Appendix 1

THEORY BEHIND MACAULAY DURATION AND ILLUSTRATIVE EXAMPLES

Definition of Duration

Macaulay duration is the weighted average of the time to each future cash flow. The weights are the present value of the corresponding cash flows. In symbols,

$$\text{Duration (D)} = \sum_{t=0} t v^t CF_t / P, \text{ where}$$

$$P = \sum_{t=0} v^t CF_t.$$

Thus, duration provides us with a sense for the average "length" of an asset or liability. Note that in this definition, we assume a flat yield curve. In a later section, we will examine the implications of a nonlevel yield curve.

Example of Duration

Consider the following example:

- Liability: $100, 16 percent, 6.9-year compound bullet (to pay $278.46 at maturity).

- o Asset: $100, 16 percent, 30-year mortgage (to pay $16.19 per annum for 30 years).

- o $D^L = 6.9 = D^A$.

- o If interest rates drop to 14 percent, market values (MV) change as follows:

$$MV^L = 278.46/(1.14)^{6.9} = 112.75.$$

$$MV^A = 16.19\ a_{\overline{30}|14\%} = 113.37.$$

The excess of asset over liability value represents a .5% gain.

- o If interest rates increase to 18 percent:

$$ML^L = 278.46/(1.18)^{6.9} = 88.87.$$

$$ML^A = 16.19\ a_{\overline{30}|18\%} = 89.32.$$

- o Again, assets now exceed liabilities by .5%.

Thus, we are now well protected against interest shifts. In fact, we appear to gain either way. We will come back to this in the next section.

It is instructive to examine this example from the maturity date viewpoint:

Value of Assets at Maturity of Liabilities, if Interest = i% to Maturity	16%	14%	18%
From cash flow reinvestment[a]	\$166.12	\$155.49	\$177.42
From liquidation of assets[b]	112.37	124.51	102.43
Total	\$278.49	\$280.00	\$279.85

a. $16.19 \times s_{\overline{6}|i} \times (1+i)^{0.9}$.

b. $16.19 \times a_{\overline{24}|i} \times (1+i)^{0.9}$.

In each case, the total value of the assets is reasonably close to the required payment of \$278.46 at maturity. Note, though, that the two pieces shown move in different directions. If interest rates fall to 14 percent, we gain on liquidation of assets (versus the 16 percent base case) but lose on the reinvestment of the annual cash flow. In the 18 percent scenario, just the opposite holds. One can thus begin to see how this works. For a better understanding, consider the following theoretical development.

Duration: Theoretical Development

Classical Immunication Theory

A_t = cash flow of assets at time t.

B_t = cash flow of liabilities at time t.

Goal of classical immunization theory is to achieve the following:

$$(1) \qquad \sum_t A_t v^t \geq \sum_t B_t v^t$$

regardless of the interest rate.

To simplify the notation, let's define:

$$P^A = \sum_t A_t v^t, \; P^L = \sum_t B_t v^t.$$

To determine the conditions that need to hold to ensure that equation 1 is true, we define

$$(2)\; f(i) = P^A - P^L.$$

If $f(i)=0$, what we want is $f(i') > f(i)$ for all interest rates i', or in other words we want $f(i)$ to be a minimum point of the function f. This will be the case if the first derivative of $f(i)$ equals 0 and the second derivative $>$ 0, so that our conditions become

$$(3)\quad \sum_t tv^t A_t = \sum_t tv^t B_t. \quad \text{(first derivative = 0).}$$

$$(4)\quad \sum_t t^2 v^t A_t > \sum_t t^2 v^t B_t. \quad \text{(second derivative} > 0\text{).}$$

The following indicators were then developed as tools for utilizing this theory:

D_1^A = asset duration = $\sum_t tv^t A_t / P^A$.

D_1^L = liability duration = $\sum_t tv^t B_t / P^L$.

D_2^A = asset spread (or convexity) of payments = $\sum_t t^2 v^t A_t / P^A$.

D_2^L = liability spread of payments = $\sum_t t^2 v^t B_t / P^L$.

Assuming that:

$P^A = P^L$ at the initial interest rate i,

Then $D_1^A = D_1^L$ and $D_2^A = D_2^L$ are necessary and sufficient to ensure that

$P^A \geq P^L$

at <u>any</u> interest rate.

Note that the notation for Macaulay duration may be D or D_1. The latter is often used to distinguish the first moment (D_1) from the second moment (D_2).

