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Warm and Fuzzy ... and Real! – Part 2

By Dave Snell

In our last issue, I introduced the idea of linguistic variables, which we all use in our daily lives, and which guide our decision process.

Most of us cross the street (or driveway or parking lot) each workday. When is it “safe” to cross the street?

We might say it is “safe” when:
There are no cars or other motorized or human-propelled vehicles as far as the eye can see; and
the weather permits you to see far enough to ensure that anything that appears will not be able to overtake you before you reach the other side; and
the path is not hazardous with potholes, slippery spots, very high winds; and
there are no nearby intersections or driveways or buildings that might be hiding a vehicle close by but out of sight; or
you could hear any such approach because the ambient noise is not too distracting; and
the curb height is not dangerous;

... and dozens more contingencies I have not included here.

No wonder some of us come into work already tired! We have been multitasking so rapidly that it is a relief to finally be “safe” at our desks.

In this article, a continuation of my article last issue (“Warm and Fuzzy ... and Real!”), I will go into a little more detail on the reasons why you should be considering fuzzy logic, why the concept seems new even though it isn’t, how fuzzy and “crisp” logic share many properties and how they differ in important ways on others. Along the way, I want to address a couple of key processes involved when using fuzzy logic: hedging, fuzzification, and defuzzification. The terms are not familiar to most people even though we perform them every day—probably thousands of times each day.

For those readers who consider intimidating formulas containing lots of Greek symbols no longer interesting, rest assured that you don’t have to learn the formulas. Most folks

drive fine without knowing how the car engine and transmission systems work. As Jeff Heaton showed in his companion article last issue (“Fuzzy Logic in R”), we have “vehicles” to make your application of fuzzy logic less tedious.

On the other hand, even though it is not necessary to know how the engine works to drive a car, virtually all race car drivers have an expert knowledge of how their engine, transmission, drivetrain, suspension, etc. work, and how they interact with each other. The more you know about the underlying theory and practice, the better you can be in your chosen field.

Learning some theory can also help us avoid “misuse” of a tool or technique. It used to be a common mistake for children who were hot on a summer day, to leave the refrigerator door open so that it would cool the room. Since a refrigerator takes heat from its interior and releases it through the fins at the bottom/back of the unit, this would have the effect of actually heating the room (and spoiling the food and raising the electric bill). Likewise, we should use the convenient tools built into packages for R and other programming languages, but understand when to use which tool.

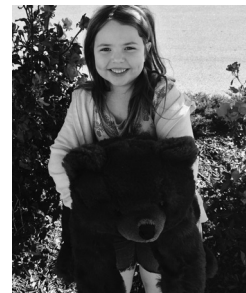
Let’s start with some history. Our mathematical heritage is a strength; and we can be proud of the longevity of rules and principles developed hundreds, and even thousands, of years ago:

Fundamental Principle	Author and Creation Date
Geometry (Elements)	Euclid of Alexandria, c. 300 B.C
Laws of Motion, ($F = ma = m * dv/dt$)	Newton, 1642-1727 C.E.
Principia Mathematica (axioms, inference rules, symbolic logic – all mathematical truths can be proven)	Whitehead and Russell 1910-1927

These (and lots of other principles) have served us well for a long time. Over time, we came to accept them as absolute truths. Yet, some of these “truths” have turned out to be not quite true.

“FUNDAMENTAL PRINCIPLE” REVISITED	DISRUPTER AND DISRUPTION DATE
<ul style="list-style-type: none"> A 0-dimensional point does not exist (or a 1-dimensional line; or a 2-dimensional plane) and dimensions do not have to be integers 	<ul style="list-style-type: none"> Real world admission, (long overdue?), plus physical anomalies such as Möbius strips Hausdorff–Besicovitch, 1918
Force = mass * acceleration → $F = m * dv/dt + v * dm/dt$	Einstein, 1905 (as velocity gets very large, mass increases)
Gödel’s incompleteness theorem shatters Principia Mathematica	Kurt Gödel, 1931

Our past knowledge can be an Impediment to understanding and modeling our world. Remember your first introduction to geometry? Your instructor said a line has length but no width. And the inquisitive child in you asked, “But I see it! How can it have no width if I can see it?” and then, in order to pass the course, you eventually conceded that it must be as the instructor said. Later, we learned that Euclid’s geometry was a small subset of the vast number of useful ways to look at geometry. The Hausdorff dimension generalizes the notion of the dimension of a real vector space.¹ Einstein’s “theory” of special relativity challenged and superseded Newton’s second “Law” of motion. Gödel proved that for any set of axioms and inference rules proposed to encapsulate mathematics, there would in fact be some truths of mathematics which could not be deduced from them.²



Author’s granddaughter, Amanda, with her warm and fuzzy friend.

