Report on Mortality Improvement Scales for Canadian Insured Lives

by

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Executive Summary

Introduction

In this work we have conducted a detailed statistical analysis of mortality improvement for the Canadian population and for the Canadian insured lives. Our objective has been to construct and calibrate a statistical model for mortality improvement. Previous analysis of mortality improvement has focused on annuitants' mortality, where prudence requires that mortality improvement is taken into consideration. In this paper we consider mortality improvement in the context of life insurance, where any allowance for improvement represents a less prudent approach than assuming current mortality rates continue. One of the questions that we consider is whether there is sufficiently strong evidence to allow for any improvement at all.

Our approach has been to model simultaneously the insured lives and the whole population. The population data is more credible, as there is more exposure and a longer history. The insured lives data behaves similarly to the population data, though not identically. We have postulated and fitted a model for the population data, and subsequently fitted a model for the relationship between the population data and the insured lives data, allowing us to take advantage of the population experience in modeling the insured lives mortality.

The model

The model we have used to project mortality is an adapted version of the Lee-Carter (1992a) model. This separates the mortality into two parts; one models the age effect and one models the time effect. We work with the central rate of mortality for each age, m_x. This is an estimate of the mid-year force of mortality, $\mu_{x+½}$. As is customary, we assume a constant force over the year of age to give an estimated survival probability $p_x = e^{-m_x}$.

The formula derived for the improvement factor is a multiple of the central rate of mortality, and is therefore applied as an exponent to the survival probabilities from the base table, which we have derived from the 2001 mortality. That is, if $p(x,s)$ denotes the one year survival probability for the year 2001+s, for a life aged x, and p_x denotes the survival probability for (x) in 2001, then the improvement factor $Is(x,s)$ is applied as

$$p(x,s) = p_x^{Is(x,s)} \qquad (0.1)$$

The improvement factor depends on the age x, the period-ahead forecast, s (measured from 2001, the final year of our data) and the confidence level k, *which* allows for a specified probabilistic margin for adverse deviation, and is explained further in the following paragraph. The equation for the improvement factor is

$$Is(x,s) = \exp\left(\hat{z}_x s + k\sqrt{\hat{u}_{1,x} + \hat{u}_{2,x} s}\right) \qquad (0.2)$$

The parameters z_x, $u_{1,x}$ and $u_{2,x}$ are derived and estimated in Section 4, with full tables of values given in Appendix D. The k factor can be taken as $k=\Phi^{-1}(\alpha)$ for some confidence level α. If α=0.75, for example, then $k=\Phi^{-1}(0.75)=0.674$ and there is, approximately, a 75% probability that the true $Is(x,s)$ will be less than the estimated value. Note that this is non-diversifiable risk; the true value of $Is(x,s)$ is the same for all lives. Using $k=\Phi^{-1}(0.75)$ gives a 75% probability that the true value is less than the estimated value for all lives, not separately for each life. The consequence is important; it means there remains a 25% chance that the mortality is underestimated for the whole portfolio.

The central estimate of $Is(x,s)$ is $\exp(\hat{z}_x s)$; applying this factor to the 25-year survival probability of a 35-year old man, using the parameters estimated in the paper, indicates a 2.5% improvement in the overall mortality – that is, with no mortality improvement we expect 93.0% of lives to survive 25 years; allowing for mortality improvement using the central estimate increases the survival probability to 95.3%. However, using k=0.674 to allow for 75% confidence, the 25-year survival probability for (35) becomes 94.7%, and using k=1.96 for 97.5% confidence, the 25-year survival probability for (35) becomes 93.6%, not much larger than with no allowance at all. We find similar results for men age 45, and for women; they are summarized in the following table:

25-Year Survival Probabilities

	No improvement	Central estimate	75% confidence	97.5% confidence
Males age 35	93.0%	95.3%	94.7%	93.2%
Males age 45	82.7%	88.8%	87.2%	83.4%
Females age 35	95.2%	97.0%	96.5%	95.5%
Females age 45	87.6%	91.2%	90.1%	87.6%

This table indicates that, at least for solvency capital, it may not be wise to make

substantial allowance for longevity improvement in life insurance. Although it seems likely from the historical data (and the model chosen) that there will be some improvement, there is around a 2.5% probability that rates do not improve by any significant amount.

Smoker status

These rates are ultimate and aggregate smoker/non-smoker. The data collected on smoker/non-smoker status of Canadian Insured lives is highly inadequate. For the earliest years, only around 2% of the exposure and death data is classified by smoker status. Even in later years, in the 2001-2 data, for example, 75% of the exposure and 89% of the deaths have undetermined smoker status. We are conscious of the industry's strong desire for a smoker/non-smoker split of the results. Nevertheless, the data available, at less than 10% of the total overall, is too sparse for any reliable inference. If the industry can improve its data collecting processes to offer reliable information on smoker status for all or almost all of the business written, then in around 5 years we might begin to have enough for some preliminary conclusions.

Selection effects

As with the smoker status, for early durations the data is sparse. In Section 4.3 we have investigated whether there is any significant selection effect on the pace of improvement, and have found there is not.

Conclusions and Recommendations

It is important when using this model to bear in mind the limitations of models in general, and of our parametric model in particular. We have taken a purely statistical approach to this project, by which we mean we have used the past to model the future. There is an argument that a sudden structural shift could render the historical experience irrelevant to the future experience; if this argument is true, then, our model will not be valid. The analysis of mortality improvement on a qualitative, subjective basis, for example, considering the possible effect on mortality of trends in obesity is outside the scope or remit of this paper. Also, by using many years of historical data, mortality shocks applying to limited ages, such as the effect of AIDS on younger male mortality in the decade from around 1986-1995 are, to some extent, smoothed out of the data. The

potential financial impact of a short term mortality shock is, however, clearly an important consideration. The probability of such a shock may be fairly low, but combined with a severity that could be very high, the overall effect may be highly significant. Such a shock is entirely possible, even within a general trend of improving mortality.

We make only very general recommendations here; the interpretation of our results for practical application belongs with the regulators in discussion with the Canadian Institute of Actuaries. Notwithstanding this, we offer some suggestions:

1. For the purpose of total balance sheet solvency, that is, including provision for adverse deviation and MCCSR, there is no strong evidence to allow for mortality improvement for insured lives.
2. If mortality improvement is to be allowed for, either in the reserves or the total balance sheet capital, then we propose the use of the adapted Lee-Carter model, using Brouhns method for utilizing the population and insured lives data, to give improvement factors applied as an exponent to the time zero survival probabilities.
3. The parameter estimation for the model should be repeated at regular, frequent intervals.
4. There should be a concerted industry initiative to improve the central collection of insured lives data, in particular with respect to the smoking status information.
5. The statistical model we have used does not take structural shifts into consideration, and may underestimate the effect of mortality shocks that would results from a flu pandemic. In addition, the necessary parametric structure of the model may indicate somewhat narrower confidence intervals than we would find with a less parametric approach; we therefore suggest that the application of mortality improvement factors beyond, say, 25 years should be regarded with very great caution.

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1. INTRODUCTION

The objective of this report is to develop a mortality improvement scale (or scales) to incorporate mortality improvement in product analysis, pricing and reserving for life insurance products of Canadian life insurance companies. Conventional mortality improvement scales assume that age-specific death rates fall by a fixed percentage per year, and this percentage is very often estimated by trending past observations, coupled with some "professional judgment". Prime examples include the AA Scale in the Society of Actuaries 1994 Group Annuity Mortality Table, the Improvement Factors in the Society of Actuaries 2001 Valuation Basic Experience Table, and the Reduction Factors in the Institute of Actuaries "92" Series Base Table. The study of mortality improvement in actuarial contexts has focused on annuitants' mortality, where allowance for improved mortality is a clear financial imperative. In life insurance, the use of mortality improvement scales may offer an opportunity to reduce costs, but actuarial prudence requires that such improvement is reasonably predictable.

Deterministic mortality improvement scales are not able to provide a holistic picture of longevity risk. As with investment risk in equity-linked insurance, longevity risk is non-diversifiable, since any change in the overall mortality level affects all life insurance policies in force. In line with the current actuarial approach to the assessment and management of investment risk in the North America (see Hardy, 2003), the revision of mortality improvement scales should be moved to a stochastic, model-based framework, allowing actuaries to assess quantitatively the underlying uncertainties in the application of the improvement scales. This offers, for example, a scientific basis for the margin for adverse deviation for the mortality assumption.

Willekens (1990) suggested that stochastic models for forecasting mortality can be roughly divided into two categories, namely, extrapolative and process-oriented. A mutual shortcoming of all extrapolative models is an entire reliance on observed past trends, and consequently a lack of information on the forces shaping the changes in mortality. Nevertheless, the implementation of process-oriented methods is always obscured by the statistical difficulties in determining the dependencies between causes of death, and more importantly the unavailability of the required individual level data – the Canadian insured lives experience available to this study contains no information on causes of death and risk factors other than smoker-status (and even the smoker status information is missing

for most records). For that reason, we shall focus on various extrapolative models. These models shall give us prediction intervals that could allow for a wide range of possible outcomes, so as to cope with our ignorance of the complex biological mechanisms underlying.

The implementation of extrapolative models is not straightforward, due primarily to the two limitations in the available Canadian insured lives experience. First, the experience is available for only 20 policy years (1982 – 1983 to 2001 – 2002), which is probably too short for a direct statistical projection, no matter which model is used. Second, in earlier policy years, the volume of data with known smoker status is extremely scanty –there is no smoker information in the ultimate data before policy year 1992 – 1993. This inevitably precludes the estimation of the mortality improvement scale separately for smokers and non-smokers.

To overcome these problems, we rely additionally on the Canadian population mortality experience, which covers a far longer period of 81 calendar years (1921 to 2002), and is much richer in the number of exposed-to-risk. Based on the fact that the Canadian population is highly insured, we believe that both mortality experiences should share common features, which means the simpler approach of fitting solely to the insured lives data is, in a sense, inefficient. In our proposed methodology, both the insured lives and population experience are projected simultaneously by means of a joint model, which consists of four fitting stages. First, at the population level, we project future values of forces of mortality. Second, we summarize the projection by some tractable mathematical formulae, which give the mortality improvement scales for the population. Third, we search for a persistent parametric relationship between the experiences of the population and the insured lives. Fourth, based on the parametric relationship, we modify the results in stage two to final improvements scales applicable to the insured lives mortality. If necessary, these scales are further adjusted for the differentials arise from the selection effect and/or the smoker-status.

The flow of this report follows the logical sequence in which the ultimate improvement scales are derived. In more detail, the structure of this report is as follows.

In Section 2, we state all sources of data, and discuss the adjustments made to the raw data at the very advanced ages.

In Section 3, we introduce various stochastic models for forecasting mortality. These models are modified to suit our purposes, and are then fitted to the population experience. We also consider the so-called cohort effect manifested in the United Kingdom.

In Section 4, we focus on the insured lives. We firstly perform a two dimensional graduation to obtain base tables and smoothed insured lives experience, which is then related to the population experience by an appropriate parametric model. This relationship, along with the results in Section 3, gives the desired mortality improvement scales.

Finally, in Section 5, we provide a brief discussion on the maximum life span, and give several recommendations to actuaries using the improvement scales.

2. SOURCES OF DATA

The revision of mortality improvement scales is based on the experiences of both the general population and the insured lives in Canada. For more informative conclusions in the analyses of the old-age mortality, the cohort effect and the exogenous interventions affecting mortality improvements, mortality data of the populations in England and Wales, the United States and Japan are also considered. Below we list all sources of data, the sample period, and the modifications made (if any).

- *Canada – general population*

  Historical death counts (D_x) and mid-year population estimates (proxy for the central exposure-to-risk, E_x) for both sexes and for every single year of age up to 89 are obtained from the Human Mortality Database (HMD) and the CANSIM (a socio-economic database provided by Statistics Canada, see Statistics Canada, 2004).

  Data from calendar year 1920 to 1997 are available from the HMD. According to the HMD documentation, complete vital statistics data back to 1921 are not available for all regions of Canada. The following are the changes in the coverage of the vital statistics during that period.

  - Deaths counts exclude Newfoundland prior to 1949, and the Territories (Yukon and Northwest Territories) prior to 1950.
  - Prior to 1944 all vital events were classified by place of occurrence. Since 1944, births and deaths are classified by area of reported residence.
  - Population counts for Canada exclude Newfoundland prior to 1949, but include the Territories from 1921. Fortunately, the fact the Yukon and NWT are included in the population counts but not in the death counts does not introduce

a significant bias in the calculation of age specific death rates, since the population in the these regions is very small – In 1921, Yukon and NWT accounted for only 0.14% of the total population of Canada (12,200 persons out of 8,787,400).

- From 1921 to 1970, except in 1961 and 1966, population counts are the estimates of population on June 1 produced by Statistics Canada.
- In 1961 and 1966, population counts are the Census populations on June 1. Starting in 1971, population counts are the population estimates on July 1 produced by Statistics Canada. These estimates include a correction for net undercount.

Rigorous outlier analyses in later part of this report indicate that the effect of these changes is not substantial.

From 1998 onwards, the required data are obtained from CANSIM. To ensure data homogeneity, we compared the overlapping portion (death counts and population estimates are available from both sources from 1971 to 1997), and we found that the data series commensurate reasonably well with each other.

- *Canada – insured lives*

The insured lives experience used in this study was collected over a 20-year period covering policy year 1982 – 1983 to policy year 2001 – 2002 from all life insurance companies in Canada by the Institute of Insurance and Pension Research (IIPR), the University of Waterloo. The mortality data are available for every single year of age up to 99, and are segregated by:

i. sex – male and females;
ii. duration – 1, 2, and so on up to 15, and ultimate (16+);
iii. smoker status – smoker, non-smoker and indeterminate (no smoker breakdown in the ultimate data prior to policy year 1992 – 1993).

In each category, D_x and E_x are given in terms of both the number of lives and the amount of insurance. We prefer "numbers" to "amounts" as Currie et al. (2004) pointed out that an undesirable statistical problem known as over-dispersion is much more substantial if stochastic mortality models are fitted to "amounts".

- *The United States – general population*

Age-specific central rates of death (m_x) for both sexes from 1900 to 2000 are

obtained from the National Center for Health Statistics (NCHS, 2004a, 2004b), Center of Diseases Control and Prevention. Unfortunately, the mortality data are presented in an abridged form, i.e., values of death rates are shown at age 0, age group 1 – 4, decadal age groups 5 – 14, 15 – 24, and so on up to 75 – 84, and the open age group 85 and over. We apply the disaggregation method proposed by Pollard (1988) to derive full life tables from the mortality data tabulated in 10-year age groups, assuming that the force of mortality varies in an exponential manner and that the population is stable within each of the age intervals.

Again, the mid-year population estimates are used as a proxy for E_x. Population estimates for every single year of age are obtained from the United States Census Bureau (2004). From 1990 to 2000, the estimates can be retrieved online, and from 1900 to 1989, the estimates are obtained by a written request.

- *Japan, England and Wales – general population*

  Historical death counts and mid-year population estimates (proxy for E_x) are obtained from the HMD. Numerical values are given for both sexes and for every single year of age from 0 to 99. The Japanese data covers calendar year 1950 to 2000, while the English and Welsh data covers calendar year 1841 to 2003.

At very high ages, the ratio of death counts to the number of exposure-to-risk may not be reliable, due partly to the inaccuracy of reported age at death, and partly to the sampling error when the number of death and the number of exposure-to-risk are small. Bourbeau and Desjardins (2002) performed a systematic verification of ages at death in Canada. Based on the verified observations, they obtained a preliminary estimation of centenarian mortality by using the extinct generation method (Vincent, 1951). They concluded that, for the time being, official statistics are not to be counted on to provide a conclusive picture of patterns of mortality at the highest ages in Canada, and mathematical techniques must still be counted on to establish the later years of the life tables.

There are a variety of mathematical models for old age mortality, for example:

i. The cubic polynomial function used by CIA 69-75 (Panjer and Russo ,1992; Panjer and Tan,1995);
ii. The old-age mortality standard developed by Himes et al. (1994);
iii. The old-age term of the Heligman-Pollard mortality model (Heligman and Pollard, 1980);

iv. The Coale-Kisker method of closure of life tables (Coale and Guo, 1989; Coale and Kisker, 1990).

