.61. We know that there is a relationship between the price and the spot rate. So, we divide the \$100 by 1 plus the two-year spot rate squared and solve for the spot rate, and we get 5.054%.

SPOT RATE TERM STRUCTURE

- One-year spot rate: 5%
- Two-year spot rate can be found using:

$$\$90.61 = \$100/(1+s_2)^2$$

- Solve to get the two-year spot rate:

$$s_2 = \sqrt{(\$100/\$90.61)} - 1 = 5.054\%$$

Now we have a two-year spot rate generated by our lattice model. Something that's very nice about doing these models in the lattice framework is that at each node in the lattice, the entire term structure is actually captured. If I were to go out three periods in the lattice, I'd have the three-year spot rate. If I went out four periods, I'd have the four-year spot rate. Again, this is assuming one-year per period in the lattice. If we move out one period in our lattice so that, instead of being at what appears to be the first period, we're out here somewhere; well, if we just regenerate the lattice with this as the initial period, we have a new lattice so to speak—one that is actually embedded in the original one. At that point in time, we also have an entire term structure implicit in the lattice. So our simple model of the short rate moving through time is actually a model of the entire term structure moving through time.

Now what does it mean to be arbitrage-free? There is a relationship between forward rates and spot rates that has to hold. Otherwise there will be arbitrage opportunities. So we need to look at this relationship in considering whether our interest rate process is realistic in some sense, or valid.

Consider two securities. Asset 1 will be $100 in two periods, and that is discounted by the sequence of two forward rates: the forward rate now, which is the one-year spot rate, and the forward rate in one year, which would be the one-year spot rate in one year. Asset 2 consists of $100 in two periods, and is discounted by the two-period spot rate.

With either security, in two years we're going to get $100. So these two assets have to be worth the same amount, otherwise we could sell one and buy the other and make a riskless profit. They have to be worth the same amount. If they are worth the same amount, then we can set the two valuation formulas equal to each other and solve for the forward rate one year from now. That is called an implied forward rate. When we solve for that value, and I just illustrated that quickly here, we have found the forward rate that we can lock in today for one year from now. If it were any other value, then we'd have arbitrage.

If we fit our model and find that there is a difference between the forward rate that our model generates and this implied forward rate, we have a problem. That's one of the things that we use when looking for problems in an interest rate model. Alternatively, if we fit a model and find that in pricing certain securities we get the "wrong" price, so that compared with the market there seems to be arbitrage between our model price and the market prices, that's another sign of a problem.

This leads us into the question of how to go about valuation. There are two kinds of broadly defined models that we can work with: equilibrium models and arbitrage-free models. Now this gets a little bit tricky because this actually has two different meanings. To the economist, an equilibrium model and an arbitrage-free model means one thing, but for an interest rate model generator the distinction is a little bit different, subtly different. To the economist, an equilibrium model is one within which you would specify a framework and all relevant factors are jointly determined endogenous to the model.

So, for example, the inflation rate, interest rates, demand, supply—all those things are endogenously determined within the model simultaneously. Out of that comes a price for discount bonds and, therefore, an interest rate model. An example in the literature is the Cox, Ingersoll, Ross model.

An arbitrage-free model assumes that certain things are exogenous; for example, the stochastic process that the short rate follows or the stochastic process that the inflation rate follows. Those things are given exogenously. Then, given those things, it's the absence of arbitrage between those fundamentals and other securities that determines the prices of the other securities. Now, in practice, they lead to the same prices. It's a bit of an academic distinction as to whether you're developing a model within an equilibrium framework or within an arbitrage-free framework because you end up at exactly the same valuation equation.

When it comes to interest rate models, Hull and White (actually Ho and Lee started it), called a model arbitrage-free for a different reason. For Hull and White, the equilibrium and arbitrage-free distinction takes on a little bit of a different meaning. Let me give you an example. These are two stochastic differential equations that are descriptions of interest rate models. The first one is the Cox, Ingersoll, Ross model, which is derived in an equilibrium framework:

$$dr = \kappa(\mu - r)dt + \sigma\sqrt{r}dz$$

where dr is the increment in the interest rate. Then there is mean reversion; that is, the short-term interest rate, r, is reverting to some level, μ, at a rate κ. So, if the difference between the current interest rate and μ is μ-r, that difference will be closed at a rate κ. Then there's the stochastic part, in which we have some volatility parameter, σ, and then the volatility of the increment process is proportional to r, and then there's this standard normal kick, dz. So this is one model of how the short rate would evolve over time. Now, in a lattice framework we would discretize this model. Instead of dr, we would have Δr, and instead of dt we would have Δt. We would have the volatility of the process determined in a way that makes the limiting distribution the same as this continuous process. We would take a set of data over time and fit these parameters as constants: κ, μ, and σ. That would be our equilibrium interest rate model. We would fit those three parameters to the data and use them as constants.

What both Ho and Lee and Hull and White call an arbitrage-free model is something like this other one.

$$dr = [\theta(t) - ar]dt + \sigma\sqrt{r}dZ$$

Again, it's a mean reversion model with the same volatility structure, but they've added this parameter θ as a function of t, which is a time-dependent function. Now they will generate a *lattice* in implementing this in discrete time, a lattice for 30 or 60 periods. Then they will fit the lattice starting from the first period, moving out to the second and third and fourth, and they will solve for the values of θ that create a lattice that exactly reprices Treasury securities. So we would have a market Treasury spot rate for one period, two periods, three periods, four periods, and we would solve for the value of θ that exactly regenerates that term structure.

