Portfolio optimization with a GMMB and risk-adjusted fees

Anne MacKay
(UQAM, Université de Sherbrooke)

joint work with Adriana Ocejo (UNC Charlotte)

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Motivation

- Variable annuity: investment product for retirement savings
  - Financial guarantees during the accumulation and payout phase
  - Also include life insurance benefits

- Investment mix typically pre-determined, static.

- Guaranteed minimum maturity benefit
  - Payoff: max(account value, guaranteed amount)
  - Put option on account value

- Financial guarantee financed by a fee from the investment account.
  - Fee rate set such that the value of the VA is fair (from a risk-neutral perspective)
Motivation

Given a fee structure, what dynamic investment mix will be the most attractive for a policyholder?

▶ Variation of Merton’s portfolio problem

▶ Non-concave utility:
  - Financial guarantee: utility is non-concave in the terminal wealth (Carpenter, 2000; Chen, Hieber, and Nguyen, 2019)
  - S-shaped utility (Kahneman and Tversky, 1979, 1986)

▶ Guarantee fee:
  - Affects returns
  - Total rate depends on investment mix
  - Investment strategy is no longer self-financing

▶ Fair pricing constraint
Related work on (constrained) non-concave utility maximization:

- Carpenter (2000):
  - Manager compensation (unconstrained) problem.

- Chen, Hieber, and Nguyen (2019):
  - Hybrid investment-insurance contract
  - No fees

  - S-shaped utility
  - Constraints.
Impact of guarantee fee

- Fee rate reduces return, affects the value of the guarantee ⇒ Impacts policyholder behaviour (M. et al., 2017)

- If a dynamic investment mix is allowed...
  - How should the guarantee fee be set up?
  - How will the fee rate(s) affect the optimal investment strategy?
  - Is there an optimal way to set up the fee structure?
Setting

- Policyholder can invest in a risky asset $S$ and a risk-free asset $P$:

\[
\begin{align*}
    dS_t &= \mu S_t \ dt + \sigma S_t \ dW_t \\
    dP_t &= rP_t \ dt
\end{align*}
\]

- VA account value process built by investing the proportion $\pi_t$ in the risky asset and deducting the guarantee fee:

\[
dF_t = \pi_t F_t \frac{dS_t}{S_t} + (1 - \pi_t) F_t \frac{dP_t}{P_t} - dC_t
\]

- GMMB rider guarantees amount $G$ at maturity $T$:

Payoff: $\max(F_T, G)$, with $G = F_0 e^{\delta T}$. 

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- GMMB rider guarantees amount $G$ at maturity $T$:
  Payoff: $\max(F_T, G)$, with $G = F_0 e^{gT}$. 
Dynamics of VA account $F$

- Consider two levels of fee:
  - $c_F$ paid on the total value of the account, and
  - Additional fee $c_S < \mu - r$ paid on the risky investment.

- Accumulated fees up to $t$ follow:
  $$C_t = \int_0^t (c_F + \pi_s c_S) F_s \, ds, \quad C_0 = 0.$$  

- VA account value has dynamics:
  $$\frac{dF_t}{F_t} = [\pi_t (\mu - r - c_S) + r - c_F] \, dt + \pi_t \sigma \, dW_t$$
  $$= [\pi_t (\tilde{\mu} - \tilde{r}) + \tilde{r}] \, dt + \pi_t \sigma \, dW_t,$$

  with $\tilde{\mu} = \mu - c_S - c_F$ and $\tilde{r} = r - c_F$. 
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  \]
  with $\tilde{\mu} = \mu - c_S - c_F$ and $\tilde{r} = r - c_F$. 
Admissible investment strategies

\( \pi = (\pi_t)_{t \geq 0} \) is in the set of admissible trading strategies \( A(x) \) for an initial investment \( x \) if:

- \( \pi_t \) is \( \mathcal{F}_t \)-measurable,
- \( F_0^\pi = x \),
- \( F_t^\pi \geq 0 \) a.s.,
- there exists a unique solution to the SDE

\[
\frac{dF_t^\pi}{F_t^\pi} = [\pi_t(\widehat{\mu} - \widehat{r}) + \widehat{r}] dt + \pi_t \sigma \, dW_t.
\]
Classical utility function: \( u(\cdot) \) strictly increasing, strictly concave, and continuously differentiable on \( \mathbb{R}^+ \) and \( u(0) = \lim_{x \downarrow 0} u(x) \).