We refer the reader to Buettner (2002) for a detailed examination of the last three models. Based on the goodness of fit, we choose the Coale-Kisker method, which is also used by Lee and Carter (1992a) in forecasting the United States mortality. In the Coale-Kisker method, it is assumed that probabilities of death are increasing with age at a linearly decreasing rate. For $x \geq 85$, define

$$k(x) = k(x-1) - R, \tag{2.1}$$

where $k(x) = \ln\left(\frac{m_x}{m_{x-1}}\right)$, and R is a constant. Extending the formula up to $x = 110$ and summing up,

$$k(85) + \ldots + k(110) = 26k(84) - R(1 + 2 + \ldots + 26). \tag{2.2}$$

Solving for R, we obtain

$$R = \frac{26k(84) + \ln(m_{84}) - \ln(m_{110})}{351}. \tag{2.3}$$

To minimize the effect of random fluctuations, $k(84)$ is replaced by $k^*(84)$, which is the arithmetic average of $k(82)$ to $k(86)$, i.e.,

$$\begin{aligned} k^*(84) &= \frac{k(82) + k(83) + k(84) + k(85) + k(86)}{5} \\ &= \frac{k(82) + k(83) + 3k(84) - 3R}{5}. \end{aligned} \tag{2.4}$$

Similarly, $\ln(m_{84})$ is replaced by $\ln(m_{84}^*)$, which is defined as

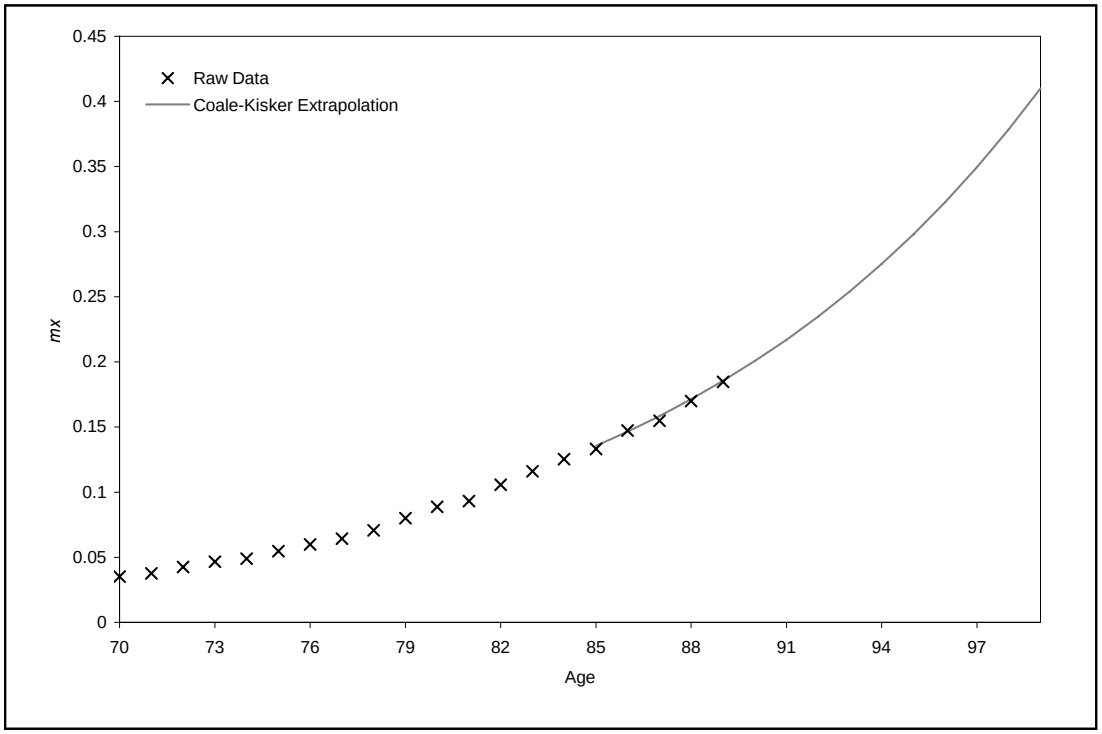
$$\ln(m_{84}^*) = k^*(84) + \ln(m_{83}^*), \tag{2.5}$$

where $\ln(m_{83}^*) = \ln(\frac{m_{82} + m_{83} + m_{84}}{3})$. Substituting $k^*(84)$ and $\ln(m_{84}^*)$ in R, we obtain

$$R = \frac{\frac{27}{5}\left[k(82) + k(83) + 3k(84)\right] + \ln\left[\frac{m_{82} + m_{83} + m_{84}}{3}\right] - \ln(m_{110})}{351 + \frac{27 \times 3}{5}}. \qquad (2.6)$$

In the implementation of Coale-Kisker method, m_{110} is assumed to be 1.0 for males. Such an assumption is based on the fact that there were almost no survivors at age greater than 110. For females, m_{110} is chosen as 0.8 so as to avoid imposing a crossover of male and female mortality at age 110.

Figure 2.1 displays the application of the Coale-Kisker method to the Canadian population data. The extrapolation shows no discernable discrepancy with the raw data from age 85 to 89, and progresses logically to the more advanced ages. To diagnose the performance of the Coale-Kisker method beyond age 89, we also utilize the method in the Japanese and English and Welsh population data, which are available up to age 99. Figures 2.2 and 2.3 indicate that the method works well in both data sets.



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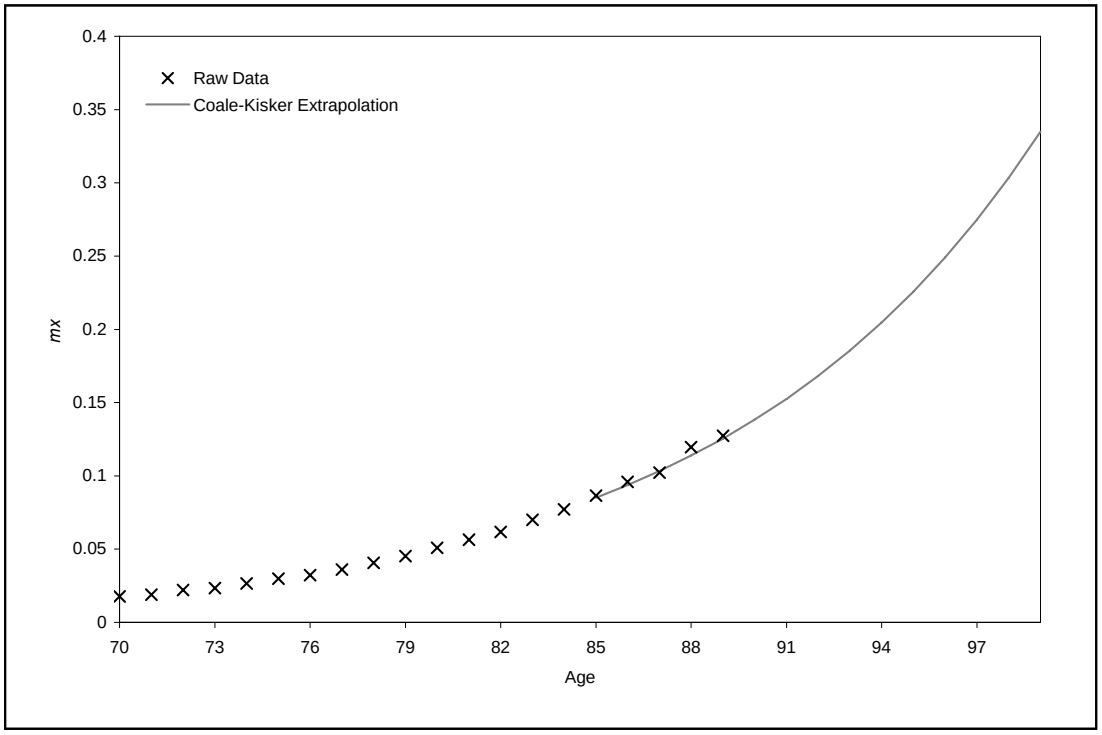

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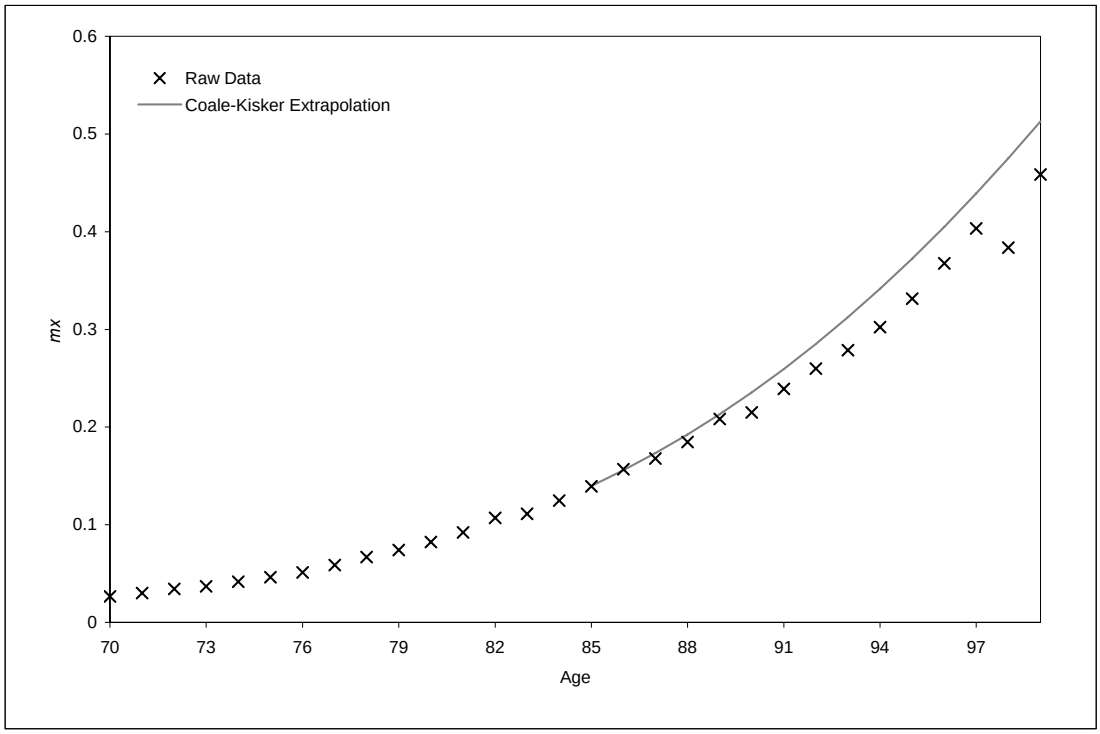
Fig. 2.1. The Coale-Kisker extrapolation, Canadian population, 1990, (a) male, (b) female.



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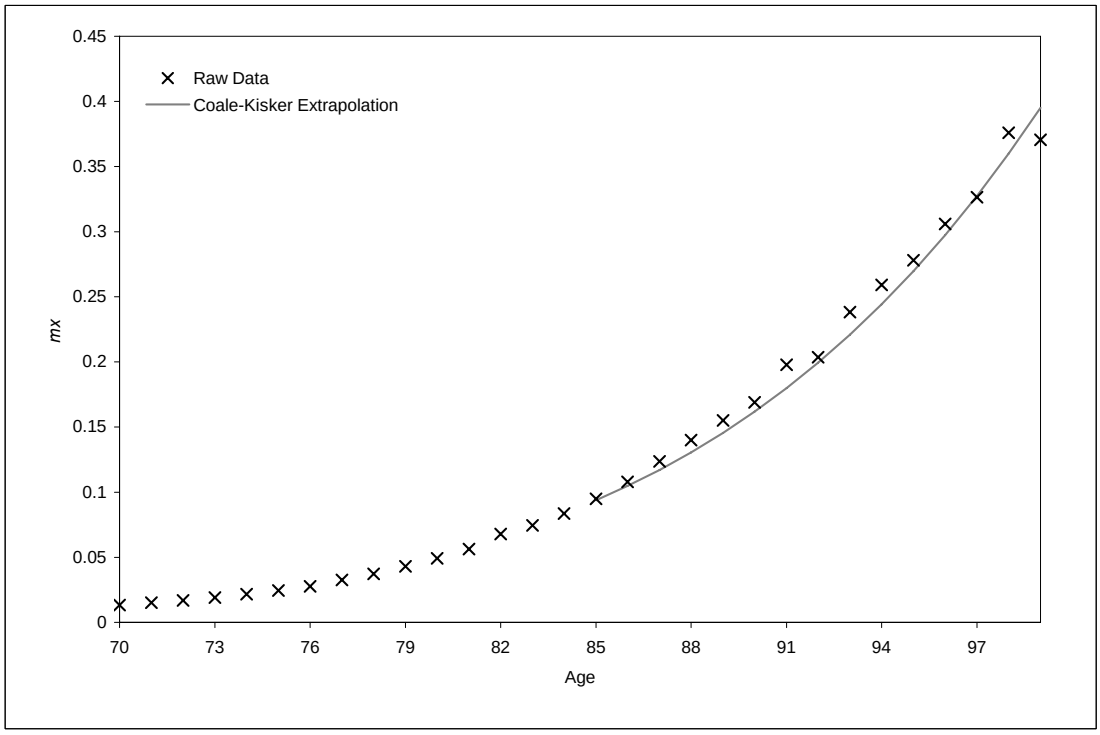
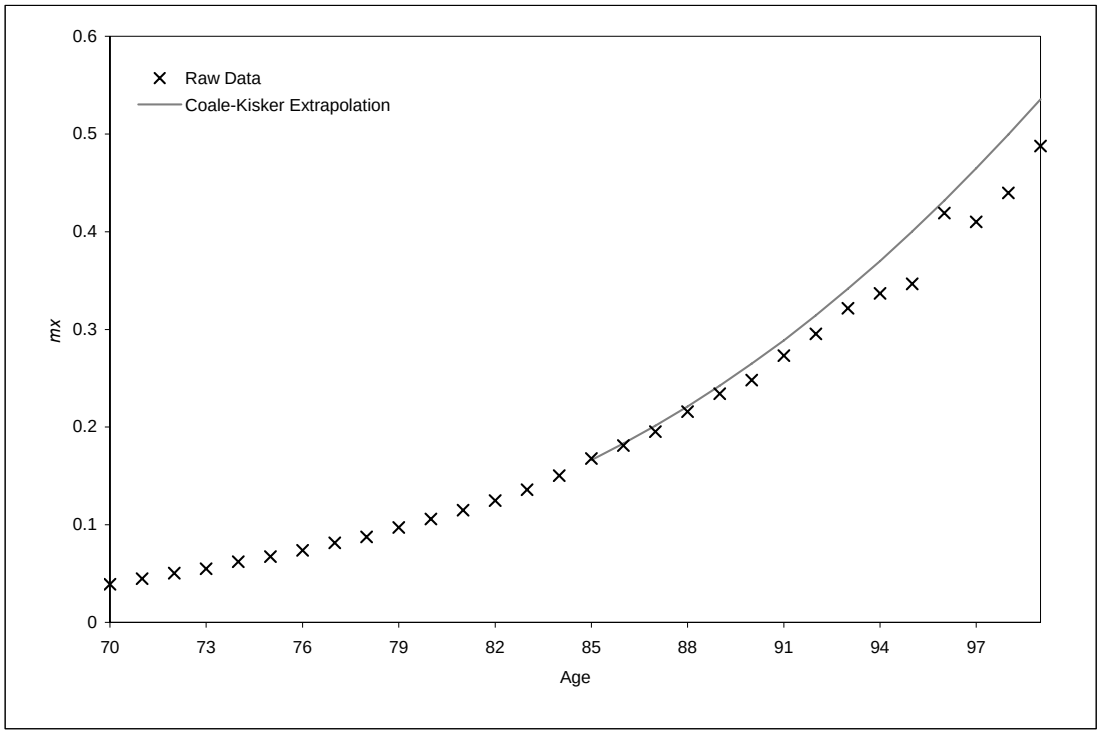


(b)

Fig. 2.2. The Coale-Kisker extrapolation, Japanese population, 1990, (a) male, (b) female.



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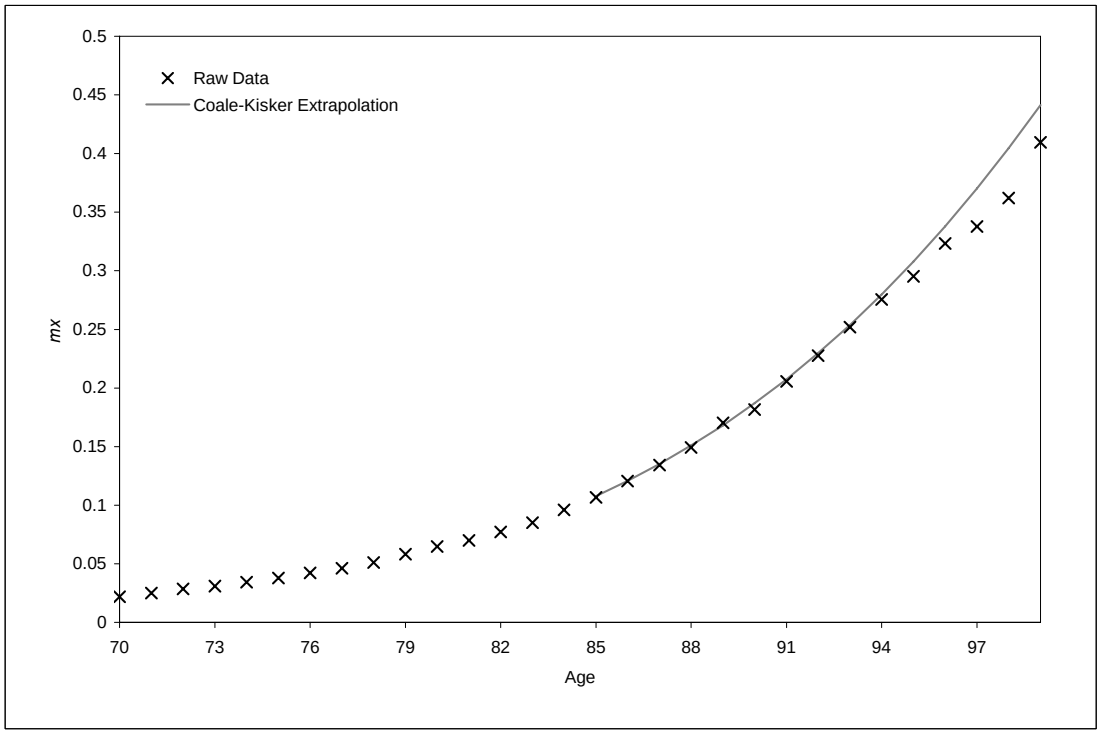


(b)

Fig. 2.3. The Coale-Kisker extrapolation, English and Welsh population, 1990, (a) male, (b) female.

3. FORECASTING THE MORTALITY OF THE CANADIAN POPULATION

In this section, we will apply two families of stochastic mortality models, namely, the penalized spline regression, *P*-splines (Currie et al., 2004), and the Lee-Carter model (Lee and Carter, 1992a), to the Canadian population mortality. These two families of models are also recommended by the Continuous Mortality Investigation Bureau (CMIB) of the Institute of Actuaries in projecting the insured lives experience in the United Kingdom (see CMIB, 2005). For each family of models, we will explain in detail how the model parameters are estimated, and how the "best estimate" and the prediction interval of future death rates are derived. We will also discuss the limitations of each family of models, and propose new variants to ameliorate.

3.1 THE P-SPLINES REGRESSION

Below we give a succinct description of the *P*-Splines regression and its relevance to mortality forecasting. We refer the reader to Eilers and Marx (1996) for detailed descriptions of the method, and Currie et al. (2004) for a recent application of the method in projecting the UK insured lives experience. In brief, the *P*-Splines method is a combination of following:

i. *Smoothing the historical mortality rates.* This is done by a two-dimensional regression with cubic *B*-splines as the basis. To avoid over-fitting, a penalty function is considered in the maximum likelihood estimation of parameters.
ii. *Extrapolating the historical mortality rates.* This is done by treating the future mortality rates as missing data. The penalty function, subjectively chosen by the forecaster, determines the "best estimate" of future mortality rates.

We shall begin with the one-dimensional case. Let $D_{x,t}$, $\mu_{x,t}$ and $E_{x,t}$ respectively be the number of deaths, the force of mortality, and the number of exposure-to-risk at age x and time t. For convenience we write $\mathbf{d}' = (D_{x,1}, D_{x,2,} \ldots, D_{x,T})$, $\boldsymbol{\mu}' = (\mu_{x,1}, \mu_{x,2,} \ldots, \mu_{x,T})$, and $\mathbf{e}' = (E_{x,1}, E_{x,2,} \ldots, E_{x,T})$, where T is the length of time-series. We use the mortality data of a 20-year-old male Canadian (i.e. $x = 20$ and $T = 81$, from 1921-2002) to illustrate the one- dimensional case.