With the equilibrium model we cannot exactly fit the term structure, except by some huge quirk of fate. With the constant parameters, there will be a little bit of an error. If you think of the term structure of time on the horizontal axis and interest rate on the vertical axis, you would have something that goes like this. With an equilibrium model, the best you can hope for is to get a good fit to that term structure. With an arbitrage-free model, in the sense that it provides an exact fit today and exactly reprices these securities, we would get something that goes from dot to dot as our term structure that's generated by the model.

There are pros and cons to each. Empirical research shows that the arbitrage-free model, if carried out for several periods beyond the point of estimation, tends to fit worse than the equilibrium model at the same distance out from the point of estimation. That is, a smooth curve tends to give a better description over some period of future time than this jagged, arbitrage-free fit is able to generate. So if our goal is to generate future scenarios, an equilibrium model might be a better choice because it tends to perform better as we project out into the future. If our goal is to price securities today and look for arbitrage opportunities or mispricing, then we are probably better off using an arbitrage-free model because that exactly captures today's market.

These are both single-factor models, and by single factor I mean that there is one stochastic factor in the model. This *dz* term is modeling the uncertainty in the market, and we have just one *dz* term; therefore, one source of uncertainty. These single-factor models tend to be inadequate. We see either more curvature in the term structure or movements that we can't capture with a single factor. People have tried to extend for multiple factors. One very easy extension to see is adding an inflation rate. So instead of trying to model nominal rates with a single factor, we model the real rates this way and then have an inflation factor so that together we get the nominal interest rate. That was explored by Cox, Ingersoll, and Ross in the same paper where they discussed this model.

Other ideas have been, for example, to take this model of the short rate that Cox, Ingersoll, and Ross have and then make one of the parameters stochastic instead of constant. For example, Fong and Vasicek have a model where V is equal to σ^2. V is the instantaneous variance of the interest rate process, and it is stochastic with the same sort of mean reversion structure as the interest rate itself. So, at each point in time, we have some change in volatility, and then that new volatility is involved in the interest rate movement; actually they are simultaneously occurring.

- Stochastic Volatility:

$$dv = \gamma(\alpha - v)dt + \xi\sqrt{v}dZ$$

Another alternative is that we could make μ, the level that the interest rate is reverting to, stochastic. So, for example, μ could follow some sort of a mean reversion process itself.

- Stochastic Mean Reversion:

$$d\mu = \gamma(\alpha + \mu)dt + \xi\sqrt{\mu}dZ$$

We have this long-run average interest rate, μ, that the short rate is fluctuating around and is always being pulled toward. But μ itself is a moving target. This *dz* is the uncertainty factor driving the interest rate movement. This over here would be a separate *dz*, a separate Brownian motion representing the uncertainty in that particular factor. Those two Brownian motions could be correlated or not, depending on the way the model is structured. Multiple factors allow for a greater variety of term structures to be fit and represented. We could have multiple humps, for example. We can allow for greater flexibility in curvature at the short end. A wider variety of interest rate term structures can be fit with a model such as that.

In practice, though, any of these models are implemented in discrete time, either by using some sort of lattice framework or by using a numerical approximation to a partial differential equation. Each of these models generates asset prices that are solutions to partial differential equations when done in continuous time. So the choice of solution method becomes important as well. An extended Vasicek model is similar to the Cox, Ingersoll, Ross model except instead of the square root of r in the volatility there's no r in the volatility term. So there is a normal distribution of interest rates. Admittedly, that allows for the possibility of negative rates, but the mean reversion is such that it is quite unlikely. Hull and White create a trinomial lattice. The trinomial lattice allows for an interest rate to go up, go in the middle, or go down. The middle is not necessarily equal to today's interest rate. We can show that, in general, this trinomial lattice approximation is identical to a numerical solution to the continuous partial differential equation, and we get a bridge between the discrete time and the continuous time approaches.

These authors have shown that this trinomial lattice is capable of fitting any arbitrary term structure. Now, again, the model that they're working with is arbitrage-free with time-dependent parameters. By using two time-dependent parameters, they are able to fit not only any arbitrary term structure of interest rates, but also any arbitrary volatility structure as well. By fitting the two time-dependent parameters as functions of the spot rates and the volatility, they are able to provide a fairly general model of interest rates.

MR. GORDON E. KLEIN: What I will talk about is a little less mathematical and more philosophical. What I'm going to talk about is—if you have several mathematical models of interest rates to choose from, and you have a particular purpose in mind, how do you go about choosing one? Theoretically, how would you go about choosing one, and then when can you really implement that theory and when can you not?

I would define a mathematical model of the term structure of interest rates as being a set of stochastic or deterministic differential or difference equations and their parameters. It says, given where we are now, what are the probabilities and directions of change, or if it's a deterministic model, where are we going to be at some point in the future? An example of a deterministic model is the law of gravity. Instead of calling it the model of gravity, we call it the law of gravity because it seems to be very close to what happens in reality. It kind of sets the standard for what we wish we could do for a model of a term structure, but I don't think that we are to that point yet.

A model does not have to be deterministic to be that accurate or to be characterized as a law. The model for radioactive decay is a stochastic model, and I think it still has that essential characteristic of being so close to reality that we refer to it as a law instead of as a model of reality. Once we believe that we have got a law instead of just a mathematical model, we're really saying that we believe that law or that model is not just an approximation to reality, but in some sense it's a blueprint of reality. We believe, for instance, that some underlying process called gravity behaves in that way; not just that this is a mathematical model that somehow approximates how this thing called gravity works, but that we've found a blueprint of the process of gravity, and we now know how that process works. So if we believe that we have such a thing in a particular case, then we have a law, and we can come as close to reality as we want.

