Article from:

The Financial Reporter

December 2014 – Issue 99

Probability-Weighted Reserves without Employing Stochastic Scenarios

By Ed Robbins

Ed Robbins, FSA, MAAA, is manager, Robbins Actuarial Advisors, LLP, based in suburban Chicago. He can be reached at e.robbins@frontier.com.

This article briefly discusses the progress the Annuity Reserve Working Group (ARWG) has made toward a "modeled reserve" as part of the development of VM22 (Reserving for Non-Variable Annuities), and it goes on to suggest certain potential refinements to the scenario generation process, with application to modeling in general.

The ARWG "modeled reserve" was discussed at length at the 2014 Valuation Actuary Symposium and is briefly described below. The potential refinements fall into three types:

1. A more sophisticated approach to the correlation between various assumptions
2. A set of resulting scenarios together with their probability weights
3. An approach to generation of continuous distributions that fit well with the group of probability-weighted scenarios.

I. BACKGROUND

The general historical approach in actuarial practice for projection of financial scenarios based on probability distributions has been through stochastic scenarios, under which a random number has been generated between 0 and 1, and played against a cumulative distribution [F(x)] for the variable (risk factor), to obtain the value of the variable (risk factor) for a given stochastic run. The cumulative distribution function has historically been based on a probability density function (pdf), such as the following:

- Normal
- Lognormal
- Other skewed (such as gamma).

In the process of developing the modeled reserve under VM22, the ARWG has developed an approach that has made significant enhancements to the following two general historical approaches to modeling reserves, required assets, or similar values:

General Historical Approach	ARWG Approach
Stochastic generation to obtain the probability-weighted scenarios	Proceeding directly to the underlying distribution to obtain the probability-weighted scenarios.
Varying only one risk factor (such as the interest rate path).	Vary as many as four or five risk factors

In the field testing process for the modeled reserve approach, the ARWG has been able to cut the number of scenarios down substantially from those that would be required for a robust set of stochastic scenarios, to a 17-scenario set as follows: if, for example, four risk factors are involved:

- Scenario 1: Hold all risk factors at their central estimates.
- Scenarios 2 through 17: For each risk factor, four "shock values" (1 and 3 standard deviations (σ) on each side of its central estimate), while holding the other three risk factors at their central estimates.[1]

To generate a margin for adverse deviation, this approach currently contemplates a "cost of capital" calculation. This would be accomplished by aggregating the risk factors to generate a capital requirement at the 3σ level and generating an approximation to the consequent cost of capital.

There has been mention at the ARWG of a possible "reality test" of the above approach. That test would be to run the necessary number of non-stochastic but probability-weighted scenarios to accommodate each of the five points for each of the four risk factors. Thus there would be 625 sets of combined values (i.e., 5 x 5 x 5 x 5), resulting in 625 scenarios. Each such scenario would have its "weighting," consisting of the probability of the value of each of the four risk factors, multiplied together to generate a compound probability, or weighting, for that scenario.

As an example, take four risk factors each with its probability of occurrence:

R Factor No.	Probability
1	0.025
2	0.08
3	0.050
4	0.12

The compound probability is 0.025 x 0.08 x 0.050 x 0.12, or 0.0012 percent. If each risk factor contains five possible values, each with its probability weighting, that would result in 625 possible compound probabilities.

II. GENERAL DESCRIPTION OF POTENTIAL REFINEMENTS

The purpose of this paper is to start from the above 625-scenario concept and describe an enhanced approach to the ARWG scenario generation methodology that contains the following approaches:

- To use the pdf directly in generating the distribution of scenarios, similarly to the ARWG approach, and to accommodate variables that might be subject to a skewed distribution (such as lognormal or gamma)
- To encompass multiple risk factors, again similarly to the ARWG approach
- To relate the risk factors to each other in some reasonable fashion, referred to below as a "Dependency Chain," wherein the variables are ordered, so that the mean of a "Level t" variable becomes a simple function of the value of all of the variables of Levels 1 through t-1.[2]

There would appear to be no need to generate all 625 scenarios each time for reserve or required asset generation, if one is only interested in the adverse tail of the distribution. Once the first 625-scenario test has been made, it should be relatively straightforward to understand which combinations of risk factors will result in "successful scenarios," so that subsequently the actuary should be able to ignore scenario generation for those scenarios, and thus bring the necessary number of scenarios down to, say, 200.