In the *P*-splines method, we assume that $D_{x,t}$ is a realization of a Poisson distribution with mean $E_{x,t}\mu_{x,t}$. A straightforward approach to model $\mu_{x,t}$ over time might be to fit a log-linear regression, say,

$$\ln\left(\mu_{x,t}\right) = a_1 + a_2 t + \varepsilon_t \,, \tag{3.1}$$

where a_1 and a_2 are the regression coefficients and ε_t is the error term. In matrix form, we can rewrite (3.1) as

$$\ln\left(\mu_{x,t}\right) = \mathbf{Ba} + \varepsilon_{x,t} \,, \tag{3.2}$$

where $\mathbf{a}' = (a_1, a_2)$ and $\mathbf{B} = (1, t)$. In the statistical literature, elements in $\mathbf{B}$ are known as basis functions. Figure 3.1 shows that the simple form of $\mathbf{B} = (1, t)$ does not fit well to the data that appears to be curved. To improve the fit, we may introduce an additional basis function, say t^2, which gives $\mathbf{B} = (1, t, t^2)$.

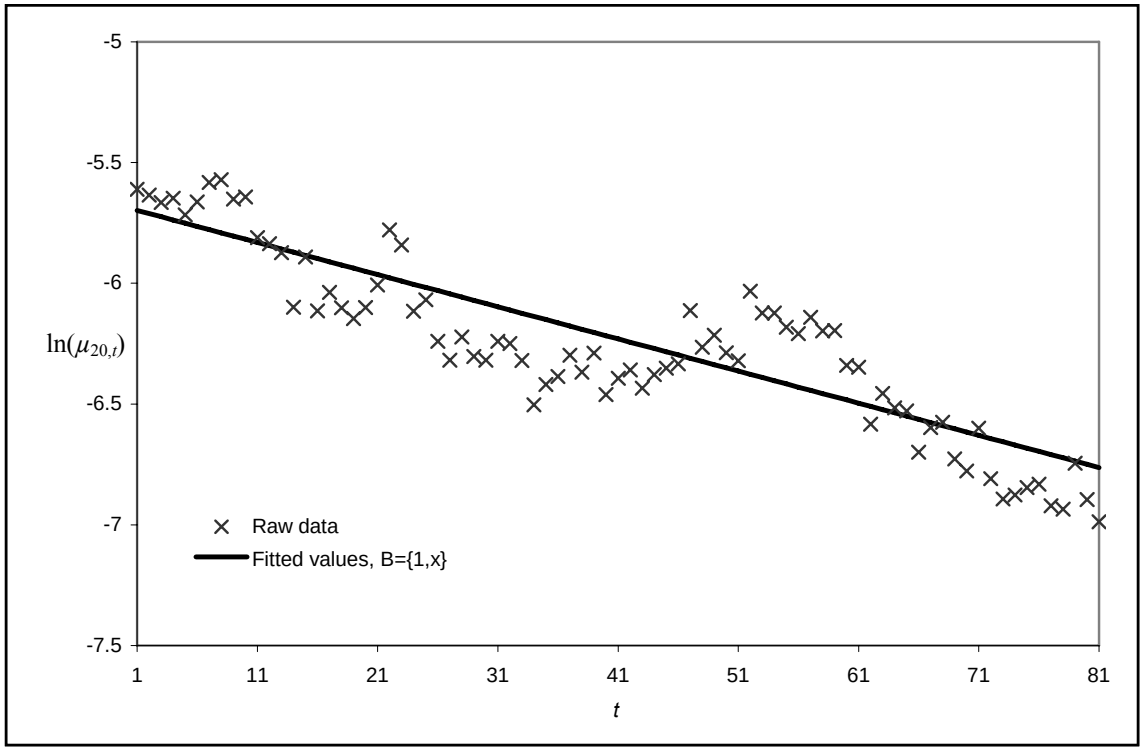


Fig. 3.1. Observed values of $\ln(\mu_{20,t})$ and fitted values using $B = \{1, x\}$.

An even more flexible basis may be provided by a set of cubic *B*-splines, i.e. $\mathbf{B} = (B_1(x), B_2(x), \ldots, B_K(x))$. Each $B_i(x)$ is a *B*-spline, which consists of cubic polynomial pieces joining together smoothly. Such a basis for $K = 19$ is shown in Figure 3.2. Note that each time point in the domain $t = \{1, 2, \ldots, 81\}$ is covered by four non-zero *B*-splines.

For instance, at $t = 18$, $B_4(18)$, $B_5(18)$, $B_6(18)$ and $B_7(18)$ are non-zero. Hence, the fitted value of $\ln(\mu_{20,18})$ under this basis will be:

$$\hat{\ln}(\mu_{20,10}) = \sum_{i=1}^{19} B_i(18)\hat{a}_i = B_4(18)\hat{a}_4 + B_5(18)\hat{a}_5 + B_6(18)\hat{a}_6 + B_7(18)\hat{a}_7 \,. \qquad (3.3)$$

The regression coefficients, a_i, $i = 1, 2, \ldots, 19$, can be estimated by maximizing the log likelihood function of the regression model, $l(\mathbf{a};\mathbf{d},\mathbf{e})$.

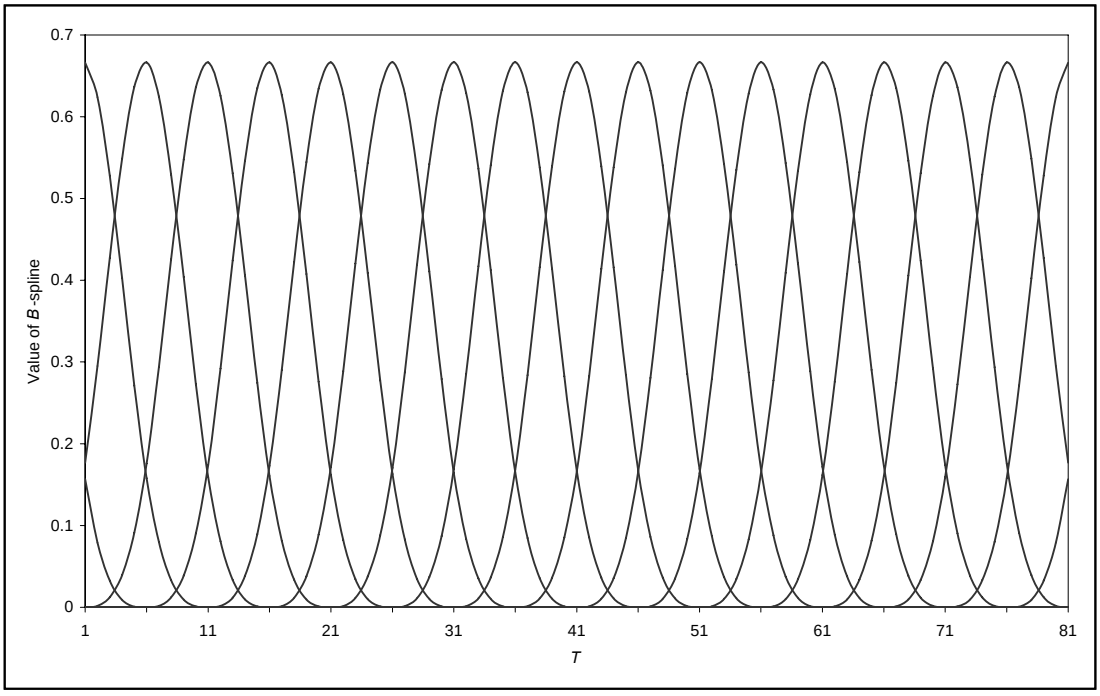


Fig. 3.2. A basis of $K = 19$ cubic B-splines.

Figure 3.3 shows the fitted model (darker line). Evidently, over-fitting exists. Eilers and Marx (1996) proposed penalizing the over-fit by placing a difference penalty on the adjacent a_i's. This penalty function is incorporated to the log likelihood function to give the penalized log likelihood:

$$l_p(\mathbf{a};\mathbf{d},\mathbf{e}) = l(\mathbf{a};\mathbf{d},\mathbf{e}) - \frac{1}{2}\lambda \mathbf{a}'\boldsymbol{\Delta}'\boldsymbol{\Delta}\mathbf{a}\,, \qquad (3.4)$$

where $\boldsymbol{\Delta}$ is a difference matrix of order n. The maximization jointly maximizes the goodness-of-fit, which is measured by the usual log likelihood $l(\mathbf{a};\mathbf{d},\mathbf{e})$, and minimizes the n^{th} order differences between the adjacent a_i's so that they tend to lie on a degree n-1

polynomial. The relative importance of goodness-of-fit and smoothness is controlled by the smoothing parameter, λ. Let $\mathbf{P} = \lambda\boldsymbol{\Delta}'\boldsymbol{\Delta}$ be the penalty matrix. The maximization of equation (3.4) gives the penalized likelihood equation:

$$\mathbf{B}'(\mathbf{d} - \mathbf{e}\boldsymbol{\mu}) = \mathbf{P}\mathbf{a}, \tag{3.5}$$

which can be solved by the scoring algorithm:

$$\hat{\mathbf{a}} = \left(\mathbf{B}'\tilde{\mathbf{W}}\mathbf{B} + \mathbf{P}\right)^{-1}\left[\mathbf{B}'\tilde{\mathbf{W}}\mathbf{B}\tilde{\mathbf{a}} + \mathbf{B}'\left(\mathbf{d} - \mathbf{e}\tilde{\boldsymbol{\mu}}\right)\right], \tag{3.6}$$

where $\tilde{\mathbf{a}}$, $\tilde{\boldsymbol{\mu}}$ and $\tilde{\mathbf{W}}$ denote current estimates of $\mathbf{a}$, $\boldsymbol{\mu}$ and $\mathbf{W}$, and $\hat{\mathbf{a}}$ denotes the updated estimate of $\mathbf{a}$. Under the assumption that $D_{x,t}$ follows a Poisson distribution with mean $E_{x,t}\mu_{x,t}$, $\mathbf{W} = \text{diag}(\boldsymbol{\mu})$.

Finally, the approximate variance of the linear predictor is given by

$$\text{Var}\left(\mathbf{B}\hat{\mathbf{a}}\right) \approx \mathbf{B}\left(\mathbf{B}'\hat{\mathbf{W}}\mathbf{B} + \mathbf{P}\right)^{-1}\mathbf{B}'. \tag{3.7}$$

Hence, assuming normality, the approximate 95% confidence interval for $\ln(\mu_{x,t})$ can be written as

$$\hat{\ln}(\mu_{x,t}) \pm 1.96\sqrt{\text{Var}\left(\hat{\ln}(\mu_{x,t})\right)}, \tag{3.8}$$

where $\text{Var}\left(\hat{\ln}(\mu_{x,t})\right)$ is the t^{th} diagonal element in $\text{Var}\left(\mathbf{B}\hat{\mathbf{a}}\right)$. Figure 3.3 (the lighter line) shows that the penalty in (3.6) lessened the extent of over-fit.

The penalized likelihood estimation is conditional on the choice of (1) the number of cubic *B*-splines, *K*, in the basis, (2) the order, *n*, in the penalty function, and (3) the smoothing parameter, λ. The choice of (1), the number of splines, is discussed in Eilers and Marx (1996), Ruppert (2002) and Currie and Durban (2002). They proposed the rule of thumb:

$$K = \max\left\{T/4, 40\right\} + 3. \tag{3.9}$$

The choice of (2), the order of the penalty function, is closely related to the form of the forecast, and this will be explained in later part of this section. Finally, the choice of (3),

the smoothing parameter, relies on the Bayesian Information Criterion, BIC, (Schwarz, 1978), which is defined as

$$BIC = Dev + \ln(n \times Tr), \tag{3.10}$$

where *Dev* is the deviance, and *Tr* is the trace of the hat-matrix **H**, which is given by

$$\mathbf{H} = \mathbf{B}(\mathbf{B}'\mathbf{W}\mathbf{B} + \mathbf{P})^{-1}\mathbf{B}'\mathbf{W}. \tag{3.11}$$

The smoothing parameter, λ, is chosen to minimize the BIC.

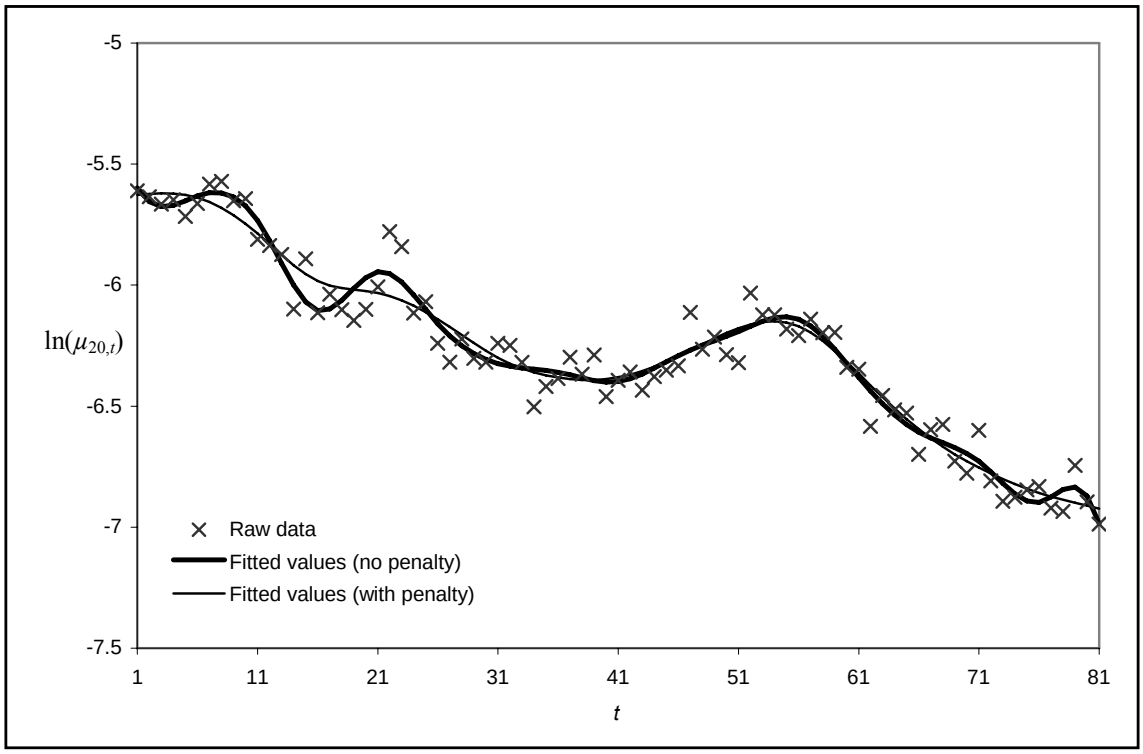


Fig. 3.3. Observed values of $\ln(\mu_{20,t})$ and fitted values under different penalties.

In forecasting, we treat the future values of $D_{x,t}$ and $E_{x,t}$ ($t > T$) as missing data, and perform model fitting and forecasting simultaneously. Suppose that we wish to forecast future death rates in S years from now. Then, we have to extend the original basis matrix **B** to $\mathbf{B}^*$ that covers $T + S$ years. $\mathbf{B}^*$ can be written as

$$\mathbf{B}^{*} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix}. \tag{3.12}$$

$\mathbf{B}_1$ and $\mathbf{B}_2$ contain the *B*-splines that cover the entire domain {1, 2, … ,*T*, *T*+1, …*S*}, and evaluate these *B*-splines at *T*+1, *T*+2, …, *S*. Define a weight matrix $\mathbf{V}$ = blockdiag($\mathbf{I}$; $\mathbf{0}$), where $\mathbf{I}$ is an identity matrix of size *T* and $\mathbf{0}$ is a square matrix of 0's of size *S*. Then, the scoring algorithm in equation (3.6) can be rewritten as

$$\hat{\mathbf{a}} = \left(\mathbf{B}'\mathbf{V}\tilde{\mathbf{W}}\mathbf{B} + \mathbf{P} \right)^{-1} \left[\mathbf{B}'\mathbf{V}\tilde{\mathbf{W}}\mathbf{B}\tilde{\mathbf{a}} + \mathbf{B}'\mathbf{V}\left(\mathbf{d} - \mathbf{e}\tilde{\boldsymbol{\mu}} \right) \right], \tag{3.13}$$

which enables us to perform fitting and forecasting simultaneously. The variance and the approximate 95% interval forecast can be obtained using equations (3.7) and (3.8).

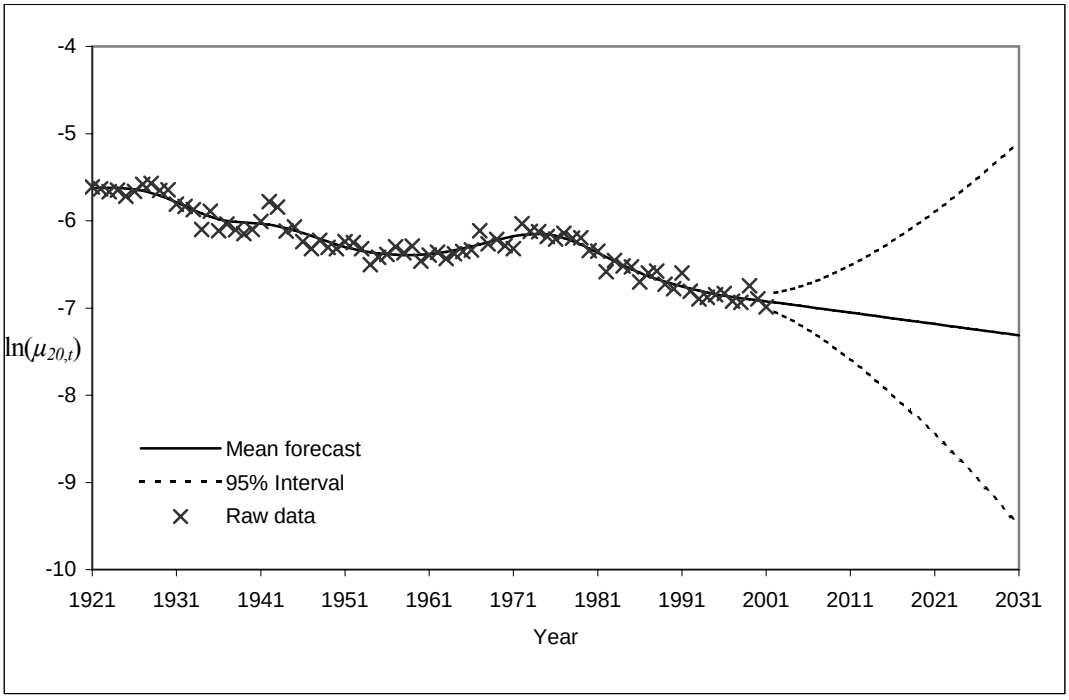


Fig. 3.4. Mean forecast and approximate 95% prediction interval of $\ln(\mu_{20,t})$. Year 1921 to 2031 corresponds to $t = 1, 2, \ldots, 111$.