For some aspects of reality that can be quantified, such as the term structure of interest rates, we don't believe that we have a law for describing how they work. No one would stand up and say, for instance, Cox, Ingersoll, Ross is the law of interest rate changes, because it can be shown that it doesn't match reality, and it is, therefore, a model of reality. That's not to say that it's useless, just that it's only a model and not a law. I think that there are several fields where we do have models of some quantifiable aspect of reality and that we do use these models. We use them in meteorology, sociology, psychology, economics, finance, and biology. In many different fields there are models that are mathematical and that are not laws.

I think it's crucial when using such models to distinguish reality from the model of reality. It's often the case that somebody takes a model of reality and finds some of the implications of this model, and it's crucial not to think that means that reality has those same implications. I think you could even go further and say that for any model of reality that is not a law, we can find some implications of that model that will be quite far from what's true in reality. We can find things that are not true and things that are absurd. The lognormal model of interest rates, for example, is frequently criticized because over a sufficiently long period of time, the probability that interest rates hit any particular level, say a million percent, is one. Now that is an absurd aspect of that model. However, I don't think that a model should be thrown out because it has absurd implications, because then we would have to throw out all our models, other than the ones like the law of gravity and the law of radioactive decay. I think it's too high a standard to say a model cannot have any absurd implications; otherwise we'd have to throw most of them out.

That brings us to the question then of how we evaluate models and decide which ones are good and which ones aren't. Is the lognormal model one that we want to use? Is another one? We have many proposed models for term structure changes, and if none of them is a law, then we have to find some way to choose among them. To evaluate a mathematical model, you first need to know what it is you're trying to model, and here we're talking about the term structure of interest rates over time or the probability distribution of how those change over time. The second thing we need to know is the particular purpose we have in mind for the model. Now this is somewhat different than if we have a law. We don't need to say what we're using the law of gravity for. It should hold in all cases. With other kinds of models, though, a model that is appropriate for one particular purpose may not be appropriate for other purposes. We may say a particular model of the term structure is great for pricing options and then find that it's lousy for cash-flow testing, for instance.

An example of what is a good model for some purposes but not for others is the well-known approximate method for converting from Celsius to Fahrenheit temperature. Of course, we know the exact model. The approximate model is to double the Celsius and add 30. For many purposes this is just fine, and so in that sense I think it's a good model.

An example of a purpose for which this would be a poor model would be if I wonder if my son has a fever. I get a Celsius thermometer and take his temperature. The reading turns out to be 37.5 degrees. Doubling that is 75; I add 30; and I conclude that his temperature is 105 degrees. But the exact value is only 99.5 degrees. So this is a good model for one purpose—for instance, should I wear a coat? It's a good model for that. But when determining whether my son has a fever, it's a poor choice.

In general, I think for any model that's not a law you would find that it's appropriate for some purpose, and you can find at least one purpose for which it's inappropriate; otherwise, you would say it's a law. So the point is you need to have a purpose in mind. If you ask what term structure model you should use, you have to finish that sentence with—for this particular purpose. I'm going to focus on cash-flow testing as the purpose for the rest of my discussion. Therefore, whatever I say may not be entirely appropriate for other purposes.

You can in some sense—at least theoretically—formalize your statement of purpose by stating a penalty function. This would be a function for a particular purpose. If you had a value of reality and a value that your model produces, this function would identify how large a cost you assign to your model and come up with a value that's different than reality or the same as reality. You might want to assign a cost of zero if your model comes up with something that's exactly what reality comes up with. That might be one trait you would want for this penalty function, or you might not. At first I wanted to say this behaves just like a metric so that it would be positive definite and obey the triangle in equality and be symmetric. But this doesn't really have to be the case. If you are trying to fool yourself in some way, and then if your model matches reality exactly, you may assign a cost to that. I don't see any reason why we can't allow for that.

This is, in some sense, a metric except that it doesn't really have to do all the things that a metric has to do. If you knew what reality was, and you knew what your model came up with, you could say, how much does it cost? So with the Celsius to Fahrenheit conversion it would cost, I guess, whatever I assigned to the panic of rushing my son to the emergency room and saying his fever's 105 and then having the hospital staff tell me that it's really 99.5. That would be a high cost that will lead to throwing out that model. It's not the fact that the model is wrong that makes us throw it out for that purpose, it is the fact that there is a high cost with a high probability.

If you knew the distribution of what was going to happen in reality, if you could model that somehow as a random process and knew the distribution, you could find the expected value of this penalty function; the expected cost due to the model being different from reality, or the same as reality in some cases. You could also assign a cost to the evaluation of the model. For instance, if you're going to have to spend a lot more for a computer to run a much more sophisticated model, that is also a cost. If a model is easier to run, then that makes its cost lower. That is the virtue of the Celsius to Fahrenheit model: double C and add 30; it's simple enough that you can most likely do it in your head. If you added up these two costs, then you'd have a cost that you could assign to each model.

Once you have come up with what it is you're modeling, in our case the term structure of interest rates, and your purpose, which I said in this case is cash-flow testing, you could then assign a metric that reflected your purpose of cash-flow testing, that would be different from the metric if you were doing option pricing or pricing of insurance or some other purpose. Once you've done all this, then this total cost will order the models from best to worst, or from worst to best, however you choose to look at it. If it has a high cost, then it is a bad model for that purpose. If it has a low cost, it's a good model.