S-shape utility function:

\[
U(x) = \begin{cases} 
- U_2(\theta - x), & 0 \leq x < \theta, \\
U_1(x - \theta), & x \geq \theta,
\end{cases}
\]

with \( U_1, U_2 \) are classical utility functions with

- \( U_1(0) = -U_2(0) \geq 0 \),
- \( \lim_{x \uparrow \infty} U_1(x) = \infty \),
- \( \lim_{x \rightarrow \infty} U_1'(x) = 0, \lim_{x \rightarrow 0} U_1'(x) = \infty, \lim_{x \rightarrow \infty} \frac{xU_1'(x)}{U_1(x)} < 1 \) (Inada and asymptotic elasticity conditions).
We want to solve

\[
\max_{\pi \in A(F_0)} \mathbb{E}[U(\max(F_T^{\pi}, G))]
\]

subject to

\[
\mathbb{E}[\xi_T \max(F_T^{\pi}, G)] = F_0,
\]

where \( U \) is an S-shape utility function and \( \xi_T \) is the state-price density.

- \( \mathbb{E}[\xi_T \max(F_T^{\pi}, G)] = F_0 \) is the fair pricing constraint.
  - Fair pricing depends on the investment strategy \( \pi \).
Economic interpretation of the problem

- What dynamic investment mix can the VA provider offer if they want to maximize (some) policyholder’s utility while keeping the contract fairly priced?

- What does the resulting payoff look like?

- What is the “best” way to set the fees?
Solving the unconstrained problem

Use martingale approach with static optimization problem:

\[
\text{arg max } \mathbb{E}[U(\max(H, G))], \quad \text{s.t. } \mathbb{E}[\tilde{\xi}_T H] \leq F_0, 
\]

with \( \mathcal{H} = \{ H : H \text{ is } \mathcal{F}_T\text{-measurable, } H \geq 0 \text{ } \mathbb{P} - \text{a.s.} \} \) and where \( \tilde{\xi}_t \) is the state price density corresponding to the “fee-adjusted” market

\[
d\tilde{P}_t = \tilde{r} \tilde{P}_t \, dt, \quad d\tilde{S}_t = \tilde{S}_t (\tilde{\mu} \, dt + \sigma \, dW_t). 
\]

Optimal payoff can always be replicated because the fee-adjusted market is complete.

Main tool for solving the static problem: concavification of the utility function (Carpenter, 2000; Reichlin, 2013; Bichuch and Sturm, 2014)
Proposition 3.1 of M. and Ocejo (2022)

Let $U(\cdot)$ be an S-shaped utility function and $M := \max(\theta, G)$. The solution to the unconstrained static optimization problem (1) is given by

$$H^* = [I(\lambda \tilde{\xi}_T) + \theta]1_{\{\lambda \tilde{\xi}_T < \hat{y}\}},$$

(2)

where

- $I(x) = \left(U_1'(x)\right)^{-1}$,
- $\lambda \geq 0$ is such that $\mathbb{E}[\tilde{\xi}_T H^*] = F_0$, and
- $\hat{y} := U_1'(\hat{x} - \theta)$, where $\hat{x} \in (M, \infty)$ is the unique root of the equation

$$U_1(x - \theta) - xU_1'(x - \theta) - U(G) = 0.$$
Solving the constrained problem

1 Static problem:

\[
\max_{H \in \mathcal{H}} \mathbb{E}[U(\max\{H, G\})], \quad \text{s.t.} \quad \mathbb{E}[\xi_T H] \leq F_0,
\]
\[
\mathbb{E}[\xi_T \max\{H, G\}] = F_0.
\]

2 Representation problem: find \( \pi^* \in \mathcal{A}(F_0) \) s.t. \( F_{T \pi^*} = H^* \).

Admissible fees

For fixed maturity \( T \), guaranteed roll-up rate \( g \), define the set of admissible fees

\[
\mathcal{P}_{T, g} = \{(c_F, c_S) : \mathbb{E}[\xi_T \max\{H^*, G\}] \geq F_0, \text{ where } H^* \text{ solves the unconstrained static problem}\}.
\]
The Lagrangian of the static problem is given by

\[ L(x; y, z) := \tilde{U}(x) - xy - z \max\{x, G\}, \quad x \geq 0, \]

where \( \tilde{U}(x) := U(\max\{x, G\}) \),

- \( y := \lambda_1 \xi_T \) (from \( \mathbb{E}[\xi_T H^*] = F_0 \)), and
- \( z := \lambda_2 \xi_T \) (from \( \mathbb{E}[\xi_T \max\{H^*, G\}] = F_0 \)).
For each $y \geq 0$ and $z > 0$, the maximizer of the Lagrangian is:

$$
\chi(y, z) = \begin{cases} 
\chi_1(y, z) = [I(y + z) + \theta] \mathbb{1}_{\{\Delta(y+z) + zG > 0\}}, & \text{if } \theta \leq G, \\
\chi_2(y, z) = [I(y + z) + \theta] \mathbb{1}_{\{0 < y+z < U'_1(\hat{x}-\theta)\}} \mathbb{1}_{\{\Delta(y+z) + zG > 0\}}, & \text{if } \theta > G,
\end{cases}
$$