As mentioned earlier, the order (*n*) in the penalty function embedded in equation (3.4) plays a crucial role in the form of the forecast. Setting $n = 1$ essentially assumes no mortality improvement in the future. Setting $n = 2$ gives a linear forecast. Setting $n = 3$

gives a forecast that follows a quadratic function. As neither the consequence of $n = 1$ nor that of $n = 3$ seems reasonable in the context of mortality forecasting, we set $n = 2$. Figure 3.4 shows both the mean forecast and the point-wise interval forecast of $\ln(\mu_{20,t})$ under the one-dimensional *P*-splines regression.

We now move on to the two-dimensional case. Let $\mathbf{D} = \{D_{x,t}\}_{X\times T}$, $\mathbf{M} = \{\mu_{x,t}\}_{X\times T}$ and $\mathbf{E} = \{E_{x,t}\}_{X\times T}$, where X is the total number of ages. We consider $x = 0, \ldots, 99$, so $X = 100$. Let $\mathbf{B}_a$ and $\mathbf{B}_y$ be the basis matrix when the *P*-splines regression is applied to the columns and rows of $\mathbf{M}$ respectively. It can be shown that the basis matrix for the two-dimensional model is

$$\mathbf{B} = \mathbf{B}_y \otimes \mathbf{B}_a , \tag{3.14}$$

where $\otimes$ denotes the Kronecker product operator. Similarly, let Δ_a and Δ_y be the difference matrix when the *P*-splines regression is applied to the columns and rows of $\mathbf{M}$ respectively. It can be shown that the penalty matrix for the two-dimensional model is given by

$$\mathbf{P} = \lambda_a \mathbf{I}_{c_y} \otimes \mathbf{D}_a^{'}\mathbf{D}_a + \lambda_y \mathbf{D}_y^{'}\mathbf{D}_y \otimes \mathbf{I}_{c_a} , \tag{3.15}$$

where λ_a and λ_y are the smoothing parameter in age and year respectively; c_a and c_y are the number of columns in $\mathbf{B}_a$ and $\mathbf{B}_y$ respectively. Let vec($\mathbf{Y}$) be the operator that converts $\mathbf{Y} = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_N]$ into $[\mathbf{y}_1', \mathbf{y}_2', \ldots, \mathbf{y}_N']'$. Replacing $\mathbf{B}$, $\mathbf{P}$ and $\mathbf{d}$ in equations (3.5) – (3.7) and (3.11) – (3.13) by equations (3.14) and (3.15) and vec($\mathbf{D}$) respectively gives the algorithm for smoothing and forecasting in the two-dimensional case.

Figures 3.4 and 3.5 show the application of the *P*-splines regression to the Canadian population mortality. A feature of the *P*-splines forecast is that the confidence intervals are very wide – In some cases, e.g. male at age 0, the interval in the far future essentially spans all possible values that the death rate can attain. In addition, except at the older ages where the dispersion is large, the flexible *P*-splines demonstrate an excellent goodness-of-fit to the historical data. Wide confidence intervals do not imply a poor model per se – it may simply be that the data is so sparse and volatile that more confident prediction is not possible.

Nevertheless, the *P*-splines perform poorly outside the region of the data. At infancy and the older ages, the mean forecasts rebound illegitimately at the forecast origin and then rise rapidly to an unacceptable level in the near future. For instance, in no more than

20 years, the mean forecast of $-\ln(m_{90})$ for male exceeds 0, or equivalently speaking, m_{90} goes over 1. This phenomenon may be attributed to the following reasons.

i. *Inappropriate model assumptions*. Strictly speaking, the *P*-splines method is merely an extension of the simple linear regression model, which means the implementation of *P*-splines assumes fundamentally that the death rates, or the "dependent variables", are not serially correlated. This assumption may not be appropriate in the context of mortality forecasting, as we know that, at least on the time dimension, death rates form a time-series and are inter-dependent.

   No different from other extrapolative forecasting methods, the *P*-splines forecast is a weighted average of previous observations. The inappropriate assumption of independence might lead to an incorrect specification of weights such that the too much emphasis is placed on the very recent observations. In Figure 3.4, we can notice that the forecast is almost completely determined by the slope of the last 5 (perhaps even fewer) observations. Consequently, a minor abnormality near the forecast origin could ruin the entire forecast.

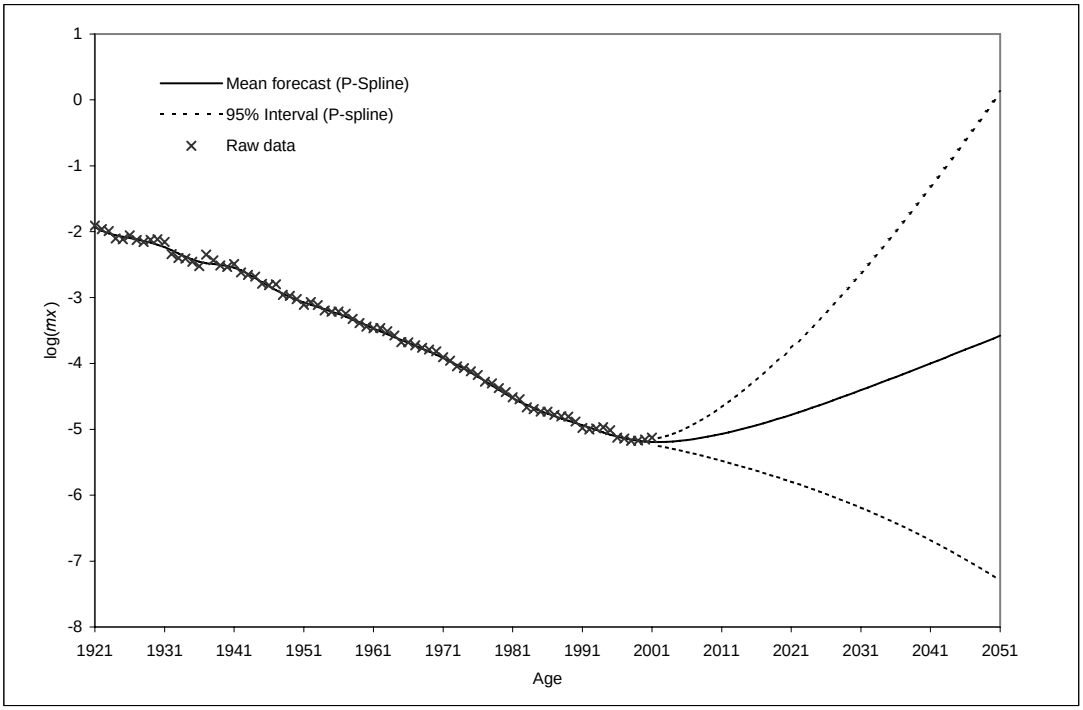
ii. *Requirements of parsimony not met.* The *P*-splines regression fails to maintain an appropriate balance between smoothness and goodness-of fit. The model selection criterion, BIC, seems to favor smaller values of l_a and l_y, which lead to less penalty on over-fitting. This situation is analogous to fitting the historical data by an arbitrarily high degree polynomial, which would be almost guaranteed to give an unreasonable "best estimate". The interval forecasts, which are computed by subtracting / adding 1.96 times the standard deviation from / to the "best estimate", consequently give no sensible meaning, no matter how wide they are.

Figure 3.6 shows the future age patterns of mortality projected by the *P*-splines regression. These patterns are rather different from the generally accepted ones. For the males, the projected pattern is almost flat from age 30 to 70, with a hump around age 70 to 80. It also appears that the 95% confidence interval does not encompass any reasonable age pattern. For both sexes, the death rates demonstrate a counter intuitive fall at around age 90. Such a fall implies the existence of crossovers in age-specific mortality trends, which is in fact the primary problem of some other older approaches of mortality forecasting, e.g. extrapolating death rates age by age (see Keyfitz, 1982).

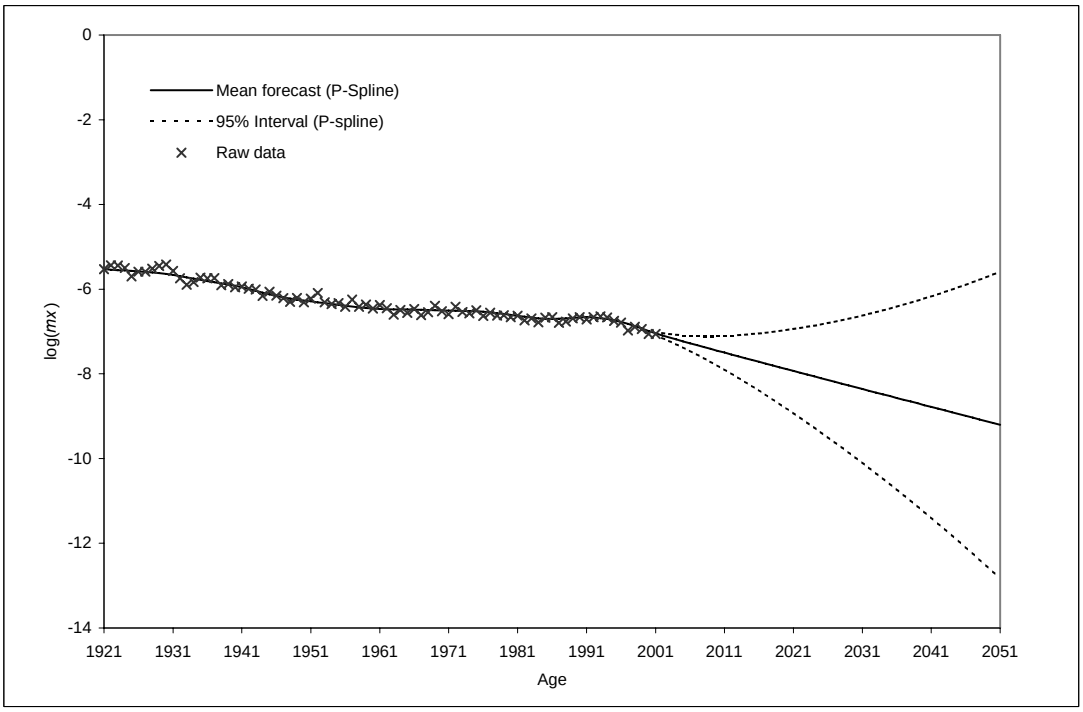
To account for the anomalous projected age pattern, we revisit the nature of the *P*-splines regression. In contrast to other extrapolative methods, the *P*-splines method is two-dimensional in the sense that the entire mortality surface is allowed to change over

time. The high degree of freedom gives the *P*-splines method the competitive advantage of a high goodness-of-fit, and allows more uncertainties in the future that leads to wide confidence intervals. On the other hand, the method imposes very few constraints on the behavior of the forecast. As time passes by, the age pattern is gradually distorted. This problem may not be noticeable in Currie et al. (2004), probably because the method was applied only to ages 16 and over – the age pattern after the hump at the younger ages is fairly linear and stable over time.

The above suggests that the *P*-splines method does an excellent job in smoothing historical data but not in projecting the future mortality of Canadians. In this study, we shall use the *P*-splines regression for the purpose of data and parameter graduation, and seek alternative stochastic models for the ultimate calibration of improvement scales.

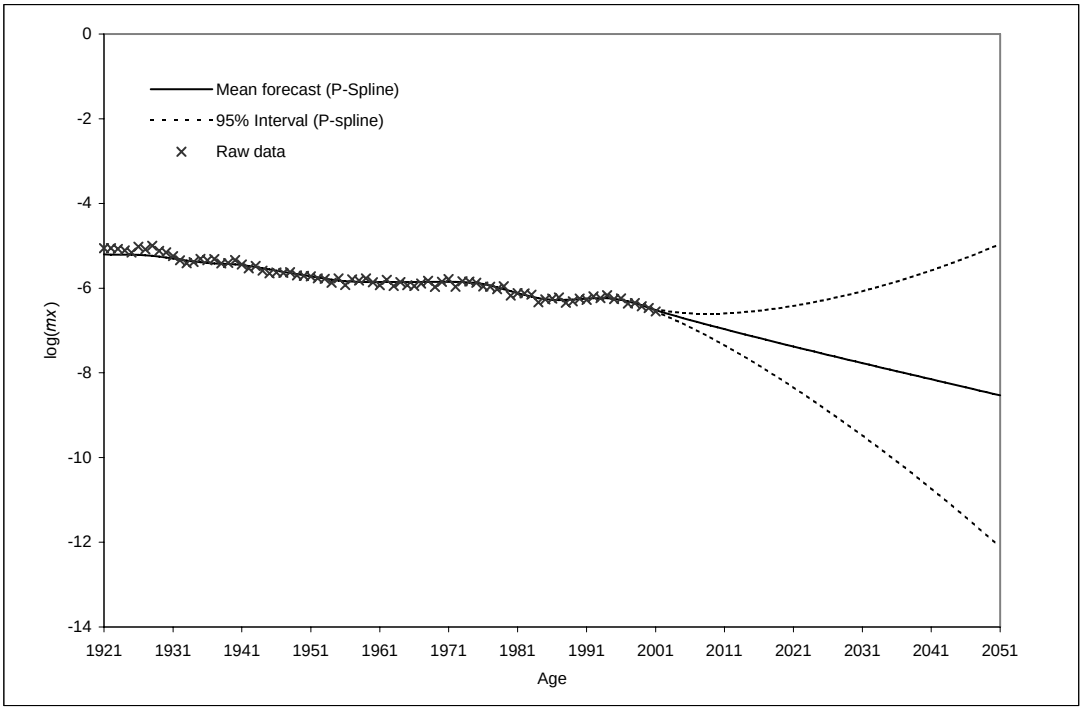


Age 0

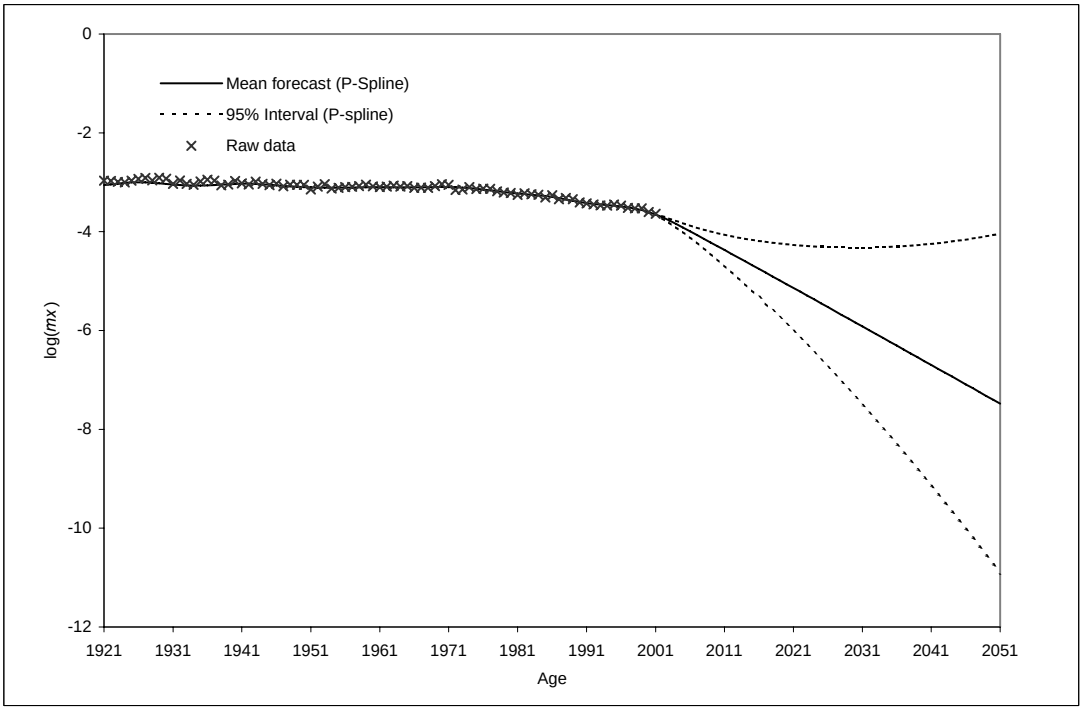


Age 30

Fig 3.4. The fitted (1921 – 2001) and projected (2002 – 2051) mortality experience using *P*-splines regression, male.