Gravity is a good model because, assuming that we want the right answer so that our metric for reality matching the model is zero, then gravity, as near as we know, always does that. The model of gravity gives us exact reality, and so that cost is zero. What's the cost of implementing the model—who knows what that is? But, whatever that is, it could be compared with other models, and we could put them in order from the worst to the best. So we have an ordering over the set of all possible models for a particular purpose.

Assume that the cash-flow testing model was perfect, and there are no problems with it other than of the selection of the interest rate model. Let's just freeze cash-flow testing; however you do it, everything else is right. Assume that your interest-sensitive lapse assumptions, your prepayments on your collateralized mortgage obligations (CMOs), and everything else are alright. Assume that at the end you're looking at the right thing—whatever that may be. Then for every interest rate model, you have an answer from your cash-flow testing model. In other words, for every interest rate or term structure model, your cash-flow test is a function that assigns that model to a particular value. So the cash-flow testing model is a mapping from interest rate or term structure models to the result that you get at the end of all the cash-flow testing. As you change the models for interest rates, you get different answers at the end.

If there were a law of term structures, like there's a law of gravity, we could then put that law into your cash-flow testing model and out would come an answer. That result would be the one that you're measuring the distance from. This is a theoretical value—whether it could ever exist, we don't know. But the law of term structures put through your cash-flow model gives you a particular value. This is reality. This is the value you are shooting for in your cash-flow testing. What you want to evaluate to tell which term structure model is good or bad is how much different the answer with your term structure model is from the answer with the law of term structures, in reality. You have this penalty function that says what the difference between the two will cost you. If you're overreserving, that has a cost. If you're underreserving, that has an associated cost. Presumably, if you were reserving at exactly the level you should be, then that would be the best outcome, so that would have a zero cost. Although if the cost of running that model were millions of dollars, then you would have to account for that, too.

I think there is an obvious problem with all this, and that is that we do not have a law of the term structure. That means that we cannot find the expected penalty function for a given model of the term structure. It is only if we know what reality is that we can find an ordering of these different models of reality. So I think for the purpose of cash-flow testing, we cannot order term structure models. We cannot say with any perfect certainty that one term structure model is better than another. I think that you cannot favor one over another necessarily because it has particular qualities—I'm going to hedge that a little bit in a second, though.

I think that the true criterion for evaluating one term structure model versus another is the total cost, which is the expected penalty or cost from being wrong, plus the cost of running the model. I think, however, we do know some things about cash-flow testing and what it does to a particular term structure model, the effect of plugging particular term structure models into the cash-flow-testing model. In particular, we know that the cash-flow-testing model is very sensitive to the tails of the distribution. If you have one model in which it's not very likely that interest rates take big jumps and another one in which it's very likely that they take big jumps, then the difference in your cash-flow-testing results will be greatly dependent on that difference in how fat the tails are. That's one thing we know.

Another thing we know is that frequency of inversions and degree to which the yield curve or the term structure inverts has an important impact. I think there may be other things that we could put in the category of not being so important as far as impact on the cash-flow-testing model. I think one would be whether the particular model is expressed as discreet or continuous. That will not have a big impact on this. So, while we cannot come up with a perfect way of saying which term structure models are better and which are worse, I think we can focus in on those things that we would like to have in a term structure model—such as it should not have fatter or narrower tails than what happens in reality.

Our next two speakers will talk about a particular model that has some particularly nice characteristics when compared historically with reality.

GENERATING STOCHASTIC INTEREST RATE SCENARIOS

MR. MARK S. TENNEY: So far, we have had an overview of two interest rate models from Craig and a discussion of evaluating models by Gordon. I want to talk about a particular model that I've described in the handout. I will not be able to go through all the things I put in the handout but I'd like to touch on the high points.

Essentially, there are two uses for a stochastic interest rate generator. These uses of a stochastic generator are first, to feed a system such as profit test system (PTS) or Tillinghast actuarial software (TAS) in which you want to do cash-flow testing or portfolio management or evaluate risk and return for portfolios and strategies, and second, to feed a set of interest rate scenarios to a software system for pricing securities, such as a mortgage or an interest rate derivative. You shouldn't use the same set of paths for one purpose as you use for the other.

For cash-flow-testing systems, you want to use what you think are the real probabilities of interest rate scenarios and yield curves moving around. For pricing, you want to put in the effect of risk adjustment. Risk adjustment basically says that bad things are more likely to occur than they really are likely to occur. So if interest rates going up is bad because that's when you lose money on an average investment bond, then interest rates going up is what's bad. For my risk-adjusted scenarios, I make it more likely that interest rates would go up. If yield curve inversions are bad, I make it more likely that yield curve inversions will occur in my risk-adjusted process. We were talking at breakfast, and Dave Becker pointed out that in risk-adjusted models, typically the probability of yield curve inversion is 50% so that your yield curve is inverted approximately 50% of the time. This obviously is different from reality. Therefore, you need a much different process for your actual process.

The requirement for a stochastic interest rate generator is essentially projecting yield curve scenarios that are inclusive. We want to get all the yield curves that could happen and in an order they could happen. We want to exclude impossible yield curves. If we're going to give up part of our return—for example, by buying a hedge or by choosing a less risky strategy—we don't want to be giving up that return for risks that cannot occur. We don't want to be looking at impossible yield curve scenarios, saying "our portfolio tanks in these, so we're going to do something less risky," because these scenarios could not occur. We want to avoid hedging phantom risks.