Children look to fuzzy for comfort. They are naturally drawn to it ... until we tell them to grow up and respect the sharp edges of life.

But life’s edges are not so sharp! Our assumed “Laws” are continually being refined or replaced by better fits based on more careful observations of the real world. We utilize fuzzy logic every day. We just refuse (most of the time) to recognize it.

Lotfi Zadeh, a Professor at the University of California, Berkeley, is considered the founder of fuzzy mathematics, fuzzy set theory, and fuzzy logic. He published his seminal work, “Fuzzy sets,” in 1965.³

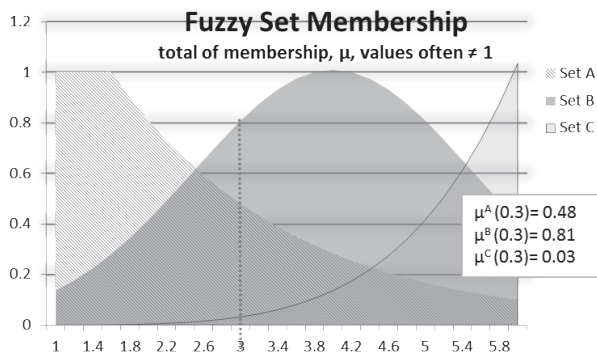
Since the introduction of fuzzy sets, the classical Boolean sets have been renamed for convenience as “crisp” sets. Crisp logic is a new name for Boolean logic (George Boole, 1847). Crisp logic is binary in nature. Crisp set membership is always 0 (false, out) or 1 (true, in). Fuzzy logic allows interim values. Fuzzy set membership can vary from 0 (completely out) through an infinite interval of the real numbers (0.2, 0.67, 0.876, ...) to 1 (all in). I described this in more detail in the article last issue, so here I’ll just summarize some similarities and some differences between crisp and fuzzy logic:

Relationships that hold for both Crisp and Fuzzy Logic	
Associative:	$(A \cap B) \cap C$ $= A \cap (B \cap C)$, $(A \cup B) \cup C$ $= A \cup (B \cup C)$
Commutative:	$A \cap B = B \cap A$, $A \cup B = B \cup A$
Distributive:	$A \cap (B \cup C)$ $= (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C)$ $= (A \cup B) \cap (A \cup C)$
De Morgan’s Law:	$\neg(A \cap B)$ $= (\neg A) \cup (\neg B)$, $\neg(A \cup B)$ $= (\neg A) \cap (\neg B)$

Fuzzy Rules
NOT $x = (1 - \text{truth}(x))$
x AND $y = \text{minimum}(\text{truth}(x), \text{truth}(y))$
x OR $y = \text{maximum}(\text{truth}(x), \text{truth}(y))$

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A key difference between crisp and fuzzy sets is that with fuzzy sets you can have varying degrees of membership in several different fuzzy sets at the same time; and the membership values do not have to sum to exactly one. You might say that X has membership 0.48 in set A, 0.81 in set B, and 0.03 in set C.



Of course, that oversimplification can lead to inconsistencies, so we also want to observe that sense prevails. If you have sets of underweight, normal, overweight, obese, and morbidly obese, then the same person could have 0.05 membership in normal, 0.80 membership in overweight, and 0.20 membership in obese (again, don't expect the sum to have to equal 1.0); but memberships of 0.5 in underweight, and 0.6 in morbidly obese, while 0.0 membership in normal seems to violate common sense.

Fuzzy Hedging

One way of addressing comparative memberships is through hedging. If a person has membership $\mu^{\text{High}} = 0.5$ in the set High, then without other data, we might assume the membership in Very High to be the square of that, since the square of a number between zero and one will be less than the original number:

$$\mu^{\text{VeryHigh}} = (\mu^{\text{High}})^2 = (0.5)^2 = 0.25 \text{ and } \mu^{\text{AlmostHigh}} \text{ to be the square root: } \mu^{\text{AlmostHigh}} = 0.707$$

Intuitively, this approach works well whenever you assume that the “very” set membership is more restrictive or

exclusive than the “normal” membership, which in turn is more restrictive than the “almost” membership.