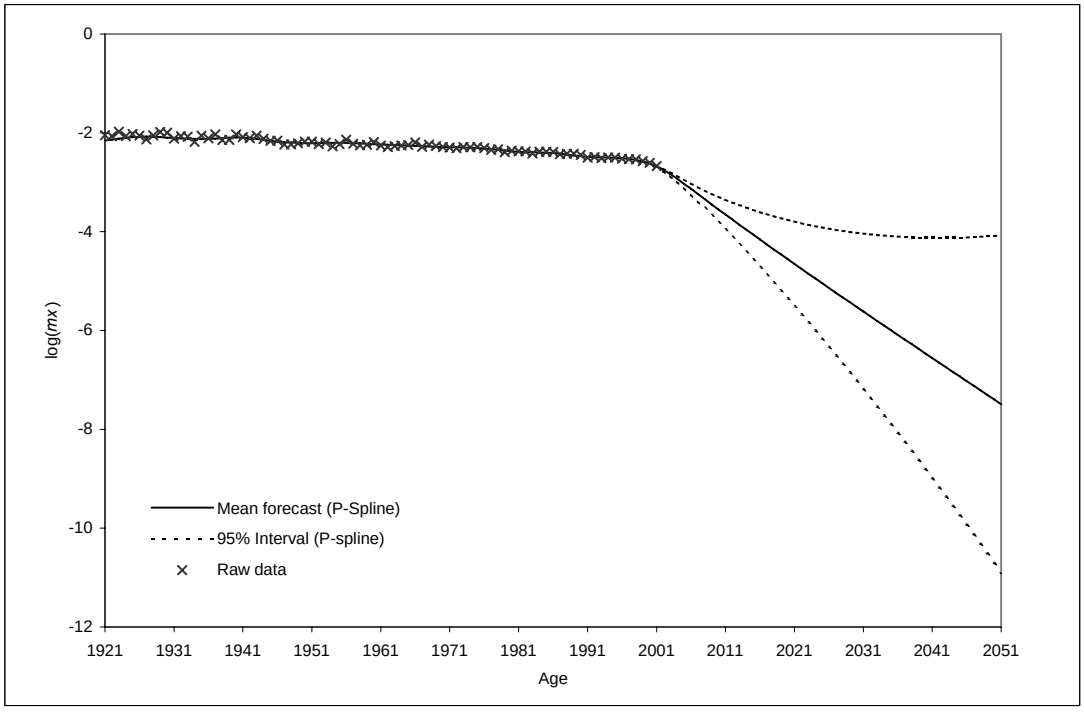


Age 50

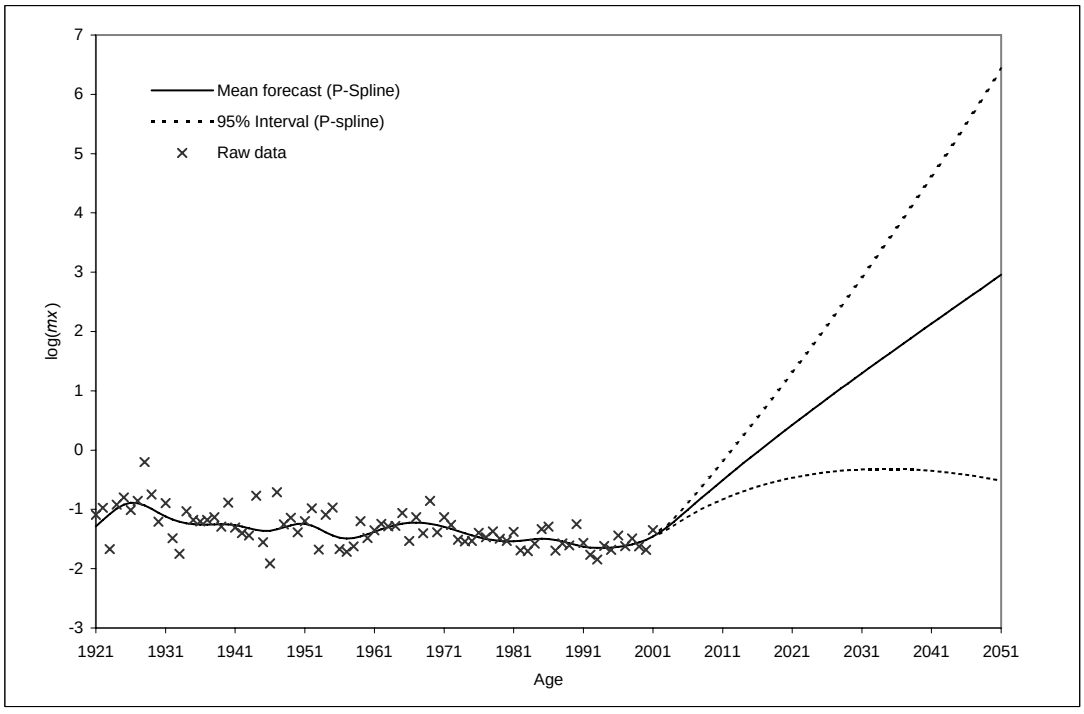


Age 70

Fig 3.4 (cont'd). The fitted (1921 – 2001) and projected (2002 – 2051) mortality experience using *P*-splines regression, male.

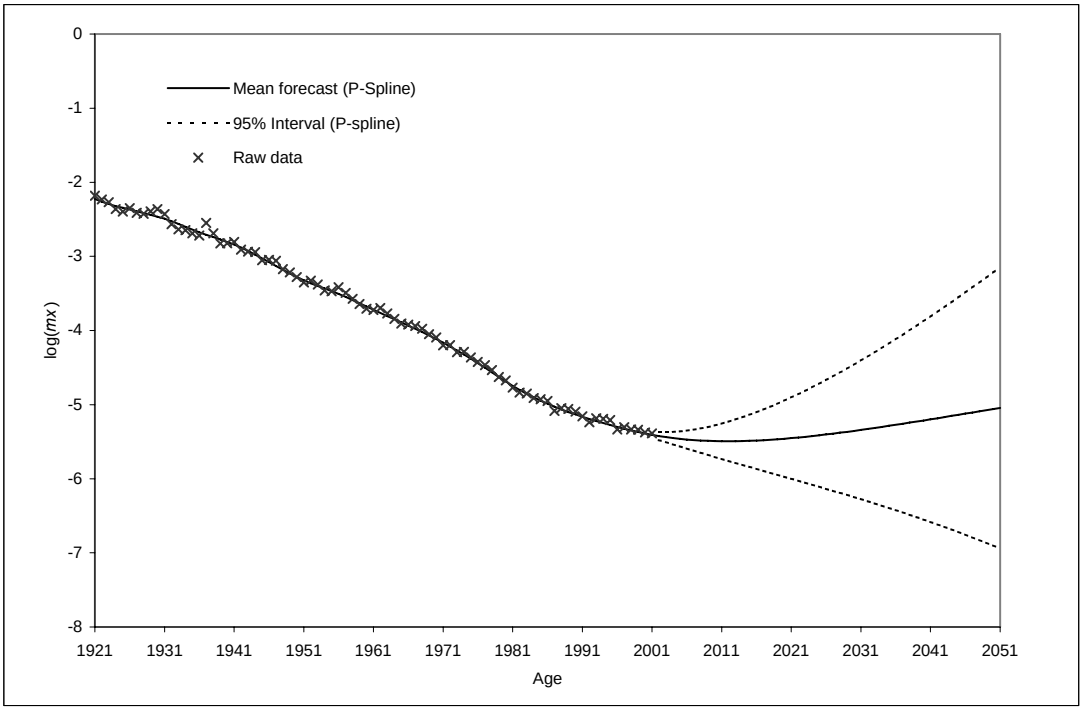


Age 80

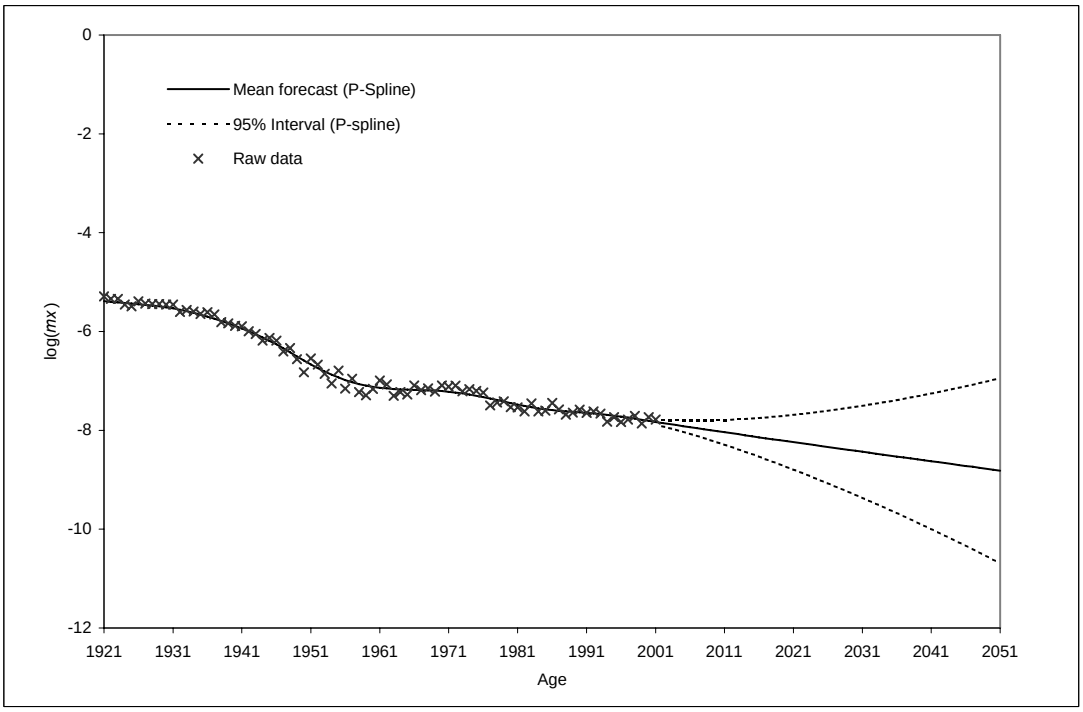


Age 90

Fig 3.4 (cont'd). The fitted (1921 – 2001) and projected (2002 – 2051) mortality experience using *P*-splines regression, male.

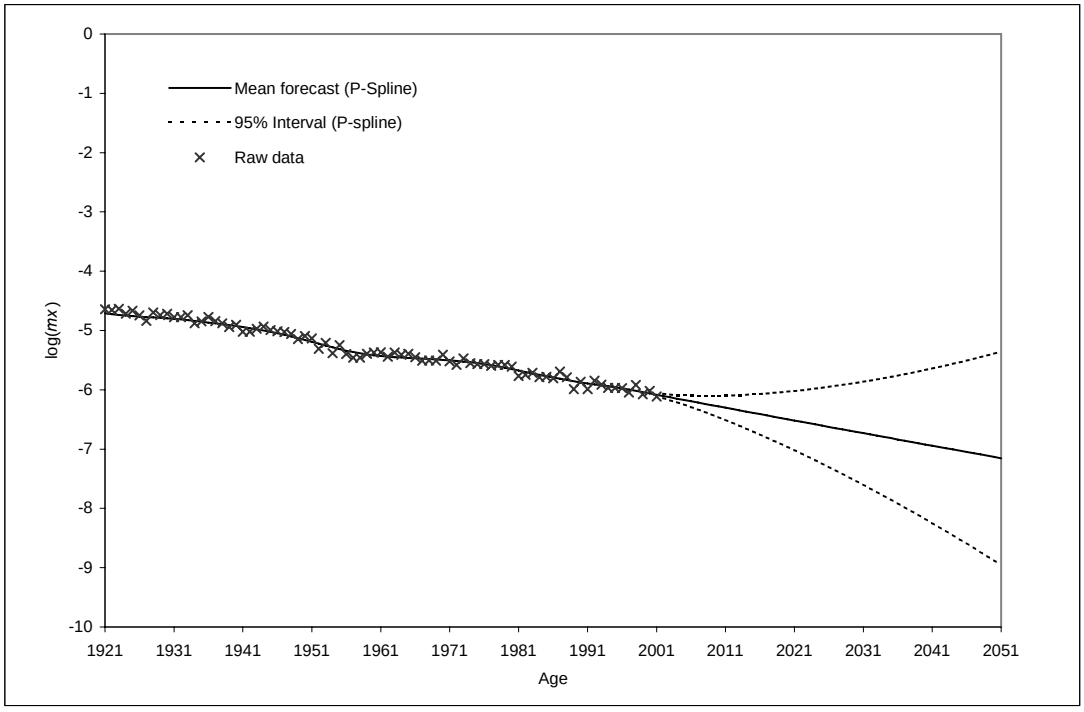


Age 0

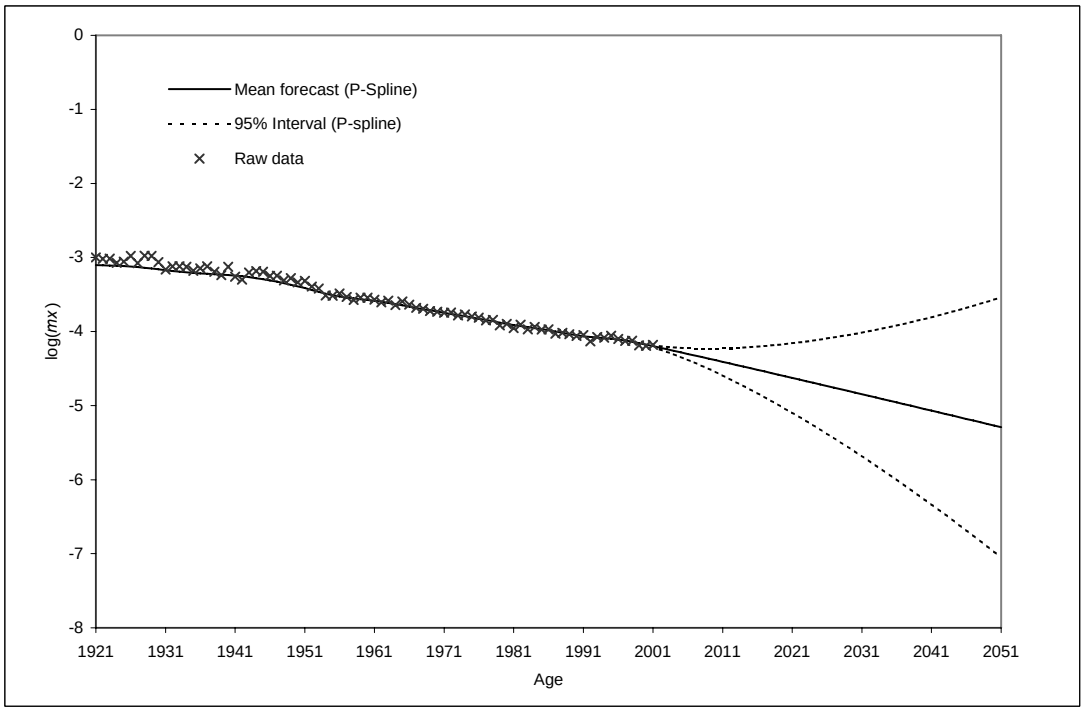


Age 30

Fig 3.5. The fitted (1921 – 2001) and projected (2002 – 2051) mortality experience using *P*-splines regression, female.

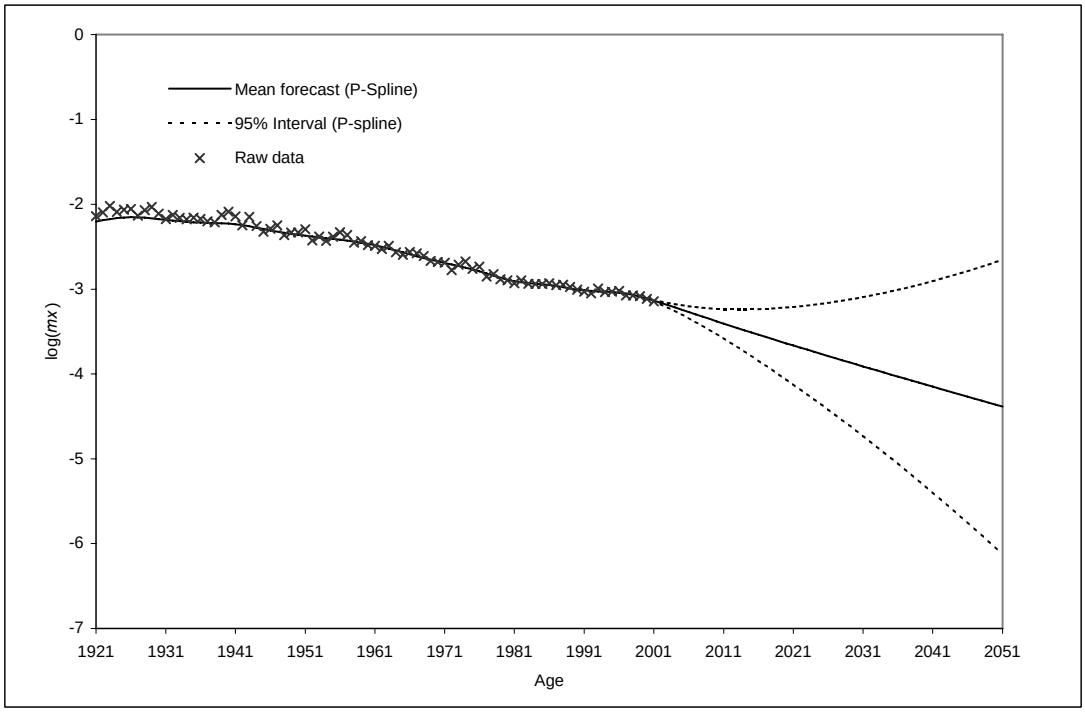


Age 50

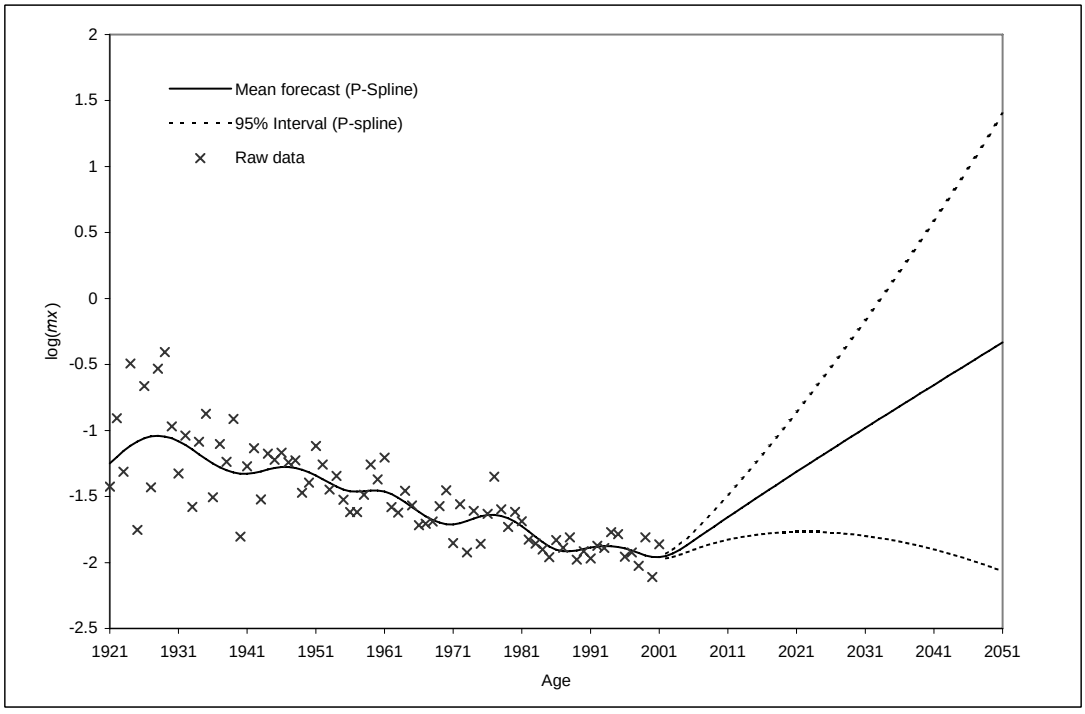


Age 70

Fig 3.5 (cont'd). The fitted (1921 – 2001) and projected (2002 – 2051) mortality experience using *P*-splines regression, female.