We want to try to reflect the correct probability of yield curve scenarios occurring. That's obviously the most difficult requirement to meet. Following along with that, we want to correctly forecast expected returns for our securities. We also want to have the joint process on the security prices and cash flows properly modeled, because that is what will determine the result of our cash-flow testing. I will focus most of my comments on cash-flow testing, as opposed to pricing, because I think that's more relevant to what many people here are interested in.

We can think about our yield curve generator and ask it to reproduce some reasonable set of facts about interest rate scenarios that are true just by casual observation. What are those facts?

1. Interest rates don't go to zero or positive infinity, as was discussed earlier by the previous speaker.

2. Interest rates can spend up to several years within a narrow band of the trading range. For example, in the early 1980s, high interest rates persisted for several years. Right now, more moderate rates have persisted for several years. In the early 1960s, very low rates persisted for several years. So there can be a period of time in which rates are at a level and are fluctuating within a relatively small band around that level.

3. Short-term and long-term rates are correlated, but not perfectly.

4. The volatility of long-term rates is less than that of short-term rates. This is familiar to all of us.

5. Yield curves can have a variety of shapes.

6. The higher the level of rates, the higher the absolute level of interest rate volatility.

Each one of these qualitative stylized facts will directly impact the results of cash- flow testing, portfolio analysis, and risk analysis. The band effect—persistence of a rate within a band for several years—will clearly have an impact. If rates go up and stay up for five years, that will be very different than if rates go up and then come back down. Likewise, if they go down and stay down for five years, that has a much bigger impact than if they just go down and come back. So the banding effect will have a critical impact on all results of cash-flow testing.

Short-term and long-term rates not being perfectly correlated will have a big impact, because typically our assets have a longer duration than our liabilities as we try to pick up the risk premium in the yield curve. Typically, we're taking on some risk by mismatching assets and liabilities, and we need to know what can happen to us because short rates could move up, and the yield curve could invert, or simply the steepness of the curve could become less. The variety of shapes adds to that risk factor. Volatility increasing with the level of rates will have another obvious impact upon the volatility of our returns.

Dave Becker has done some historical research on what we can call quantitative stylized facts. As I said before, Becker's quantitative stylized facts included some of these probabilities: the frequency of yield curve inversion, the frequency of rates being between 10% and 11%, and so forth. If we look at those events during the last 40 years, we get some numbers, and, as the previous speaker pointed out, we want those numbers to correspond closely to what our generator produces to get meaningful results. This basically boils down to the fact that to figure out the probability of a sequence of yield curves is a very difficult problem. It is difficult because we have only one sequence, the historical one over the last 40 years. So how will we decide what the probabilities of yield curve sequences are?

The only way we can do that is to try to develop some smaller set of reality of which we can address the probability, things such as the probability of the yield curve being inverted. We can observe that something happened 11% of the time. So we can adjust our model to reproduce that. Or, how long has the ten-year yield been between 10% and 11% during the last 40 years? We can adjust our model to reproduce that as well. By matching a number of facts simultaneously, we can then address what the probabilities of interest rates are. If our model has a reasonable coherence and logic to it, then those probabilities may very well translate into the probability of the yield curve sequences. Of course, if the model doesn't have much logic to it, then that probably won't happen.

Let me tell you what my model is. It's the double mean reverting process, or DMRP™, and it builds from the lognormal interest rate process.

The risk-adjusted DMRP is given by defining the short term rate of interest, $r=e^u$, and

$$du=\kappa_1(\theta-u)dt+\sigma_1 dz_1$$

$$d\theta=\kappa_2(\bar{\theta}-\theta)dt+\sigma_2 dz_2$$

where dz_1 and dz_2 are pure Brownian motions with a correlation factor of ρ.

Craig discussed another way of formulating a double mean reverting process earlier. The formulation given here is essentially one that I developed. We'll go through the use and the rationale for it, but let me first define what is does.

The first thing we do is take the logarithm of the interest rate r—call it variable u—to take the advantages of the lognormal model and have them for our model as well. These advantages are, first, that interest rates are always positive. We know that because the interest rate, r, is equal to e^u, which is a positive function. Then you get the fact that for higher levels of the interest rate you get higher volatility. We get those two things out of using the lognormal.

We now have a double mean reversion process in these two equations. The first equation tells us that u is attracted toward θ, which is a target but is disturbed by a random component dz_1. In turn, θ is attracted to $\bar{\theta}$. So, if u is less than θ, θ-u is positive, so u tends to go up. If u is bigger than θ, θ-u is negative, u tends to go down, and the same is true as far as θ relative to $\bar{\theta}$. Now by calibrating to historical rates, we find that κ_1 is a number in the range of 0.5–0.7—so there is a 50–70% return toward the target per year—and that κ_2 will be around 5–15% per year.

What this means is that u is strongly attracted toward $\bar{\theta}$. So, θ will establish the trading range. But the trading range established by θ can move slowly back toward θ. So a trading range can persist for several years, just as we've observed empirically. The DMRP can model the banding effects that are so obvious as we look at the last 20 or 30 years of U.S. interest rates.

Next I'd like to introduce the risk-unadjusted version of the DMRP model of interest rates. The risk-unadjusted DMRP™ is given by defining the short term rate of interest, $r=e^u$, and

$$du=\kappa_u(\theta-\lambda_u)dt+\sigma_u dz_1$$

$$d\theta=\kappa_\theta(\bar{\theta}-\lambda_\theta-\theta)dt+\sigma_\theta dz_2$$

where dz_1 and dz_2 are pure Brownian motions with correlation factor of ρ.