In the example previously given, $D_2^A = 82.2$ and $D_2^L = 47.6$. Thus, $D_1^A = D_1^L$ and $D_2^A > D_2^L$, which assures us that the present value of the asset cash flows will equal or exceed the present value of the liability cash flows at all interest rates. As we did see, this inequality held for various interest rates.

The requirement involving the D_2's implies that the absolute "length" or spread of the assets is somehow "longer" than that of the liabilities. In the example given, we saw that this permitted losses upon liquidation to be offset by gains upon cash flow reinvestment, or vice versa. Thus, we can see that the D_2 requirement is necessary. In fact, if we had switched the asset and liability in this example, then even though durations matched, we would have (small) losses upon interest rate moves. From the calculus, we can see that f(i) would be at a maximum of i rather than a minimum.

The phenomenon where one can gain if interest rates move anywhere was noted by I. T. Vanderhoof,[1] who suggests that in real life, the structure of interest rates (that is, yield curves) prevent such all-win situations.

Duration as Measure of Change in Price per Change in Interest

Now consider an alternate demonstration. Assume

C_t = cash flow at time t.

P_o = purchase price at time o

$$\sum_t C_t/(1+i)^t.$$

We can obtain the marginal change in price due to a change in interest rate by taking the derivative of P_o with respect to (i); that is,

$$dP_o/di = -\sum t\, C_t/(1+i)^{t+1} = \frac{-1}{(1+i)} \sum tv^t C_t$$

or

$$dP_o = \frac{-di}{(1+i)} \sum tv^t C_t$$

Dividing both sides by P_o yields

$$\frac{dP_o}{P_o} = \frac{di}{(1+i)} \times \frac{tC_t v^t}{P_o} = \frac{di}{(1+i)} \times -\sum \frac{tv^t C_t}{P_o}$$

---

1 I. T. Vanderhoof. "Immunization with (Almost) No Mathematics." The Actuary 16, no. 4 (1982): 1, 4, 7.

or

% change in price = % change in (1 + i) x (-duration).

One implication of this result is that if the durations of the assets and liabilities are identical, then a change in interest rates will produce comparable changes in market values. In other words, immunization is achieved.

It is important to note, though, that exact matching is ensured only for infinitesimal changes in interest, that is changes by di. In fact, once interest rates change by di, the new duration based on this new interest rate will likely no longer match; however, as we saw in the last section, if $D_2^A > D_2^L$ at the original interest rate, the asset market value will still exceed the liability value with further changes in interest rates.

For example, assume $100 is invested in a 30-year mortgage yielding 16 percent; the following table compares the percentage change in market value produced by the duration formula just given to the actual percentage change in price under various interest rates:

	Duration Formula			Actual Change in Market Value		
	(1)	(2)	(3)	(4)	(5)	(6)
Interest Rate	Duration at 16%	%Change in (1 + i) from 1.16	-(1) x (2)	Market Value	% Change from 16% Value	Ratio of (3) to (5)
16%	6.90	0.00%	0.00%	$100.00	0.00%	—
17	6.90	0.86	-5.93	94.38	-5.62	105.5%
18	6.90	1.72	-11.87	89.32	-10.68	111.1
20	6.90	3.45	-23.81	80.61	-19.39	122.8

Thus, as interest rates move significantly from the base rate, duration becomes less useful as a measure of price change.

Other Properties of Duration

It is often necessary to combine duration for several assets and/or liabilities. Fortunately, duration has a simple aggregation property, developed here:

Let A_t and A_t^1 denote cash flows for two separate assets, and D and D^1 denote their respective durations. The duration for the combined asset, D^C, equals

$$\begin{aligned} D^C &= \sum t v^t (A_t + A_t^1) / \sum v^t (A_t + A_t^1) \\ &= (\sum t v^t A_t + \sum t v^t A_t^1) / (P + P^1) \\ &= (DP + D^1 P^1) / (P + P^1) \\ &= D \frac{P}{P + P^1} + D^1 \frac{P^1}{P + P^1}. \end{aligned}$$

Thus, if the durations and present values of the separate assets are known, it is easy to compute the duration of the combined asset. Furthermore, this relationship is useful for deriving additional properties of immunization and duration.

For example, it is usually the case that when working with statutory liabilities and the associated assets, that P^A exceeds P^L. Recall that equality is assumed in the classical immunization requirements. We can modify the traditional requirements in one of two ways, depending on our goals.