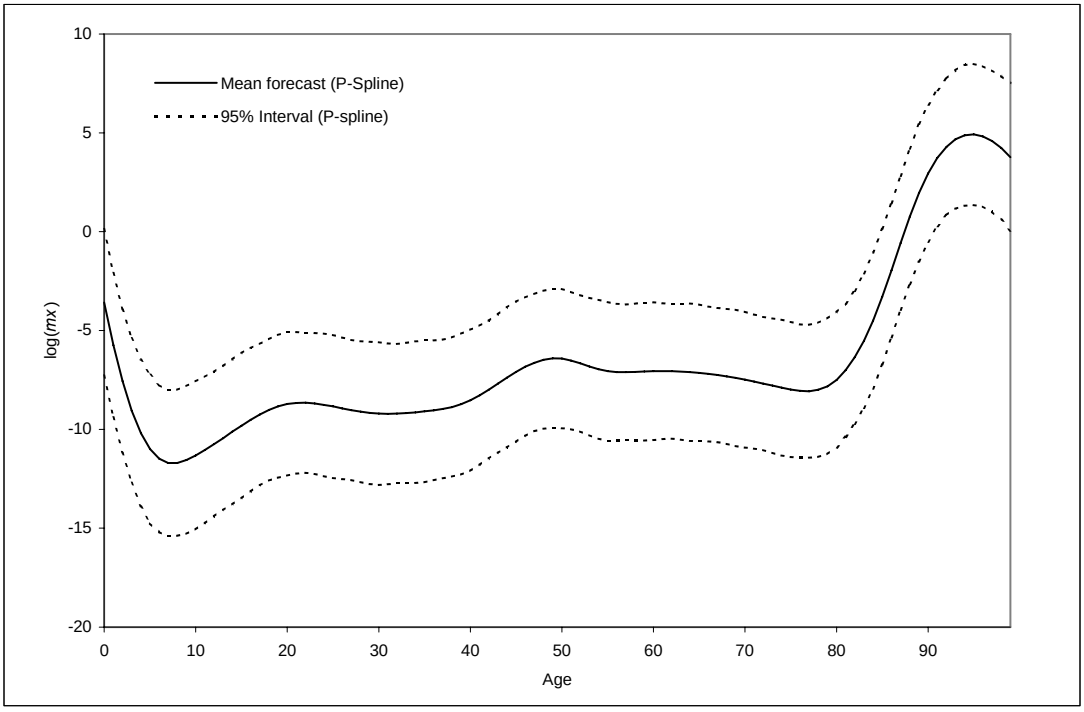


Age 80

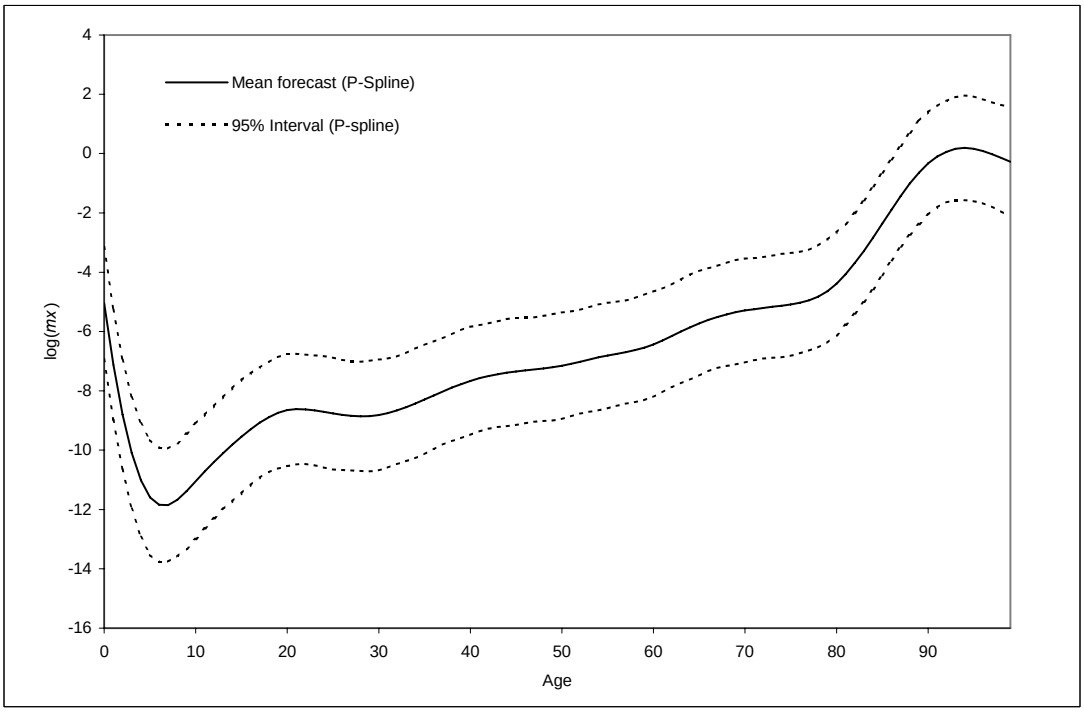


Age 90

Fig 3.5 (cont'd). The fitted (1921 – 2001) and projected (2002 – 2051) mortality experience using *P*-splines regression, female.



(a)



(b)

Fig. 3.6. The age pattern of mortality projected by the *P*-splines regression, (a) male and (b) female, 2051.

3.2 The Lee-Carter Model

The Lee-Carter model has been extensively used by actuaries for a wide range of purposes, ranging from the assessment of retirement income adequacy (Chia and Tsui, 2003) to the projection of mortality patterns at very old ages (Buettner, 2002). Tabeau (2001) provides a comprehensive review of the model.

The model describes the logarithmically transformed age-specific central rate of death ($m_{x,t}$)[1] as a sum of an age-specific component (a_x) that is independent of time, and the product of a time-varying parameter (k_t, also known as the mortality index) that summarizes the general level of mortality and an additional age-specific component (b_x) that represents how rapidly or slowly mortality at each age varies when the mortality index changes. Mathematically,

$$\ln(m_{x,t}) = a_x + b_x k_t + \varepsilon_{x,t} . \qquad (3.16)$$

The final term, $\varepsilon_{x,t}$, is the error term that captures all the remaining variations. It is known that equation (3.16) is over-parameterized. To stipulate a unique solution, a_x is usually taken as the arithmetic mean of the $\ln(m_{x,t})$ over time, while the sums of b_x and k_t are normalized to unity and zero respectively.

Mortality forecasting in the Lee-Carter framework consists of two phases. Historical mortality data firstly determines parameters a_x, b_x and k_t. After that, the fitted values of k_t are further modeled by an autoregressive integrated moving average (ARIMA) time-series model, whose specification is determined by the orthodox Box and Jenkins (1976) approach. An order of (0,1,0), equivalent to a random walk with drift, usually gives a good fit, although sometimes an additional AR or MA term may give a slightly better one. The extrapolation of k_t through the fitted ARIMA model gives the forecast of future central rates of death.

Note that as all parameters on the right hand side of equation (3.16) are unobservable, fitting by the ordinary least squares method would be impossible. This can be resolved by using either the singular value decomposition (SVD) method proposed by (Lee and Carter, 1992a), or the maximum likelihood estimation (MLE) method suggested by Wilmoth (1993) and Brouhns et al. (2002). As the differences in their assumptions

[1] In some literatures, the model is defined in terms of the force of mortality ($\mu_{x,t}$). Nevertheless, under the assumption of constant force of mortality in fractional age, the central rate of death and the force of mortality are equivalent (see Bowers et al., 1997).

often lead to dissimilar results, an in-depth understanding of both methods is important. Below we give detailed descriptions of both methods.

- *Singular Value Decomposition*

  The SVD method involves two stages. In the first stage, we set

$$a_x = \frac{1}{T}\sum_{t=0}^{T} \ln\left(m_{x,t}\right), \tag{3.17}$$

  where T is the length of the time-series. Then, we apply SVD to the matrix of $\mathbf{X} = \{\ln(m_{x,t}) - a_x\}$, which breaks $\mathbf{X}$ down into a product of three matrices, denoted, $\mathbf{U}$, $\mathbf{S}$ and $\mathbf{V}'$. The columns of $\mathbf{U}$ are known as the left singular vectors and that of $\mathbf{V}$ are known as the right singular vectors. $\mathbf{S}$ is diagonal matrix and its elements are known as singular values. The first left singular vector gives an estimate of b_x and the first right singular vector gives an estimate of k_t. The first singular value is distributed to b_x and k_t so that they sum to one and zero respectively.

  As the SVD method is merely a mathematical approximation, the estimated number of deaths computed from the model fitted by SVD can be very different from the observed number of deaths. Lee and Carter (1992) suggested an *ad hoc* second stage estimation. In this stage, the fitted values of k_t are re-estimated by solving for k_t such that

$$D_t = \sum_x \{\exp(a_x + b_x k_t) E_{x,t}\}, \tag{3.18}$$

  where D_t is the total number of deaths in time t. This is to ensure that the mortality schedules fitted over the sample years will reconcile the total number of deaths and the population age distributions.

  Suppose that the model is fitted to the historical data and an appropriate ARIMA model is estimated for the fitted values of k_t. Let $\hat{k}_{t+s}$ be the s-period ahead forecast of parameter k_t. Then the s-period ahead forecast of the logarithm of the central rate of death rate is given by

$$\hat{\ln}(m_{x,T+s}) = \hat{a}_x + \hat{k}_{T+s}\,\hat{b}_x, \tag{3.19}$$

  Assuming that the model specification is correct, the true value of $\ln(m_{x,t+s})$ can be expressed as

$$\ln(m_{x,T+s}) = (\hat{a}_x + \alpha_x) + (\hat{k}_{T+s} + u_{T+s})(\hat{b}_x + \beta_x) + \varepsilon_{x,T+s}, \qquad (3.20)$$

where α_x and β_x are the errors in estimating a_x and b_x, u_{T+s} is the error in the s-period ahead forecast of k_t, and $\varepsilon_{x,T+s}$ is all the remaining variations. Combining equations (3.19) and (3.20) gives an expression for the forecast error of the logarithm of the central rates of death:

$$E_{x,t+s} = \alpha_x + \varepsilon_{x,T+s} + (\hat{b}_x + \beta_x)u_{T+s} + \beta_x \hat{k}_{T+s}. \qquad (3.21)$$

In the computation of the interval forecast, we have to assume that the parameters are independent of each other. Under this assumption, the variance of the forecast error will be

$$\sigma^2_{E,x,T+s} = \sigma^2_{\alpha,x} + \sigma^2_{\varepsilon,x,T+s} + \left(\hat{b}_x^2 + \sigma^2_{\beta,x}\right)\sigma^2_{k,T+s} + \sigma^2_{\beta,x}\hat{k}^2_{T+s}, \qquad (3.22)$$

where $\sigma^2_{\alpha,x}$, $\sigma^2_{\beta,x}$ are the error variance of $\hat{a}_x$ and $\hat{b}_x$ respectively; $\sigma^2_{\varepsilon,x,T+s}$ is the variance of $\varepsilon_{x,T+s}$; and $\sigma^2_{k,T+s}$ is the variance of the error in the s-step ahead forecast of k_t. The estimate of $\sigma^2_{\alpha,x}$ is the variance of $\ln(m_{x,t})$ over time divided by T, and that of $\sigma^2_{\varepsilon,x,T+s}$ is the variance of the error in fitting age group x. The estimation of $\sigma^2_{\beta,x}$ requires bootstrapping (see Lee and Carter, 1992). The form of $\sigma^2_{k,T+s}$ depends on the order of the fitted ARIMA model. However, the numerical values of $\sigma^2_{k,T+s}$'s can be easily obtained from standard statistical software, such as SAS.

Assuming further that normality holds, the approximate 95% point-wise interval forecast of $\ln(m_{x,t})$ can be written as

$$\ln\left(m_{x,T+s}\right) = \ln\left(\hat{m}_{x,T+s}\right) + 1.96\sigma_{E,x,T+s}. \qquad (3.23)$$

- *Maximum Likelihood Estimation*

In MLE, the forecaster is required to specify a probability distribution for the number of deaths. Brouhns et al. (2002) assumed that the observed number of deaths is a realization of the Poisson distribution with mean equal to the expected number of deaths under the Lee-Carter model. Mathematically,

$$D_{x,t} \sim \text{Poisson}\left(E_{x,t}\exp\left(a_x + b_x k_t\right)\right). \qquad (3.24)$$

Under equation (3.24), we can determine parameters a_x, b_x and k_t by maximizing likelihood function, which is given by

$$L(\mathbf{a},\mathbf{b},\mathbf{k}) = \prod_{x,t} \frac{\left(N_{x,t}\exp(a_x + b_x k_t)\right)^{D_{x,t}} \exp\left(-N_{x,t}\exp(a_x + b_x k_t)\right)}{D_{x,t}!}, \qquad (3.25)$$

where **a**, **b** and **k** are vectors of the parameters a_x, b_x and k_t. The maximization can be done by Newton's method., which contains the iterative steps shown below.

$$\hat{a}_x^{(v+1)} = \hat{a}_x^{(v)} - \frac{\sum_t \left(D_{x,t} - \hat{D}_{x,t}^{(v)}\right)}{-\sum_t \hat{D}_{x,t}^{(v)}}, \quad \hat{b}_x^{(v+1)} = \hat{b}_x^{(v)}, \quad \hat{k}_t^{(v+1)} = \hat{k}_t^{(v)}; \qquad (3.26)$$

$$\hat{k}_t^{(v+2)} = \hat{k}_t^{(v+1)} - \frac{\sum_x \left(D_{x,t} - \hat{D}_{x,t}^{(v+1)}\right)\hat{b}_x^{(v+1)}}{-\sum_x \hat{D}_{x,t}^{(v+1)}\left(\hat{b}_x^{(v+1)}\right)^2}, \quad \hat{a}_x^{(v+2)} = \hat{a}_x^{(v+1)}, \quad \hat{b}_x^{(v+2)} = \hat{b}_x^{(v+1)}; \qquad (3.27)$$

$$\hat{b}_x^{(v+3)} = \hat{b}_x^{(v+2)} - \frac{\sum_t \left(D_{x,t} - \hat{D}_{x,t}^{(v+2)}\right)k_t^{(v+2)}}{-\sum_t \hat{D}_{x,t}^{(v+2)}\left(k_t^{(v+2)}\right)^2}, \hat{a}_x^{(v+3)} = \hat{a}_x^{(v+2)}, \quad \hat{k}_t^{(v+2)} = \hat{k}_t^{(v+1)}; \qquad (3.28)$$

where $\hat{a}_x^{(v)}$, $\hat{b}_x^{(v)}$ and $\hat{k}_t^{(v)}$ are the estimate of a_x, b_x and k_t in step v and

$$\hat{D}_{x,t}^{(v)} = E_{x,t}\exp\left(\hat{a}_x^{(v)} + \hat{b}_x^{(v)}\hat{k}_t^{(v)}\right). \qquad (3.29)$$

At each update, we impose respectively the scaling constraint and the location constraint to $\hat{b}_x^{(v)}$ and $\hat{k}_t^{(v)}$ for parameter uniqueness. The starting values $\hat{a}_x^{(0)}$, $\hat{b}_x^{(0)}$ and $\hat{k}_t^{(0)}$ are arbitrary, although a faster convergence can be achieved if the SVD estimates are used. The iteration stops when the change in the logarithm of equation (3.25) is less than 10^{-6}. Note that equation (3.26) implies the equality between the observed number deaths and the fitted number of deaths on convergence, and hence, the *ad hoc* second stage estimation involved in the SVD method can be

circumvented.

Having estimated the model parameters, we follow the usual procedure to project future values of k_t through a properly identified ARIMA model. The s-period ahead forecast of the logarithm of the central rate of death rate is again given by equation (3.19). Brouhns et al. (2005) proposed computing the interval forecast by the parametric bootstrap method, which can be summarized as follows.

i. Simulate N realizations from the Poisson distribution with mean equal to the fitted number of deaths under the Lee-Carter model. We use the transformed rejection method for generating Poisson random variables (Hörmann, 1993).

ii. For each of the N realizations:
   (1) Re-estimate the model parameters a_x, b_x and k_t using MLE.
   (2) Specify a new ARIMA model for the re-estimated k_t. For simplicity, we keep the ARIMA order constant and change only the parameter values.
   (3) Simulate future values of k_t under the newly specified ARIMA model.
   (4) Compute future values of central rates of death using the re-estimated a_x and b_x, and the simulated future values of k_t under the re-specified ARIMA model.

iii. Step (ii) gives an empirical distribution of $\ln(m_{x,t+s})$ for all x. The 2.5^{th} and the 97.5^{th} percentiles of the empirical distribution respectively yield the lower and upper limit of the 95% interval forecast of $\ln(m_{x,t+s})$.

The above algorithm allows both the sampling fluctuation in the model parameters and stochastic error in the forecast of k_t be included in the prediction interval. Also, Step (iii) implies that an asymmetric confidence interval is possible.

Figures 3.7 and 3.8 contrast the two methods of fitting the Lee-Carter model. At most of the selected ages, the mean forecasts are very similar. Nonetheless, in some cases, for example males at age 30 and females at age 80, the fitted values obtained by MLE seemingly better capture the pattern of the historical death rates. This might be due to the avoidance of the *ad hoc* second stage estimation required in SVD. In all cases, the width of the interval forecasts are approximately equal, since the method of fitting does not alter the number of effective parameters, which means the same degree of freedom is given to the forecast.