That first process I presented was the process on the actual rate. However, we can re-interpret θ (instead of being the actual target rate) as being a risk-adjusted target rate. Then we reinterpret those first two equations as being a risk-adjusted process to use for pricing.

These two equations are the risk-unadjusted process for the risk-adjusted θ. I know that's sort of a convolution to get through, but the idea is that we can set up this model here where θ now refers to essentially a variable determining the shape in the yield curve. So it will be a variable typically higher than the short-term rate.

Look at what the first of these two new equations tells us. It tells us that there's a variable, λ_u, which is standing between u and θ. So when we simulate, on average, the difference between θ and u would be λ_u. Now, because θ is the risk-adjusted target rate measuring the slope of the yield curve, we know that on average θ is bigger than u. So, λ_u should be a positive number corresponding to the historical frequency of the yield curve being positively sloped by some amount.

The second equation reflects the same idea. We have θ being an average distance from $\overline{\theta}$. $\overline{\theta}$ is now a risk-adjusted target rate. That represents where the long-end of the curve is trending toward.

There's an average distance again between θ and $\overline{\theta}$ that represents the idea that θ is really an intermediate point on the yield curve to some extent; not exactly that, but it determines that structure. Then the θ to $\overline{\theta}$ difference also helps determine the average slope of the yield curve.

One other thing I'd point out here is when you go to the risk-adjusted form, all you do is set λ_u and λ_θ to zero, and then you'll see what happens. U, on average, goes toward θ, which means 50% of the time it's bigger than θ and 50% of the time it's less. Although it is not exactly the same thing as yield curves, you can view it in that way.

This means 50% of the time the yield curve is inverted and 50% of the time it is not, corresponding closely to what Dave has experienced with risk-adjusted models. So we see that these risk-adjustment parameters are key to making our interest rate process correspond to historical fact about yield curve inversion and the probability of rates.

We estimated this model by fitting the interest rate model parameters to the yield curve over different subperiods and the overall period from 1970 to 1994.[1] You can see by looking at the 1970-94 results in Table 1, that κ_u is about 69%. Now this is the risk-adjusted process that κ_u refers to.

This indicates a 69% per annum speed of mean reversion. The κ_θ is 3.7% per year, so this is very low for the risk-adjusted process. We have a very high risk-adjusted target

[1] The estimation is reported in the paper, "The Double Mean Reverting Process," by Mark S. Tenney, Working Paper, Mathematical Finance Company, 4313 Lawrence Street, Alexandria, Virginia 22309.

rate of 10.68%, and then we have the volatility and so forth, and we have a variation within the subperiods.

TABLE 1
DMRP MODEL ESTIMATION AND DESCRIPTION FOR RISK UNADJUSTED PROCESS IN FIRST FORM

Parameter	1970-79	1980-89	1985-94	1970-94
λ_u	0.1350	0.1290	0.3761	0.1982
λ_θ	0.0669	0.2595	0.1728	0.2830
κ_u	0.76	0.97	0.64	0.69
κ_θ	0.10	0.037	0.054	0.037
$\overline{\theta}$	7.64	12.62	9.37	10.68
σ_u	0.33	0.31	0.21	0.30
σ_θ	0.23	0.22	0.23	0.21
ρ	0.20	0.45	0.46	0.32
N	999	1080	1080	2619
σ	15.0	15.9	13.5	16.0

Table 2 simply discusses the process where we had a target θ, which is the risk-unadjusted target. In this case, the κ's are kept the same. Now I'm going to talk about a case where κ's become different. With the risk-adjusted κ's, we find we can't represent the quantitative stylized facts about interest rate movements that Dave has analyzed.

TABLE 2
DMRP MODEL ESTIMATION AND DESCRIPTION FOR RISK UNADJUSTED PROCESS IN SECOND FORM

Parameter	1970-79	1980-89	1985-94	1970-94
κ_u	0.76	0.97	0.64	0.69
κ_θ	0.10	0.037	0.054	0.037
$\overline{\theta}^w$	6.229	8.555	5.411	6.602
σ_u	0.33	0.31	0.21	0.30
σ_θ	0.23	0.22	0.23	0.21
ρ	0.20	0.45	0.46	0.32
N	999	1080	1080	2619
σ	15.0	15.9	13.5	16.0

In particular, our value for κ_θ of 0.037, or 3.7% per year, is too low, which allows interest rates to go high. It allows the probability of interest rates being above 15% or

20% to be higher than what's historically been observed. Those larger, fatter tails have a significant impact on cash-flow-testing, even though it may be just a few scenarios. Just a few extra scenarios of high interest rates can bias the results of the cash-flow-testing run or risk analysis run and give potentially misleading results.

We can estimate a new κ_θ for the risk-unadjusted process that is higher; 0.15, or 15% a year. κ in this case falls down to 50% per year. I then adjusted the value of λ_θ as well to calibrate to Dave's stylized facts. This translation from risk-adjusted to risk-unadjusted gives me some parameters I can use to do this calibration. I still have my same fit to the Treasury curves that I had before. That was a fit where I had a standard deviation of 16 basis points measured in yield terms, model yield minus market yield. This is a good fit over the last 25 years, and essentially tells us that we can model the historical set of yield curves. I can retain that good fit for my pricing, and at the same time I can have good fits to the quantitative stylized facts for my cash-flow testing, just by adjusting these κ's and λ's for the risk-adjusted coefficient.