Assume first that we want the dollar amount of "economic surplus" ($S = P^A - P^L$) to be at least as great in all other interest scenarios. Stated differently, we want to immunize the dollar amount of surplus.

Relative to the calculus development used earlier, we still want the function $f(i) = P^A - P^L$ to be at a maximum at i, so that we want $f'(i) = 0$, and $f''(i) > 0$. Therefore,

$$f'(i) = - \sum tv^{t+1}A_t + \sum tv^{t+1}B_t = 0$$

$$\Rightarrow \sum tv^tA_t = \sum tv^tB_t .$$

$$\Rightarrow P^AD^A = P^LD^L, \text{ or}$$

(1) $D^A = P^LD^L/P^A$ and

$$f''(i) = \left[\sum t(t+1)v^{t+2} A_t - \sum t(t+1)v^{t+2}B^t\right] > 0$$

$$\Rightarrow \sum t^2v^tA_t + \sum tv^tA_t > \left[t^2v^tB_t + \sum tv^tB_t\right]$$

$$\Rightarrow P^AD_2^A + P^AD^A > P^LD_2^L + P^LD^L$$

$$\Rightarrow P^AD_2^A + P^A(P^LD^L/P^A) > P^LD_2^L + P^LD^L$$

$$\Rightarrow P^AD_2^A > P^LD_2^L, \text{ or}$$

(2) $D_2^A > P^LD_2^L/P^A$ **(all assuming condition 1 holds).**

Thus, if $P^A > P^L$ and conditions 1 and 2 hold, the value of surplus, or f(i), will be at least as great at other interest rates. Note that $P^A = P^L$ is just a special case of this more general case.

Some may instead wish to immunize the surplus - to - liability ratio (S/P^L) so that it will be at least as great at other interest rates. It turns out that this is achieved by the original conditions, $D^A = D^L$ and $D_2^A > D_2^L$, when $P^A > P^L$.

To see this, define $P^S = P^A - P^L$, and consider the following:

- We can carve out representative assets whose present value, P^{AL}, equals P^L at the initial interest rate. By representative, I mean that

  $D^{AL} = D^A = D^L$ and

  $D_2^{AL} = D_2^A > D_2^L$. Let P^S equal the present value of the

  Let P^S equal the present value of the remaining assets.

- Let P_1^{AL}, P_1^L, and P_1^S denote the present values at a different interest rate.

- From the duration conditions, we know that $P_1^{AL} > P_1^L$.

- This implies that:

$$\frac{P_1^S}{P_1^{AL}} \leq \frac{P_1^S}{P_1^L}$$

- Since we "carved" out representative assets, $P_1^S/P_1^{AL} = P^S/P^{AL} = P^S/P^L$ for any interest rate, so that

$$P_1^S/P_1^L \geq P^S/P^L.$$

Shortcomings of Duration

There are a few practical and theoretical difficulties in actually using duration. One problem is that we can be matched ($D^A = D^L$) today, but mismatched tomorrow, even without any changes in interest rates. Consider the following example:

- Liability: \$100, 16 percent, 6.9 - year compound bullet (to pay \$278.46 at maturity).

- Asset: \$100, 16 percent, 30 - year mortgage (to pay \$16.19 per annum).

- At issue, $D^L = 6.9 = D^A$ ($= \sum_{t=1}^{30} tv^t / \sum_{t=1}^{30} v^t$) .

$$D_2^L = 47.6. \quad D_2^A = 82.2.$$

- One year later,

$$D^L = 5.9.$$

D^A depends to some extent on how the first payment was invested. If invested in a cash equivalent instrument,

$$D^A = \left[1 + \sum_{t=1}^{29} t \times v^t\right] \Big/ \left[1 + \sum_{t=1}^{29} v^t\right] = 6.04.$$

If the first payment is reinvested in a 29-year mortgage,

$$D^A = \sum_{t=1}^{29} t \times v / \sum_{t=1}^{29} v^t = 6.85.$$

Note especially that D^A may change only slightly, where as D^L changed rather significantly. What this implies is that if durations are matched today, and interest rates change today, we should be in good shape. However, if one year passes and then interest rates change, we are no longer protected. The solution to this problem is active portfolio management.

Another interesting problem is that the values of D_1 and D_2 usually vary with interest rates. Thus, if interest rates shift, durations may no longer match, which may or may not be a problem. If the shift occurs immediately after we determined $D_1^A = D_1^L$ and $D_2^A > D_2^L$ at the original i, then we are still protected against further interest rate changes that may occur immediately.