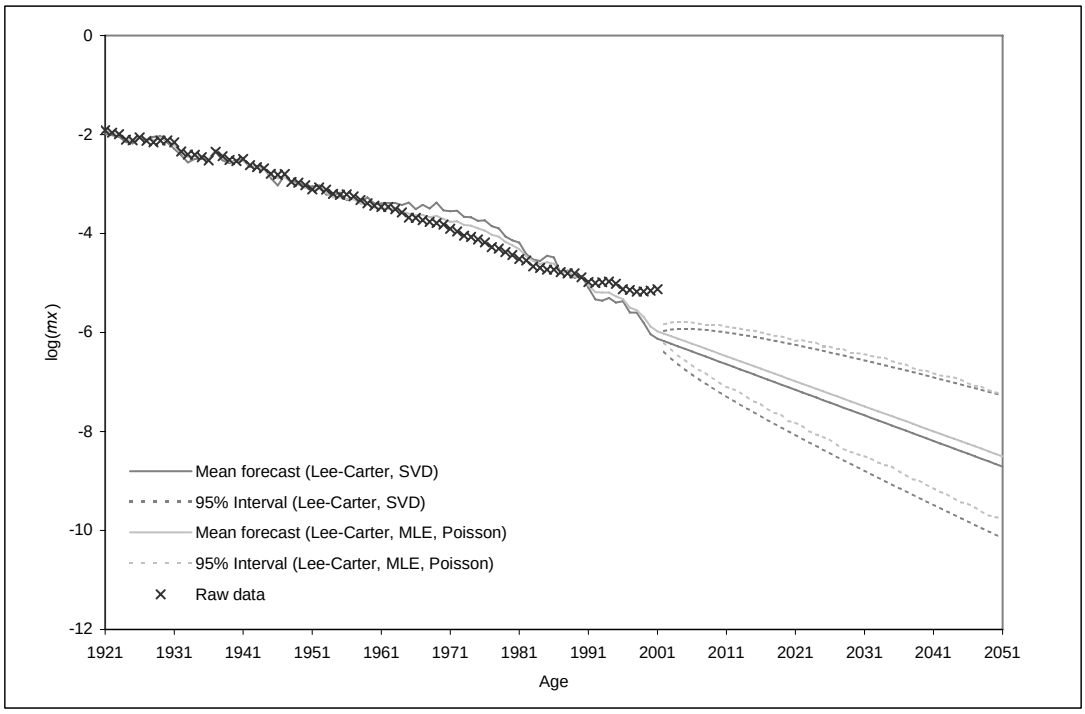
Occasionally, (for example, see the males at age 0) the fit near the forecast origin is imperfect. This is primarily due to the rigid model structure, which does not allow the

model to fit every part of the historical data perfectly. On the other hand, the rigid model structure keeps the principal of parsimony firm, and consequently, the forecasts are less vulnerable to the abnormalities near the forecast origin. In addition, as the Lee-Carter is heavily time-series based, the weights on previous observations have explicitly taken into account the autocorrelations in the historical rates, and this avoids an over-emphasis on the very recent observations.

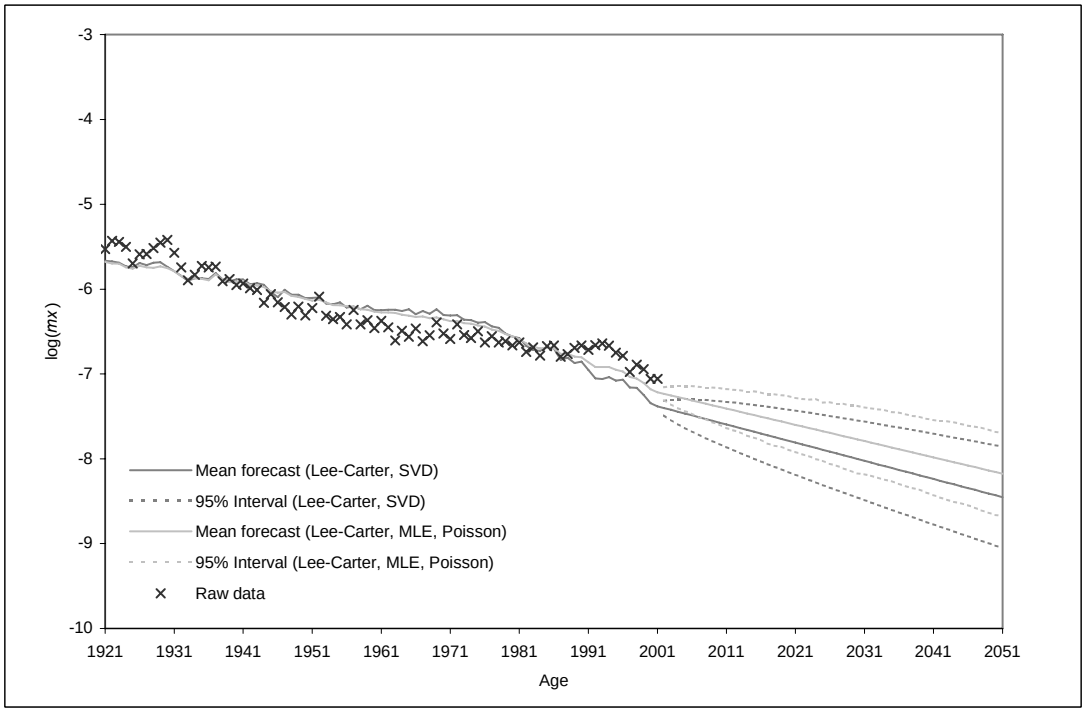
Figure 3.9 shows that the Lee-Carter model gives reasonable projections of future age pattern of mortality, which could be again attributed to the model structure. In Lee-Carter, mortality improvement is assumed to be additive (in log terms) to the fundamental age pattern of mortality, specified by parameters a_x's, which are assumed to be time invariant. As a result, even though mortality rates at different ages are allowed to improve at various paces, the age pattern in the far future could not be too much distorted.

Though the rigid model structure gives the above appealing features, it might overly restrict the behavior of the future rates and hence might give confidence intervals that are too narrow to reflect the true level of uncertainty. In Section 3.4, we propose a relaxed version of the Lee-Carter, hoping that the relaxation could give wider confidence intervals for more possible outcomes, and at the same time could keep all the nice properties in the original version.

Finally, in fitting the model, we might have smoothed out several discrepant observations that might arise from non-repetitive exogenous interventions, such as pandemics or wars. In Section 3.3, we conduct an outlier analysis to discern how these events might have affected the mortality series, and to predict how the future death rates series will respond if these or similar interruptive events recur.

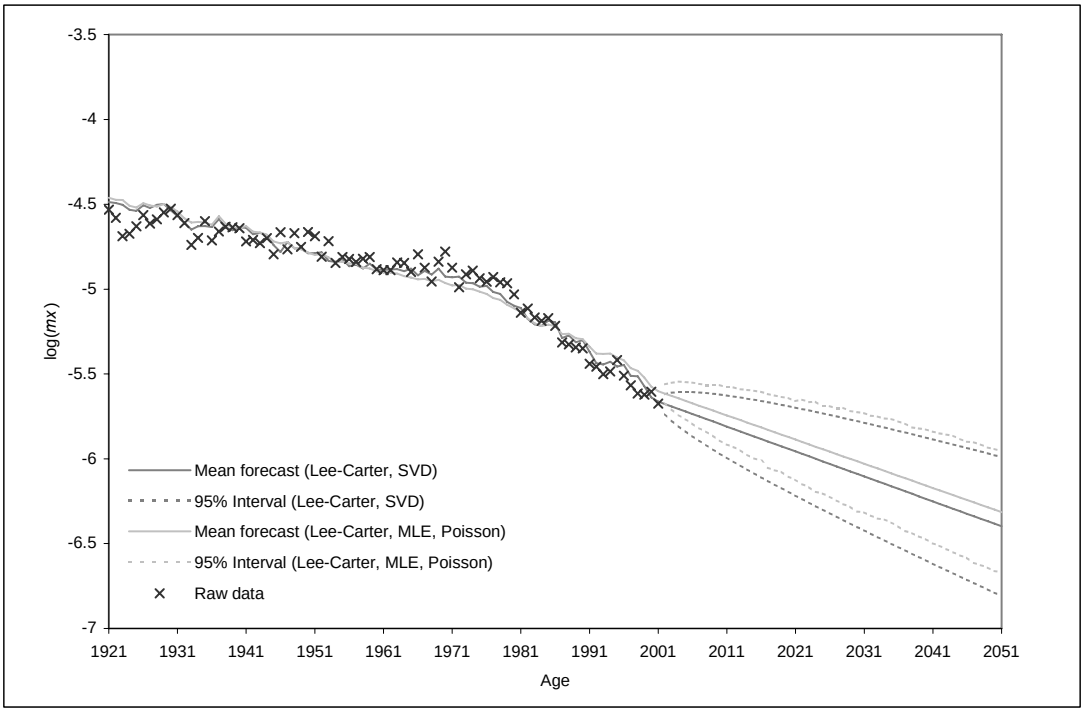


Age 0

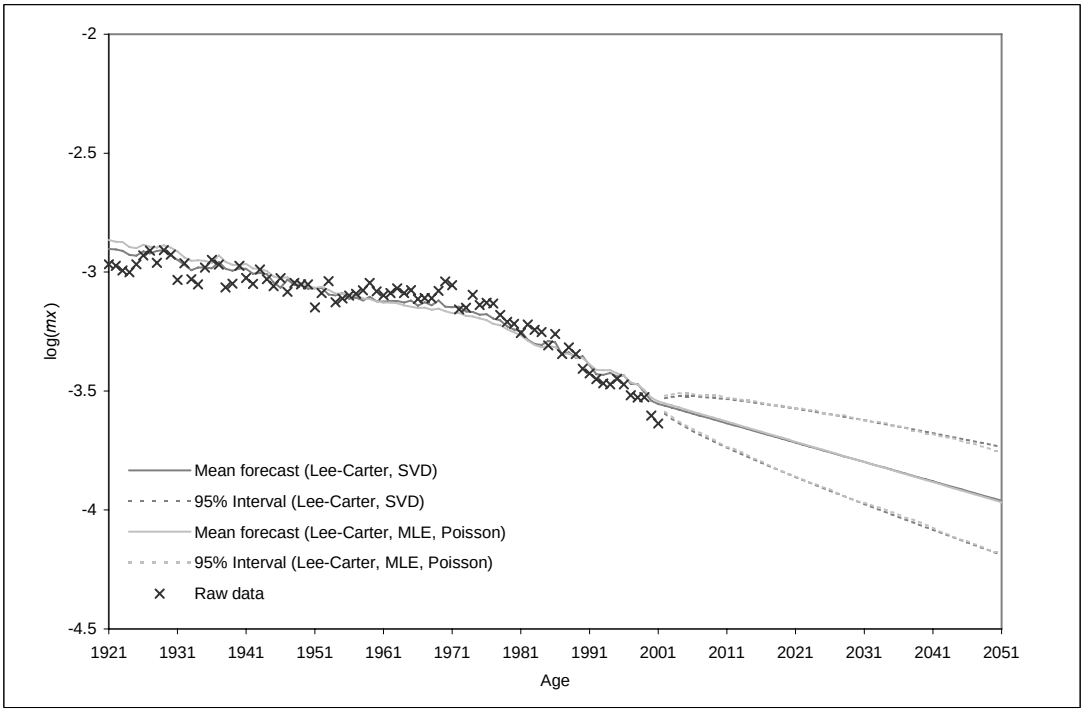


Age 30

Fig 3.7. The fitted (1921 – 2001) and projected (2002 – 2051) mortality experience using the Lee-Carter method, SVD and MLE with Poisson likelihood, male.

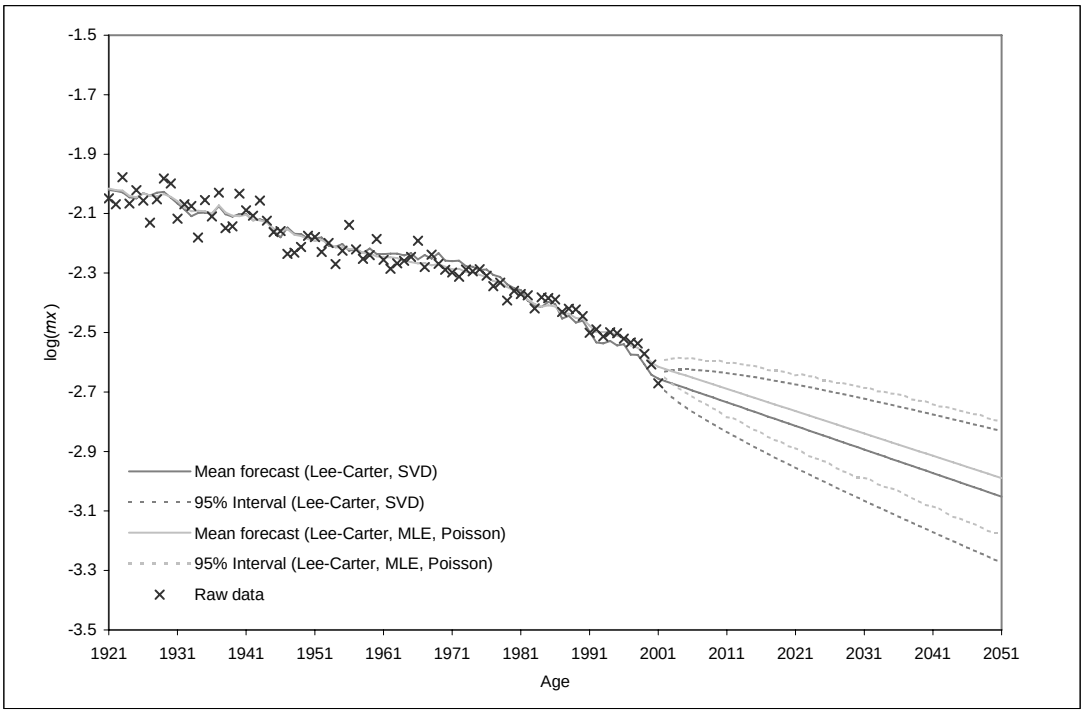


Age 50

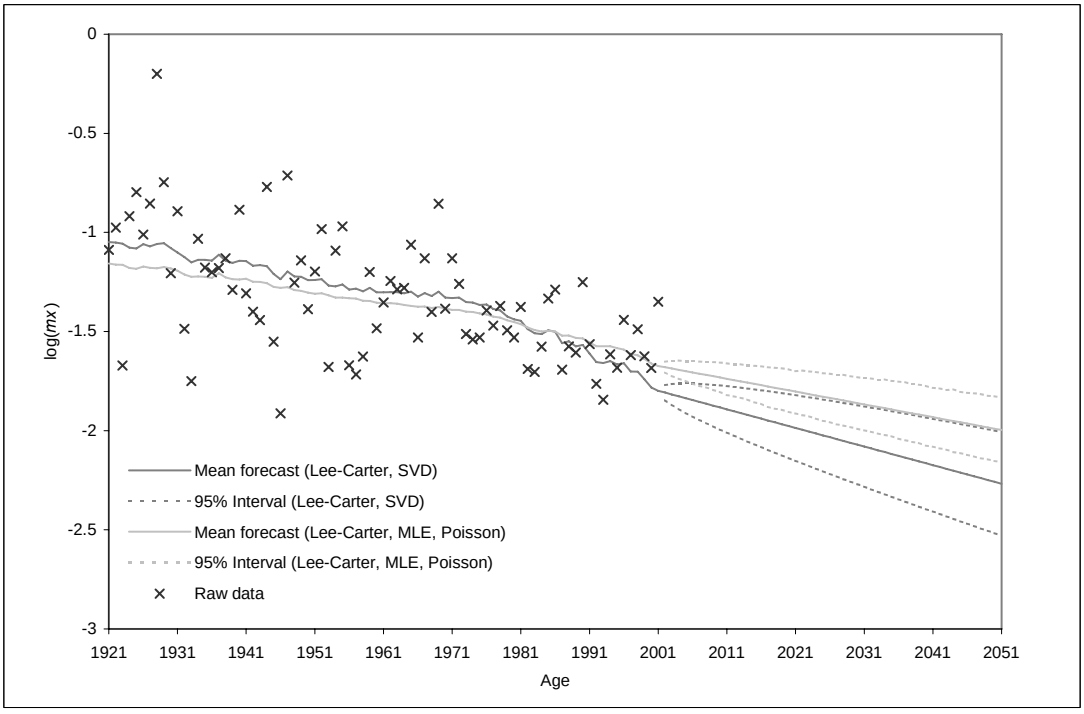


Age 70

Fig 3.7 (cont'd). The fitted (1921 – 2001) and projected (2002 – 2051) mortality experience using the Lee-Carter method, SVD and MLE with Poisson likelihood, male.