When I do this, I essentially end up with saying that the $\overline{\theta}$, in terms of what my average interest rate will be, is now referring to the risk-adjusted process of 10.68%. When you work your way down through λ_θ and λ_u, λ_θ is 0.47. So we go from 10.68% and reduce that by 47%. That gives me an average θ of around 6.6%. Then I have λ_u, which is 19%. So, we take 6.6% and take off about 20%, which gets us down to about 5.5%. So my average value for the short-term interest rate will be 5.5%. My average value for θ will be 6.6%. So I wind up with a positively sloped yield curve for the risk-unadjusted process. I also have a value of the average interest rate of 5.5%, which is considerably less than the risk-adjusted target rate average of 10.68% in the risk-adjusted process.

My main message is essentially that we want an interest rate process that reflects the qualitative stylized facts I described early on and that does a good job of describing the quantitative stylized facts that represent reasonable requirements for an interest rate model. These quantitative stylized facts have a direct translation into whether we get reasonable results for cash-flow testing. I was able to get a good fit to Treasury curves over the last 25 years, within 16 basis points in that overall period with a fixed set of parameters. That tells me that I can do a good job fitting the set of curves that could occur because that will vary from period to period. The fact that I'm restricting myself to a two-factor model that has reasonable qualitative properties and that can fit reasonably the quantitative stylized facts tells me there's not much freedom left to reproduce impossible yield curves, such double-humped yield curves, or whatever.

The DMRP model is producing obviously not the true set of yield curves, but something relatively close; something that Gordon described as getting us close enough for use in a reasonable set of analytic applications.

MR. DAVID N. BECKER: Part of my presentation will be a follow-up to Edger E. Peters' speech at the Investment Section Council breakfast yesterday. I have been working, off and on, in that area since 1991 when Edger's first book on using chaos theory techniques in the capital markets was published. A more detailed presentation of some of the items I will mention here will be in an article in the next issue of *Risks and Rewards*. ("Some Observations on U.S. Treasury Interest Rates: 1953–1988," No. 24, December 1995, pp. 7–12).

I would like to lead off with the observation that, in the work we do, which has tremendous financial implications, we often make explicit assumptions. We may not always be comfortable about having to make those or choosing how to set them, but we make those explicit assumptions. Always remember that, quite often in the work you do, you may be making hidden assumptions as well—things that you do or accept automatically. So, whenever you get results that don't look right, go back and look for the hidden assumptions in what you're doing—things that were just assumed, that you didn't even realize. Sometimes you can find that this thing—this hidden assumption—makes a significant difference.

Last, what do you want to look at? What are the implications of the assumptions you made? In particular, with regard to this morning's topic, we want to try to estimate how good, in some sense, our assumptions are as measured by comparison to model output—I should say *model output characteristics* in the case of term structures and interest rate movements—to the characteristics of historical rates. We are not trying to reproduce historical rates, but we are trying to say what the characteristics of historical rates are. What are the characteristics of the rates that we get out of our models and, therefore, how comfortable or not are we with regard to those assumptions we made that generated those items?

I want to continue with a brief background on a paper that I wrote back in 1989 or so. As you'll notice, almost all interest rate generators use the lognormal process at their foundation. The lognormal process with a mean or drift of the distribution of zero is a typical assumption, except for when you make the process arbitrage-free by finagling so that it is no longer drift-free. The standard deviation or volatility is typically assumed to be constant. The error terms are considered to be independent, which means that when you look at the realization of rates that emerge out of your model, the emergence is independent. There is no correlation among the rates this period and the rates last period or two periods ago, or ten periods for that matter.

I tested several different data sets because I wanted to make sure that any conclusions that I might reach were not going to be influenced by the particular data set I used. The first result of these statistical tests showed that you cannot reject that the mean is equal to zero.

Second, you can reject that the standard deviation is constant. So, volatility is not constant over the period from the end of 1953 through the end of 1988. For a lot of the subsequent work, I've taken the analysis up to the end of 1994. I used a particular test called Laird's test because Laird's test is robust for nonnormal distributions. Some significance tests for volatility depend on the underlying distribution being normal. If the underlying distribution is not normal, you can get erroneous conclusions. So, I wanted a test that was robust, and I used that. I found I can reject the hypothesis that the volatility or standard deviation is constant.

Third, I tested the hypothesis that the distribution is lognormal by testing the log of the ratios in the rates. That was easier to do. Two useful tests for this are called *standardized skewness*, which tests distribution symmetry, and *standardized kurtosis*, which tends to measure how peaked the distribution is at the center and how fat the tails are compared with the normal distribution. Both those tests showed that you can reject the hypothesis that distribution of rates is, in fact, lognormal.

Last, the assumption that rates emerge independently over time was also rejectable by examining autocorrelation coefficients. This last item has some interesting implications, and I will go into that in a minute.

People often talk about alternatives to the lognormal assumption when trying to explain the observed shape of the distribution of interest rate movements. If I assume a lognormal distribution, but with nonconstant standard deviation, I can match the shape of historic rates and their probability distribution and density functions. If I assume a lognormal distribution with autocorrelation in the generation of the rates, that, too, can reproduce the distribution actually observed by looking at rates during 1953 and 1988. If you assume a lognormal distribution with autocorrelated rates and constant volatility, you still reproduce the distribution. So, you can combine both those assumptions.

Finally, you can use the stable Paretian assumption, which Gordon mentioned, and which has been the basis of recent work in financial chaos theory; it does the same job. Note that the implications behind these different ways of explaining the same data are nontrivial and have some deep significance.