Fuzzy Logic Can Help You Get Quantitative Results!

Let's say we want to predict lapse rates as a function of interest rate and the unemployment rate. We might feel that if the unemployment rate is low and the interest rate is low, then policyholders will be inclined to keep their life insurance policies, and the lapse rate will be low. On the other hand, if interest rates are high, or very high, and so is the unemployment rate, then more policyholders will lapse their policies for cash needs or for arbitrage.

In order to predict lapse rates under these scenarios, we need to be able to determine the criteria that will allow us to say interest rates are high (versus low, normal, very high, etc.)

One approach is the old “crisp” approach—set precise ranges. Under that approach, we might say

- 3.001 percent to 5 percent is normal,
- 5.001 percent to 7 percent is high, and
- 7.001 percent to 9 percent is very high.

A disadvantage of this crisp approach is its focus on precision rather than accuracy. Do we really think there will be a big jump in lapse rates if the interest rate moves from 5 percent to 5.001 percent? A more realistic way to handle this classification situation might be to utilize fuzzy logic and to assume a Gaussian distribution, and this might mean that the move from 5 percent to 5.001 percent just increases the membership in High and decreases the membership in Normal.

Fuzzification

Fuzzification involves the classification of several quantitative values into linguistic variables. The programming language R⁴ has several libraries available to handle fuzzy logic. One popular one is “sets.” You can add this library to your R workspace with two simple statements:

```
Install.packages("sets")
#5 this is a one-time download to your computer
library(sets)
#this makes the many functions within “sets” available
```

Once you have sets available, define a universal set for computing the membership grades.

```
sets_options('universe',
+ seq(from=1, to=9,by=.5))
#note: the + signs, indicate a line continuation;
```

For this example, with just the three variables of Interest (int), Unemployment (unemp), and Lapse (lapse), we'll use membership values that fit into our defined universe. If the relative size of our variables were vastly different though, we'd normalize them for visual clarity in our plots.

Now that the housekeeping setup is done (at most three statements ... not too bad so far!), let's define the three variables and the linguistic variables we are assigning to them:

```
vars<-set(
+ int=fuzzy_partition(varnames
+ =c(low=2,norm=4,hi=6,vhi=8),sd=1),
+ unemp=fuzzy_partition(varnames
+ =c(low=3,norm=4,hi=5,vhi=6),sd=.8),
+ lapse=fuzzy_partition(varnames
+ =c(low=3,med=5,hi=9),sd=2))
#sd=standard deviation
```

Next, we'll create a couple of rules (in a real model, you would likely have lots more rules):

```
> rules<-set(
+ fuzzy_rule(int %is% low
+ && unemp %is% low, lapse %is% low),
+ fuzzy_rule((int %is% hi
+ || int %is% vhi)
+ && (unemp %is% hi
+ || unemp %is% vhi), lapse %is% hi))
#rules take the form fuzzy_rule(antecedent, consequent)
i.e., condition(s) and the resulting implication(s) from the condition(s).
```

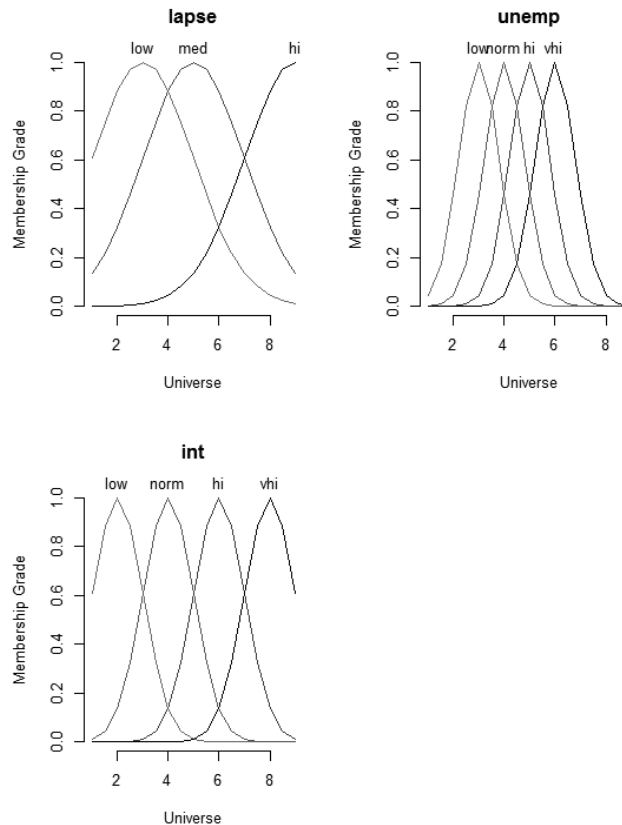
Finally, we connect our rules to our variables:

```
sys<-fuzzy_system(vars,rules)
```