The advantages of this approach over traditional stochastic scenario generation, if it can be made practical, would include:

- Increased speed of data generation from the stochastic scenario approach
- A more logical relationship between the assumptions
- A potentially more robust "area under the extreme values of the tail" (especially if the set of scenario results is converted into a continuous distribution)
- A more refined approach to generation of prudent margins in the reserve.

Moreover, this general approach is not susceptible to the purely statistical deviation generated by random number generation, as this approach would follow the underlying probability distributions directly and precisely.

III. DEPENDENCY CHAIN TO RELATE THE ASSUMPTIONS (RISK FACTORS)

It is unduly simplistic to assume that the sum of the variances of the risk factors equals the variance of the sum. That is only true if the risk factors are independent of each other. The ARWG has made some adjustments for correlated variables, but it would appear that the Dependency Chain concept may work better.

Illustratively, let's take a traditional deferred annuity portfolio.

The Dependency Chain concept would make the mean value (and possibly the standard deviation) of a "Level

CONTINUED ON PAGE 8

t" risk factor dependent in some simple form on some or all of the values of the risk factors in Levels 1 through "t-1". An example for a traditional deferred annuity is below:

Thus Level 1 includes the yield curve and its pdf (to determine the probabilities shown below), and Level 2 is the industry (or competitor) interest crediting rate. Let the five points for Level 1[4] be:

S	Level (s)	Possible Risk Factor(s)[3]
1	Yield curve, unemployment rate, inflation rate, contract terms	
2	Competitor credited rate for contract type	
3	Company credited rate	
4	Lapse rate scale	

- Level 1 Central Estimate minus 3σ, with probability 0.02
- Level 1 Central Estimate minus 1σ, with probability 0.20
- Level 1 Central Estimate, with probability 0.55
- Level 1 Central Estimate plus 1σ, with probability 0.22
- Level 1 Central Estimate plus 3σ, with probability 0.01

The Level 2 risk factors will then have a base pdf and five mean values, one for each of the five Level 1 points above. Perhaps the Level 2 mean values equal the Level 1 three-year swap rate less 0.015 for each of the five given Level 1 points.[5] With that, we can now derive the five Level 2 points for each of the five Level 1 points (i.e., 25 points in all).

To obtain the probability distribution (weightings) of a given five-point set, we know the pdf, the mean, and the standard deviation of that small discrete set, and we should be able to solve for the parameters of the underlying pdf that generate the same mean and standard deviation. The next step is to assign weightings to the five points of each of the five Level 2 sets, such that the discrete probability distribution of the five points has the same mean and standard deviation as the underlying distribution. See Exhibit 1 for a further explanation of this process and two alternative approaches.

Continue on through risk factors of Level 3 and Level 4, with the same Dependency Chain linkage concept. Following this logic, we will have 125 data sets at Level 3 and 625 data sets at Level 4.

IV. ASSIGNMENT OF COMPOUND PROBABILITIES OF ASSUMPTION (RISK FACTOR) COMBINATIONS

If for a moment we assume that we will be running all 625 possible combinations (each consisting of one of five points within one of four risk factors), the definition of a given set of risk factors to run a scenario will consist of the following:

- Level 1 value and Level 1 probability (weighting)
- Level 2 value and Level 2 probability (weighting)
- Level 3 value and Level 3 probability (weighting)
- Level 4 value and Level 4 probability (weighting).

From this set we can define the probability (weighting) of each of the 625 scenarios as the product of the four risk factor weightings (i.e., the resulting "compound probability").

V. GENERATION OF FINANCIAL PROJECTIONS (SCENARIOS) GIVEN THE COMPOUND PROBABILITIES

Given each of those 625 sets of assumptions, along with their respective compound probabilities, the 625 corresponding scenarios can now be run.

As can be seen above, each of the 625 scenario generations will thus derive from a unique set of Level 1, Level 2, Level 3, and Level 4 assumptions (risk factors). Those scenario results can now be ordered by size, along with their respective "compound probabilities" of occurrence, in order to generate a reserve with the desired confidence levels.