Example:

- At time 0:

Liability: \$100, 16 percent, 6.9 - year compound bullet (to pay \$278.46 at maturity).

Asset: \$100, 16 percent, 30 - year mortgage (to pay \$16.19 per annum).

$$D^L = 6.9 = D^A;\ D_2^L = 47.6;\ D_2^A = 82.2.$$

o At time 0, assume interest rates shift to 18 percent; evaluate at time = 0:

$$P^L = 278.46 \times v_{18\%}^{6.9} = 88.87.$$

$$P^A = 16.19 \times a_{\overline{30}|18\%} = 89.32.$$

$$D^L = 6.9 \text{ years.}$$

$$D^A = \sum_{t=1}^{30} t v_{18\%}^t / P_{18\%}^A = 6.3.$$

Note that durations and market values no longer match. This is not a problem, since at 16 percent we have seen that all necessary conditions for immunization are satisfied. However, it can lead to a misinterpretation of results.

If, at a given interest rate, our three conditions are satisfied, we are protected. If not, it is not always clear where we stand. Consider the following example:

- o Liability: \$100, 16 percent, 8 - year compound bullet (to pay \$327.84 at maturity).

- o Asset: \$100, 16 percent, 30 - year mortgage (to pay \$16.19 per annum).

- o At 16%

  - o $P^A = P^L = \$100.00$.

  - o $D^A = 6.9 \neq D^L = 8.0$.

  - o $D_2^A = 82.2$ $D_2^L = 64.0$.

- o Since $D^L > D^A$, one might guess that interest rate increases will cause a gain, whereas interest rate declines should cause losses. This is almost, but not quite, true:

i	P^A	P^L	%
8%	182.26	177.12	+2.8%
10	152.62	152.94	0.2
12	130.41	132.41	-1.5
14	113.37	114.93	-1.4
16	100.00	100.00	0.0
18	89.32	87.22	2.4
20	80.61	76.25	5.7
22	73.40	66.80	9.9

- It turns out that at 12.75 percent, $D_1^A = D_1^L = 8.0$, $D_2^A = 107.4$ and $D_2^L = 100$,

  but $P^A = 123.51$ and $P^L = 125.52$. Interestingly, then, $i = 12.75$ is the point of maximum loss. If we could increase our assets by 1.6 percent (125.52/123.51), all conditions would be satisfied (at 12.75 percent) and P^A would then equal or exceed P^L at all other interest rates.

The conclusion of all this is that if we can find any interest rate such that

$$P^A \geq P^L,$$

$$D_1^A = D_1^L, \text{ and}$$

$$D_2^A > D_2^L,$$

then we are protected against immediate changes in interest rates. If not, we need to restructure the assets and/or liabilities to satisfy these three conditions.

Call and Withdrawal Options

The theoretical justification for duration assumes that cash flows are independent of the interest scenario. If, instead, the assets call when interest rates decline, or policyholders withdraw when interest rates rise, then losses could result, regardless of the duration tests. This is a major shortcoming of simple duration satistics.

Term Structure of Interest Rates

The term structure of interest rates is the structure of yields on debt instruments that differ only in the time remaining to their maturity dates.

One purpose of this theory is to explain the relationship between the yield curve and investors' expectations of future yields.[2] Notation used is as follows:

$R_{n,t}$ = Actual yield per period at time t on an investment maturing n periods from time t,

$r_{1,t}$ = Expected yield per period at time t on an investment maturing in 1 period from time t.

Thus, the $R_{n,t}$'s are the yields one would observe in a standard yield curve at time t. The $r_{1,t}$'s are 1-year yields that investors anticipate will apply to future periods.

---

[2] This subject is covered in Chapter 14 of Security Market by Garbade.

The Term Structure Theory holds that these two yields are closely interrelated. In fact,

$$R_{n,t} = \frac{1}{n}(R_{1,t} + r_{1,t+2} + \ldots + r_{1,t+n-1})$$ (See footnote 3.)

Numerical examples of the r's versus R's follow:

	Case A		Case B	
k	Expected Future Yields $r_{1,t+k}$	Term Yields at t, $R_{k,t}$	Expected Future Yields $r_{1,t+k}$	Term Yields at t, $R_{k,t}$
1	16.0%	16.00%	12.0%	12.00%
2	15.5	15.75	12.5	12.25
3	15.0	15.50	13.0	12.50
4	14.5	15.25	13.5	12.75
5	14.0	15.00	14.0	13.00

We can now modify the Classical Immunization Theory to accommodate this non-level yield curve. Assume the following:

A_t = cash flow of assets at time t.