All the hard work (for us) is now done. R can do the heavy lifting from here going forward. For example, say you wish to see a plot of your variables. Simply ask for it via

```
Plot(sys)
```

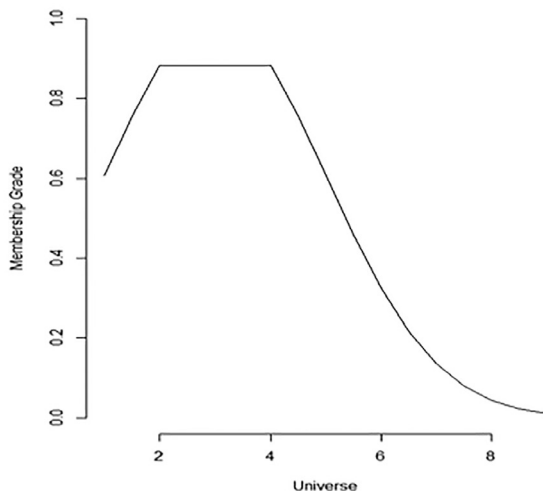
And voila! We get:



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It gets even better. Now, let's assume we want to see what kind of inference we get from an interest rate of 2.5 percent and an unemployment rate of 3 percent:

```
#fuzzify
fz_inf<-fuzzy_inference(sys,
+ list(int=2.5, unemp=3))
plot(fz_inf)
```



This shows us a graph of membership values for lapse rates at the various lapse rate possibilities. Clearly, the lapse rate is not likely to be high (9) since that membership, $\mu^{\text{lapse}}(9)$ is zero. The high membership values are clearly in the low area (low is centered at 3).

We have a graph, but we are not done yet. It still would be nice to quantify this.

Defuzzification

We'll defuzzify this graph to a single number with another function:

```
gset_defuzzify(fz_inf,'centroid')
[1] 3.496817
```

Our answer is a lapse rate of 3.496817; but how was that calculated? Let's make sure we don't just have the refrigerator door open!

The centroid method was used here, and the centroid can be computed by the integral

$$C = \frac{\int xg(x) dx}{\int g(x) dx}$$

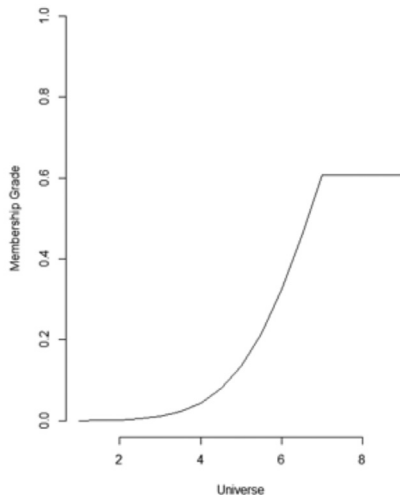
where the integrals are taken over the whole space, and g is the characteristic function of the subset, which is 1 inside X and 0 outside it. A physical interpretation of the centroid would be the point where you would place a pin and a string in order to balance the area. In this example, the centroid is probably appropriate, but this package offers a few more options for you for situations where the centroid might not be a good choice:

```
gset_defuzzify(fz_inf,'meanofmax')
#returns 3 for this example
gset_defuzzify(fz_inf,'smallestofmax')
#returns 2 for this example
gset_defuzzify(fz_inf,'largestofmax')
#returns 4 for this example
```

Again, I urge you to investigate the many ways to defuzzify (there are several) to choose the best fit for your particular situation.

Before we leave this example, let's try one more input combination:

```
fz_inf<-fuzzy_inference(sys,list(int=7,  
+ unemp=5))  
plot(fz_inf)
```



This time we input a high interest rate and a high unemployment rate, and accordingly, the graph shows a high lapse rate is likely.