Exhibit 2 is an illustration of this type of ordering and compound weighting. Exhibit 2 also illustrates the calculation of both the 70 percent confidence level and the CTE(70) amount.

If the scenarios are reasonably normally distributed, then the standard deviation is a reasonably easy approach to generate confidence levels. On the contrary, for a skewed distribution the standard deviation is not a particularly valid measure of the confidence level. In this latter case, one can approach this issue in one of two ways:

- Compile the "sumproduct" of the scenario values, as in Exhibit 2, column (4), i.e., each such scenario value multiplied by its respective compound probability of occurrence (weighting) and sum the weighted scenario values up to the confidence level being sought, as depicted in column (3).[6]
- Fit those 625 resulting values to a continuous skewed pdf and thus derive another appropriate measure of the confidence level. See Section VI below.

VI. FITTING THE DISTRIBUTION OF SCENARIO VALUES TO A CONTINUOUS DISTRIBUTION

Various techniques exist for fitting a set of discrete points to a continuous distribution. We understand that mathematicians generally consider accuracy to the first four moments to be sufficient replication. If, for example, accuracy to four moments is desired, and where the pdf formula contains at least two parameters,[7] a weighted average of two pdfs can be used to obtain the scenario distributions mentioned above, each such pdf being given a 50 percent weighting, thus resulting in four parameters. With calculation of the first four moments of the pragmatic distribution of scenarios and, for example, using the first four moments of a continuous distribution with four parameters, one can set up four equations with four unknowns. Thus by the Method of Moments, a continuous pdf can be developed whose first four moments are equal to the corresponding first four moments of the pragmatic probability distribution of the scenarios.

Fitting to a continuous distribution whose first four moments are equal to the first four moments of the pragmatic distribution can provide other valuable information, such as insight into extreme "tail values" and interpolated values.

SOME DISCLAIMERS

The derivation of the underlying pdfs of the assumptions (risk factors) was not covered in this work. Extensive work was done by the ARWG in this regard. (This is especially true of variables such as interest rate paths and equity paths.) Rather, this article covers the process once those pdfs are concluded upon.

Second, it appears that the gamma pdf is somewhat difficult to apply to the above "five-point" distribution. That is, it is difficult to match the mean and variance of the theoretical continuous gamma pdf with the mean and variance of the five-point model for compound probability generation. (This does not appear to be a problem for the normal pdf.) That said, however, the gamma pdf appears to be a relatively appropriate approach to take when fitting the pragmatic financial scenario value results to a continuous distribution. The reasons why this is so are:

1. The successive moments of the gamma pdf are simple in form, thus easy to calculate. See Exhibit 3.
2. The minimum value of a gamma pdf is zero, and it is skewed and asymptotic to the right. Thus it has the general form of a required asset model.
3. Scenario results for reserves and required assets are generally not normally distributed, and a skewed pdf such as the gamma pdf is a better fit than the normal pdf.

Thus a conclusion was reached to recommend the normal and/or lognormal pdfs for generating the various assumptions while leaning toward the gamma pdf for generating a continuous distribution of final scenario values.

Third, whether to use a conditional tail expectation

CONTINUED ON PAGE 10

(CTE) approach versus a cost-of-capital approach to establish the margin has been subject to much discussion. This methodology can accommodate either approach.

Exhibit 1

Alternative Approaches to Generation of a Five-Point Distribution that, with Its Weightings, Generates the Mean and Standard Deviation of the Underlying Probability Density Function (pdf)

ALTERNATIVE 1

Step 1: Assign the five points as $\mu - 3\sigma$, $\mu - \sigma$, μ, $\mu + \sigma$, $\mu + 3\sigma$. Given the underlying pdf, find the pdf value that is represented by each point.

Step 2: The sum of those pdf values will only coincidentally be equal to 1. Normalize those values by dividing each of the five values by the sum. That derives the five weightings. For example, if the sum of those pdf values comes out to 0.86, then divide each of the five values by 0.86, to obtain a sum of the weightings equal to 1.

Step 3: See if the discrete probability distribution represented by those Step 2 values and weightings results in the same μ and σ as the underlying pdf. μ should be very close, while σ may not be. Stretch out or bring in the shock values until the σ of the discrete distribution replicates the σ of the underlying distribution.