I have a pattern that is somewhat useful to discern in that it shows the correlation between the rate at month *t* and the rate at the prior month, *t*-1. This is the relationship between the changing of rates between *t* and six months prior and seven months prior. The autocorrelation coefficients beyond seven months were not statistically different from zero, so you don't see them. For lags for months two, three, four, and five back, they, too, were not statistically different from zero. But there's a clear pattern for lags of six and seven months. You see these numbers because they're different from zero, and they're statistically significant, often at more than the 0.01 level of significance. The interesting thing is how there's a tendency for all these to be in the same range. It says that the change in rates this month will be positively correlated with the change in rates last month, but with regard to the change in rates six and seven months ago, the change will be negatively correlated.

Here's some empirical evidence of mean reversion. Notice again that these are all about the same order of magnitude, although there is a tendency to decline as you get toward the longer ten-year maturity. There's a declining order of magnitude as you get to the longer maturity. So the degree of negative correlation with sixth- and seventh-month back changes in interest rates diminishes as the maturity of the security gets longer. That pattern will come out rather interestingly in another way.

For those of you who heard Edger Peters' talk (this is now where I will mention things that specifically will be in much more detail in the upcoming *Risks and Rewards* article), you know there is a global analysis technique called rescaled range, or *R/S* Analysis. It was developed by a hydrologist by the name of Harold E. Hurst back in the 1920s and 1930s. That technique was later stumbled and elaborated upon by Bernard Mandelbrot. He and other people have since used this as a very interesting tool. Essentially, it's a global level analysis, and it looks for characteristic behavior in an observed series of a process over time. It asks, are there meaningful characteristics at a global level? The autocorrelation coefficients we discussed, however, tend to tell you about a local level of characteristics.

I don't really want to spend a lot of time here. The key on *R/S*—and it's really better done in that article—is that a quantity called the rescaled range can be represented in equation form as some constant times *n*, which represents the length of subintervals of the time series that you are decomposing, raised to the *H* power, where *H* is the Hurst coefficient:

$$(R/S)_n = cn^H$$

You can determine what that *H* factor is by essentially finding the slope of the log-log plot of the rescaled range for *n*. You look at intervals of sublength *n* compared to the whole interval. Basically, you solve for the regression slope of the line. If *H* turns out to be 0.5, that says your process is a pure Brownian motion. However, if *H* is between 0.5 and 1, we call that a persistent process. a persistent process is one in which if you had an uptick last time, then you're more likely to have an uptick this time. An antipersistent process is one in which if you had an uptick last time, you're now more likely to have a downtick. Conversely, if last time was a downtick, you're more likely to have an uptick this time.

Applying *R/S* to Treasury rates between 1953 and 1988, I found that when I plotted the log-log plot, it wasn't a nice, straight line. The straight line appeared to have somewhat of an elbow or bend. I was basically just playing with the statistics of it, trying to find or fit a line over a given period, and then fitting a line over the balance of the period. I found a natural break at five years, and that natural break seemingly was very consistent for all the maturities. The Hurst coefficient, *H*, found by taking the regression line for each of these maturities over the first five years, turned out to be the coefficients 0.70, 0.70, 0.57, and 0.64. Note that these are all greater than 0.5, and they are all about in the same order of magnitude.

Remember, a Hurst coefficient greater than 0.5 means a persistent process. That means that an increase in rates tends to be associated with a later increase in rates, or a decrease is associated with another decrease. But the five-year issue here means that the memory effect appears to only last five years. But when I fit the regression line by using subintervals greater than five years, the coefficients are less than 0.5; except the last one, and I'll return to that. This tends to say that the memory effect after five years tends to be antipersistent. An increase now tends to increase the probability of a decrease five years later.

One thing to note is that these coefficients start out here at 0.31, 0.34, 0.39, 0.43, and 0.51. Basically, that says for the ten-year rate that the effect is not statistically different than that for the five-year. But it also says that the strength of the global indicator of mean reversion or antipersistence is the strongest at the 90-day rate. It's strong, but it's a little weaker at one year and weaker still at three years. This is barely significant at the four-year rate, and this is right at the edge of significance for the five-year. By the end of five years there is no memory effect at all. This is interestingly very parallel to the pattern of those autocorrelation coefficients that we saw earlier.

If we think about this rather curious set of facts, what might account for it? Very crudely, what occurs approximately over five-year periods? The business, industrial, or production cycle. What happens in booms? Interest rates rise. What does the Fed do? It tries to dampen the cycle. It puts pressure on the rates to raise them to slow things down. When the boom ends and turns into a recession, interest rates tend to fall. Then what does the Fed do? It doesn't just let them fall naturally. It actually tries to aggressively drive them down to stimulate capital investments so the recession doesn't become too deep, and the economy tends to come back. So the business cycle runs about five years. Federal Reserve Board activity follows it trying to help tune the economy to avoid too high a recession with a lot of inflation and high interest rates and avoid busts where basically production is too low. So it lowers interest rates, making it more favorable for companies to borrow and get production going. That's a good connection.

When the Federal Reserve Board does its thing, where does it do it? At which end of the yield curve? Can the Federal Reserve Board have any meaningful impact at the ten-year term? No. But it can have a lot of effect at the 90-day term. We saw that antipersistence was strongest at the 90-day maturity. But changing the short end of the yield curve does have repercussions all along the yield curve. But, you would expect those repercussions to dampen. The pattern of these numbers suggests a dampening over time.

We can see numerically here a pattern of a main effect at the short end of the yield curve, which ripples out in a dampening way to the long end. By the time you get to the ten-year, it no longer has a statistical significant effect. Some very interesting ideas are presented and elaborated on in more detail in the *Risk and Rewards* paper.