I'll quantify it according to all four defuzzification methods of this "sets" package:

```
> #defuzzify  
> gset_defuzzify(fz_inf,'centroid')  
#output is 7.322127  
> gset_defuzzify(fz_inf,'meanofmax')  
#output is 8  
> gset_defuzzify(fz_inf,'smallestofmax')  
#output is 7  
> gset_defuzzify(fz_inf,'largestofmax')  
#output is 9
```

Our range of returned values is 7 to 9, with a centroid of 7.322127 which is consistent with our rule that high unemployment and high interest rates would likely result in high lapse rates. If instead of just 3 to 9 percent, your product has lapse rates that might range from 5 percent to 75 percent, just normalize them to fit the universe we set for the other variables, then scale up after you have your answer.

The R vehicle is pretty easy to drive—especially for someone with a statistics background. Keep in mind though that the normal distribution is not always the correct function to use! If, instead of interest rates, we were fuzzifying wait times for a bus to arrive when we reach the bus stop at random times, the more appropriate probability distribution would be the exponential distribution.⁶ Functions available include `fuzzy_trapezoid` (you specify the corners), `fuzzy_cone` (you specify the radius), etc. See Jeff Heaton's article (in Recommended Reading) for examples using other fuzzy shapes.

A Ferrari is not the best vehicle choice for hauling a trailer home; nor is a Humvee going to get you optimal fuel economy in city traffic. A Tesla, although quiet and powerful, would be risky for a trip over 300 miles through a remote area. Your knowledge of the subject matter of the problem and of the applicability of various statistical distributions will help you choose a better tool for the task at hand.

Fuzzy Logic can be a useful way to improve many actuarial models:

- It can be a closer match to the way humans think.
- Linguistic variables introduce both clarity and flexibility.
- Fuzzification can handle incomplete and inconsistent data.
- Rules sets can be cleaner and fewer in number.
- Defuzzification produces quantifiable results.

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Consider getting fuzzy to get more real!
Recommended Reading

- Shang, Kailan and Hossen, Zakir [2013] “Applying Fuzzy Logic to Risk Assessment and Decision-Making,” CAS/CIA/SOA Joint Risk Management Section.
- L.A. Zadeh, “Outline of a new approach to the analysis of complex systems and decision processes,” IEEE Trans. Syst., Man, Cybernetics, SMC-3 (1973), pp. 28–44.
- Snell, David,[2014] “Warm and Fuzzy ... And Real!,” F&F newsletter Issue 9.
- Heaton, Jeff, [2014] “Fuzzy Logic in R,” F&F newsletter Issue 9.
- Klir, George and Yuan, Bo [1995], “Fuzzy Sets and Fuzzy Logic – Theory and Applications,” Prentice Hall PTR, Upper Saddle River, New Jersey,1995.
- Ross, Timothy [2010] “Fuzzy Logic with Engineering Applications,” Third Edition, John Wiley and Sons, Ltd., UK.
- Search for fuzzy logic on the SOA website for a current list of actuarial papers. ▼



Dave Snell

Dave Snell, ASA, MAAA, is technology evangelist at RGA Reinsurance Company in Chesterfield, MO. He can be reached at dave@ActuariesAndTechnology.com

ENDNOTES

- ¹ http://en.wikipedia.org/wiki/Hausdorff_dimension ... and we can't even rest assured that we live in 3 dimensional space! String theory suggests perhaps 11 dimensions; or even 26-dimensional Bosonic space.
- ² http://en.wikipedia.org/wiki/Kurt_G%C3%B6del
- ³ http://ac.els-cdn.com/S001999586590241X/1-s2.0-S001999586590241X-main.pdf?_tid=5f5a4f04-49eb-11e4-9b4c-00000aacb362&acdnat=1412223779_25c3ce3d1619da26ab8fa6440a8fcb9 “Fuzzy Sets,” by Lotfi Zadeh, June, 1965, Information and Control, Volume 8, pp338-353.
- ⁴ In this example, I am using a free graphical user interface for R, called R Studio. Get it at www.rstudio.com.
- ⁵ Note that in R, # denotes a comment. Anything after the # on a line is ignored.
- ⁶ Doing Data Science, by Cathy O’Neil and Rachael Shutt], pp 128-129.