ALTERNATIVE 2

Take the underlying pdf and establish the five points desired, such as:

- X_1, Central Estimate minus 3σ
- X_2, Central Estimate minus 1σ
- X_3, Central Estimate
- X_4, Central Estimate plus 1σ
- X_5, Central Estimate plus 3σ

Beginning with X_1, find the interval (a to b) in the pdf for which X_1, is the expected value. That is, find points a and b such that:

$$X_1 = \int_a^b x*pdf(x)dx \,/\, \int_a^b pdf(x)dx,$$

and a is the low point of the pdf.

Proceed through X_2 through X_5, finding intervals "b to c," "c to d," "d to e," and "e to f," respectively (where f is the high point of the pdf). The five respective weightings will then be:

For X_1, $\int_a^b pdf(x)dx$.

…

For X_5, $\int_e^f pdf(x)dx$.

Those five weightings will theoretically sum to 1.000. Moreover, the "sumproduct" of the five points and their weightings will produce the expected value (first moment) of X over the entire distribution.

Two practical issues deserve mention. First, it may provide better control to begin with X_3 (the central estimate) and solving for the related points c and d, moving next to points X_2 and X_4, and finally to X_1 and X_5.

Second, the sum of the weightings may not be exactly 1.00, due to approximate integration techniques and to cutting off asymptotic tails; the solution would be to divide each weighting by that initial sum of the weightings to move that sum to 1.00.

Exhibit 2
Illustrative Scenario Framework

Scenario No. (t) *	(1) Scenario Value	(2) Compound Probability	(3) $(3)_{t-1}+(2)_t$	(4) (1)*(2)
1				
2				
3				
4				
etc…				
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
520			0.700	
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
624				
625			1.000	
Total		1.000		

* Ascending order.

If we sum up Column 4 from Scenario 520 to Scenario 625, and divide by 0.3, that would be the CTE(70) amount.

By the same token, this structure enables one to develop the confidence level desired, in order to develop a margin based on the cost of capital.

Exhibit 3
Gamma Probability Density Function (pdf)

$pdf(t) = e^{-(\lambda t)}*(\lambda^{\alpha})*(t^{\alpha-1})/\Gamma(\alpha)$, where, for positive integral values of α, $\Gamma(\alpha)= (\alpha-1)!$ Parameters are thus α and λ.

The n^{th} moment, $E(t^n)$, is conveniently calculable as $(\alpha+n-1)^{(n)}/(\lambda^n)$.
[Note: $t^{(n)} = (t)*(t-1)*(t-2)*\ldots*(t-n+1)$.]

The gamma pdf is not directly integratable by typical means, but there exists a feature in EXCEL titled "GAMMADIST," which efficiently provides the pdf and cumulative distribution functions. ■

ENDNOTES

1 This process is undergone in order to eventually obtain the aggregate reserve margin over a central estimate. Each of the five points for a risk factor would have its "weighting" or probability, the sum of the weightings of those five points equaling 1.000.

2 Variables that are independent of other variables can be easily accommodated. They can simply be outside the Dependency Chain (or listed as "Level 1 risk factors").

3 It is possible to have several risk factors at a particular level.

4 It may be anticipated that the weightings of the five points should be "symmetric," e.g., that "Central Estimate plus 3 σ" would have the same weighting as "Central Estimate minus 3 σ" However, for skewed pdfs that may not yield that result when using the σ values to generate the points.

5 Assuming that the actuary desires the standard deviation to also vary as a simple function of the values of the lower level risk factors, the Level 2 standard deviation (σ) for each of the five underlying Level 1 points might be the "base σ" for Level 2, multiplied by (Level 1 Value/Level 1 Central Estimate)^0.5.

6 For example, for a 90 percent confidence level, you would want to take those weighted scenario values in column (4) that are reflected for those cases where Column (3) shows a value of 0.90. The sum of the column (4) values from Scenario 1 to that level gives you a confidence level of 0.90.

7 Such as the normal and gamma pdfs. The normal pdf contains parameters μ and σ, while the gamma pdf contains parameters α and λ. Parameters α and β are the symbols used in the EXCEL feature "GAMMADIST" (see Exhibit 3).