B_t = cash flow of liabilities at time t.

---

[3] Both $R_{n,t}$ and $r_{1,t}$ are defined as continuous interest rates. Thus, $e^{R2,t} = e^{R1,t} \times e^{r1,t+1}$, so that $R_{2,t} = 1(R_{1,t} + r_{1,t+1})/n$. The theory also provides for the existence of risk premiums in reality that serve to create equilibrium in the market between supply and demand for issues of different maturities.

We want to achieve the following equation, regardless of the interest rate or rates:

$$\frac{A_1}{(1 + r_{1,0})} + \frac{A_2}{(1 + r_{1,0})(1 + r_{1,1})} + \ldots \geq \frac{B_1}{(1 + r_{1,0})} + \frac{B_2}{(1 + r_{1,0})(1 + r_{1,1})} + \ldots .$$

It turns out that if:

(1) $$\sum_{t=1}^{\infty} \frac{tA_t}{\prod_{i=0}^{t-1} (1+r_{1,i})} = \sum_{t=1}^{\infty} \frac{tB_t}{\prod_{i=0}^{t-1} (1+r_{1,i})} \quad (\text{or } D_1^A = D_1^L)$$

(2) $$\sum_{t=1}^{\infty} \frac{t^2A_t}{\prod_{i=0}^{t-1} (1+r_{1,i})} > \sum_{t=1}^{\infty} \frac{t^2B_t}{\prod_{i=0}^{t-1} (1=r_{1,i})} \quad (\text{or } D_2^A > D_2^L)$$

then if interest rates change, such that

$$\frac{d(1 + r_{1,t})}{d(1 + r_{1,t})} = \frac{d(1 + r_{1,0})}{(1 + r_{1,0})}, \quad \text{for all } t,$$

then P^A will equal or exceed the P^L at the new interest rate or rates. This additional requirement implies that there cannot be drastic changes in the slope of the yield curve. The following example will illustrate.

Example:

- Expectations ($r_{1,t}$): $r_{1,0}$ = 12 percent incremented by ½ percent per year to 16.5 percent at t = 9.

- Liability: $100, 12.92 percent, 4.43 - year compound bullet (to pay $171.31 at maturity).

- Assets: $100, 13.30 percent, 10 - year mortgage (to pay $18.65 per annum).

- $D_1^L = 4.43 = D_1^A$.

  $D_2^L = 20.$ $D_2^A = 27.$

- If $(1 + r_{1,t})$ decreases by 5% ($r_{1,0}$ = (1.12) x 0.95 - 1 = 6.4% $r_{1,1}$ = 6.9% , . . . , $r_{1,9}$ = 10.7%), then

  P^L = $125.75.

  P^A = $127.18 (1.1% gain).

- If $(1 + r_{1,t})$ increases by 5% ($r_{1,0}$ = (1.12) x 1.05 - 1 = 17.6%, $r_{1,1}$ = 18.1% , . . . , $r_{1,9}$ = 22.3%), then

  P^L = $80.72.

  P^A = $81.31 (0.7% gain).

o If the slope steepens to $r_{1,0}$ = 12% ($r_{1,1}$ = 13% , . . . , $r_{1,9}$ = 21%),

then

P^L = \$96.87.

P^A = \$95.42 (1.5% loss).

o If the slope becomes flat at 16% ($r_{1,t}$ = 16% for all t), then

P^L = \$88.76.

P^A = \$90.14 (1.6% gain).

o If the slope declines ($r_{1,0}$ = 12%, $r_{1,1}$ = 11.5% , . . . , $r_{1,9}$ = 7.5%),

then

P^L = \$107.36.

P^A = \$111.50 (3.9% gain).

This example demonstrates that if the $r_{1,t}$'s change by a constant percentage, the P^A will exceed the P^L. Note, though, that in some cases, even where the $r_{1,t}$'s changed by other than a constant percentage, we still gained. Only in the case where the slope became more than proportionately steeper did we lose.

I believe that this may be a general truth although we have not yet rigorously proved it. The requirement of the D_2's implies that the spread of the assets is greater than that of the liabilities. Thus, a steeper yield will likely reduce the market value of assets more than it will reduce the market value of liabilities. A decline in the slope will have the opposite effect.

In any case, this represents another shortcoming in the concept of duration. Even if we meet the conditions of the D_1's and D_2's, we may still lose, should the yield curve change shape dramatically.