- $\Delta : [0, \infty) \mapsto \mathbb{R}$ is defined by $\Delta(a) := U_1(I(a)) - a[I(a) + \theta] - \tilde{U}(0)$.
- $\hat{x} \in (\theta, \infty)$ is the unique root of
  $$
  U_1(x - \theta) - (x - G) U'_1(x - \theta) - \tilde{U}(0) = 0.
  $$

In both cases, the maximizer is either larger than $\max(\theta, G)$ or equal to 0.
Idea of the proof

1. **Split** the Lagrangian in two:

\[
\sup_{x \geq 0} L(x; y, z) = \max \{ \sup_{0 \leq x < G} L(x; y, z), \sup_{x \geq G} L(x; y, z) \}.
\]

2. For \( x \in [0, G) \), \( L(0; y, z) = \tilde{U}(0) - zG \) is the supremum.

3. For \( x \in [G, \infty) \), write \( w := x - G \) and \( V(w) := \tilde{U}(w + G) \).

   - If \( \theta \leq G \), \( V(w) \) is concave \( \Rightarrow \) use first-order condition.
   - If \( \theta > G \), \( V(w) \) is not concave \( \Rightarrow \) use concavification techniques as in the unconstrained problem.
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Theorem 3.1 of M. and Ocejo (2022)

For \((c_F, c_S) \in \mathcal{P}_{T,g}\), the solution to the constrained static problem is given by

\[
H^* = \chi(\lambda_1 \tilde{\xi}_T, \lambda_2 \xi_T),
\]

where \(\lambda_1 \geq 0, \lambda_2 > 0\) are such that

\[
\mathbb{E}[\xi_T \max(H^*, G)] = F_0
\]

and either \(\lambda_1 = 0\) or \(\mathbb{E}[\tilde{\xi}_T H^*] = F_0\).
Existence of multipliers


- Can split the fee rate vectors in 3 categories:
  - Fee rates are not in $\mathcal{P}_{T,g}$, utility is maximized by the solution to the unconstrained problem;
  - $\lambda^*_1 = 0$: budget constraint is not binding, contract is fair, no solution to the representation problem;
  - $\lambda^*_1, \lambda^*_2 > 0$: both constraints are binding and the constrained dynamic portfolio problem has a solution.
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Existence of multipliers

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  2. $\lambda_1^* = 0$: budget constraint is not binding, contract is fair, no solution to the representation problem;
  3. $\lambda_1^*, \lambda_2^* > 0$: both constraints are binding and the constrained dynamic portfolio problem has a solution.
Numerical illustration

- Variable annuity contract with $T = 10$, $F_0 = G = 1$.

- Market parameters: $\mu = 0.04$, $r = 0.02$, $\sigma = 0.2$, $S_0 = 1$.

- $U_i(x) = x_i^{\gamma_i}/\gamma_i$, $\gamma_1 = 0.2$, $\gamma_2 = 0.4$.

▶ Some remarks:
  - Fair fee rate if $\pi_t \equiv 1$: $c_F + c_S = 2.45\%$.
  - Constrained dynamic portfolio problem has a solution for all fee rates considered.
Impact of $c_S$ on optimal payoff, $\theta < G$

Optimal payout $\max(H^*, G)$ as a function of $S_T$, $\theta = 0.95F_0$ (left: $c_F = 1\%$, right: $c_F = 2\%$)
Impact of $c_S$ on optimal payoff, $\theta > G$

Optimal payout $\max(H^*, G)$ as a function of $S_T$, $\theta = 1.05 F_0$ (left: $c_F = 1\%$, right: $c_F = 2\%$)
Optimal expected utility $E[U(\max(H^*, G))]$

\[\theta = 0.95F_0\]

\[\theta = 1.05F_0\]
Impact of $c_S$ on distribution of payoff

The figure shows the distribution of the optimal payout $\max(H^*, G)$ for different values of $c_S$. The horizontal axis represents $\max(H^*, G)$, and the vertical axis represents the count. The graph compares two values of $c_S$: 0.002 and 0.016. The distribution appears to be skewed towards higher values of $\max(H^*, G)$ for $c_S = 0.016$ compared to $c_S = 0.002$. This suggests that an increase in $c_S$ leads to a wider and higher distribution of payoffs.
Impact of $c_s$ on the optimal investment strategy

$c_s = 0.2\%$

$c_s = 1.6\%$
Unconstrained vs constrained optimal payout

\[ \max(H^*_T, G) \]

\[ c_s \text{ (in %)} \]

Unconstrained

\[ \theta = 1.05 F_0, \ c_F = 0.02448 - c_s \]

Constrained

\[ \theta = 1.05 F_0, \ c_F = 0.02448 - c_s \]
Concluding remarks

- Constrained, non-concave utility maximization.
- Use of auxiliary market to account for fee outflow.
- Utility of policyholder maximized with lower fees (linked to more conservative payouts).
- VA products maximize policyholder’s expected utility by offering dynamic investment strategies, especially if fees are low.
Thank you for your attention!