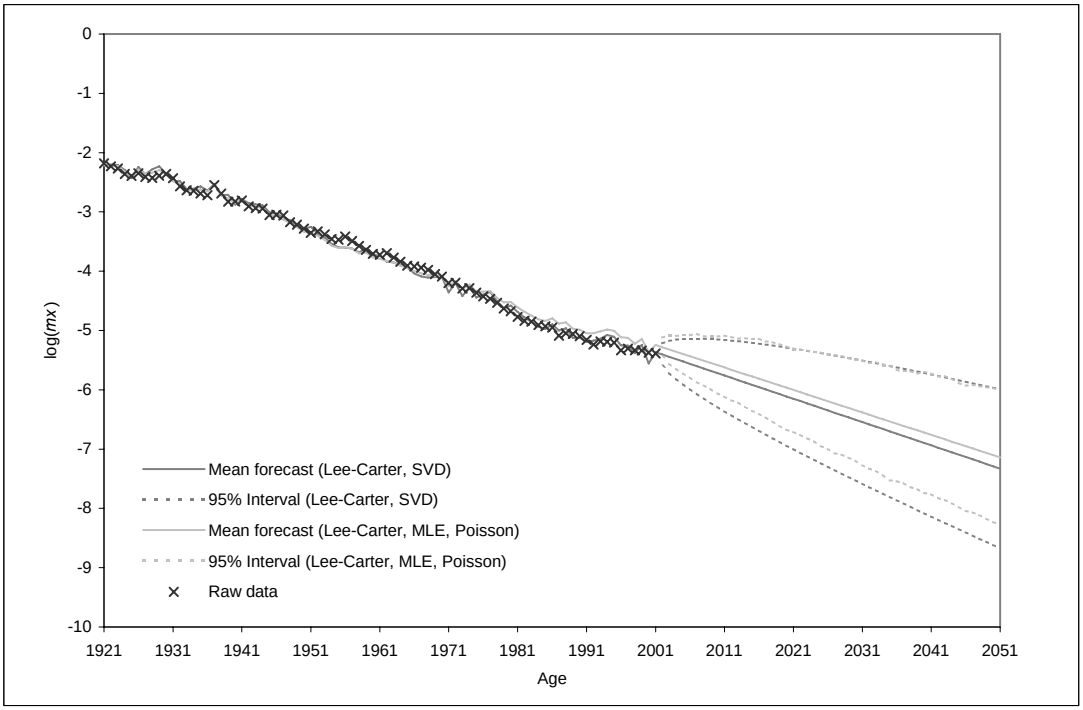


Age 80

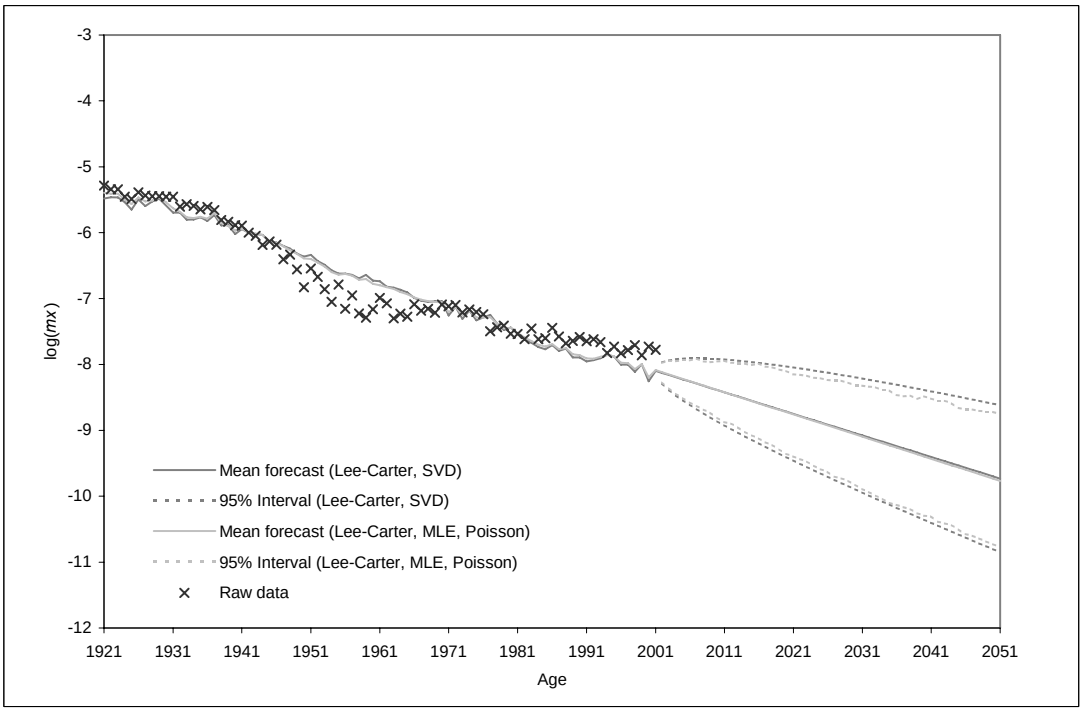


Age 90

Fig 3.7 (cont'd). The fitted (1921 – 2001) and projected (2002 – 2051) mortality experience using the Lee-Carter method, SVD and MLE with Poisson likelihood, male.

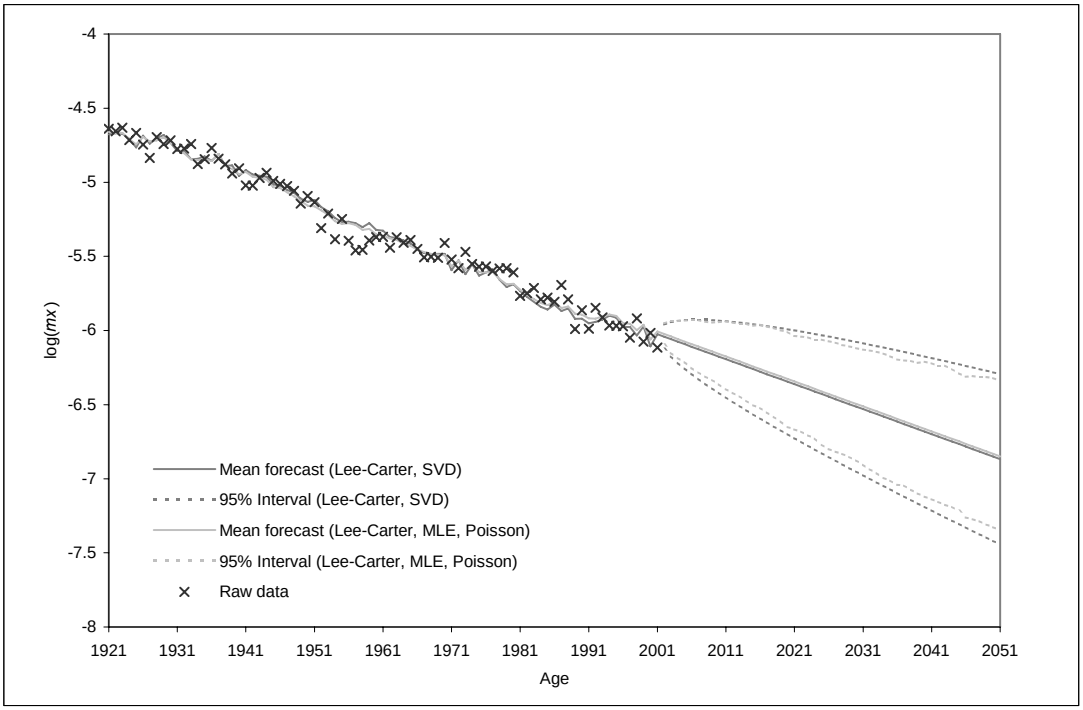


Age 0

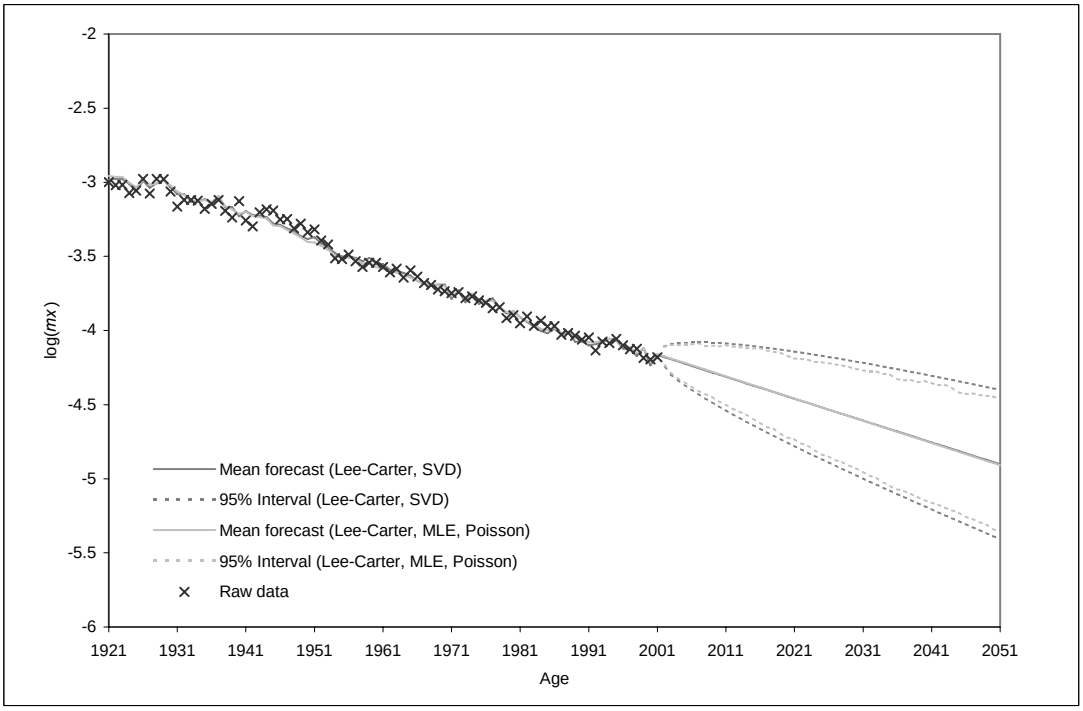


Age 30

Fig 3.8. The fitted (1921 – 2001) and projected (2002 – 2051) mortality experience using the Lee-Carter method, SVD and MLE with Poisson likelihood, female.

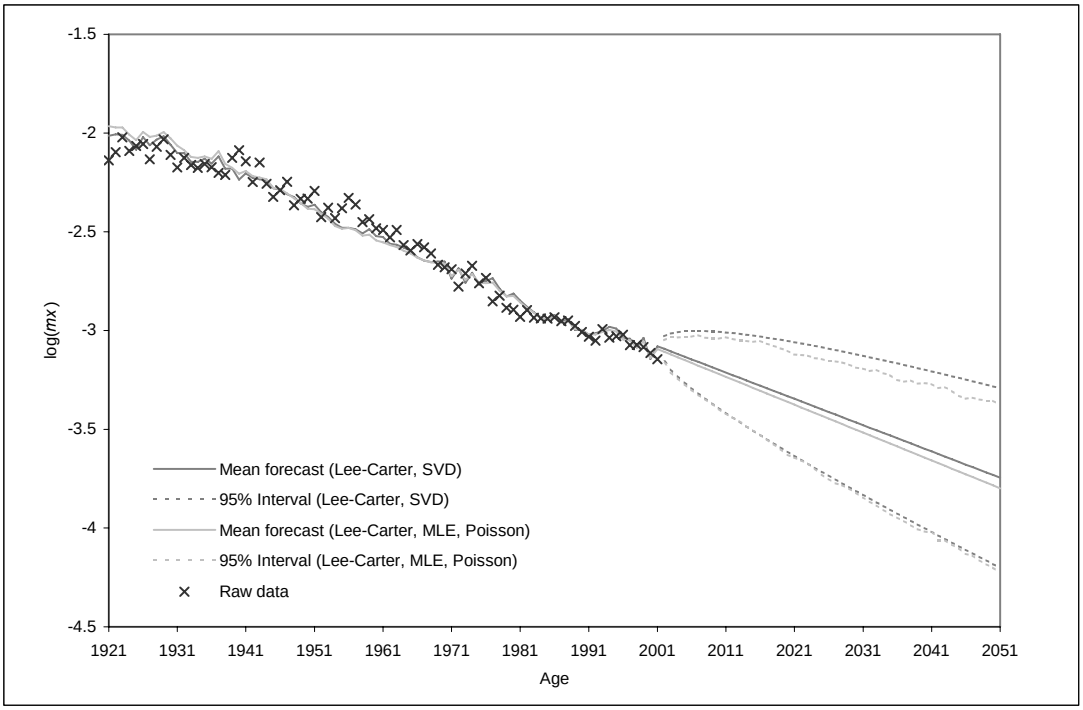


Age 50

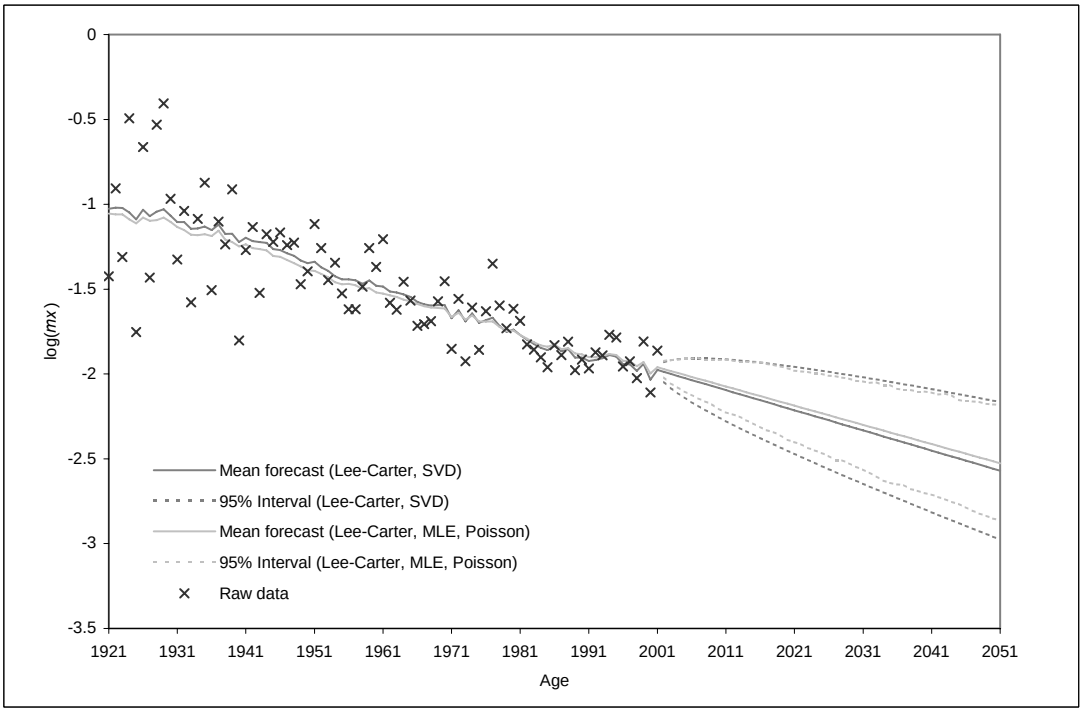


Age 70

Fig 3.8 (cont'd). The fitted (1921 – 2001) and projected (2002 – 2051) mortality experience using the Lee-Carter method, SVD and MLE with Poisson likelihood, female.

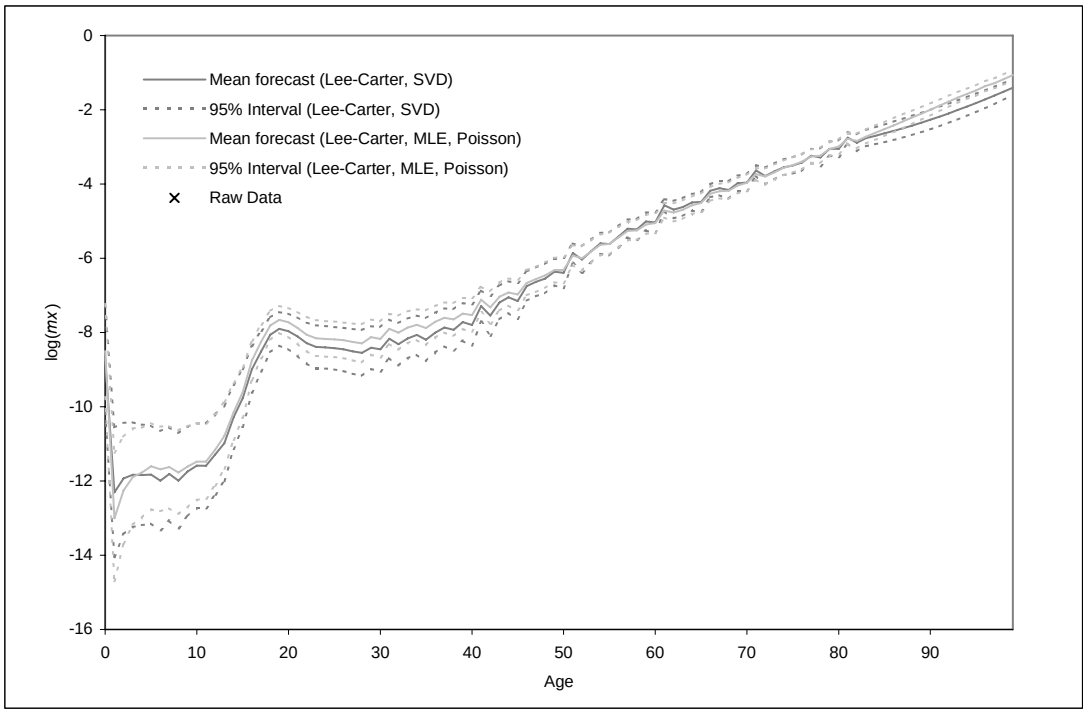


Age 80

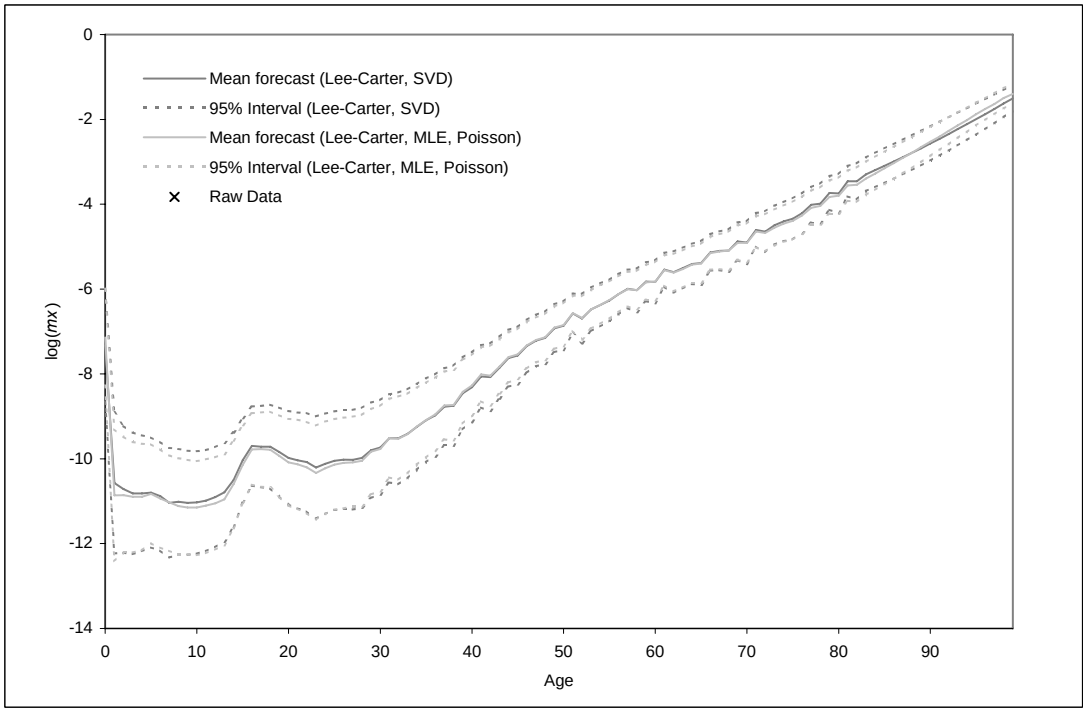


Age 90

Fig 3.8 (cont'd). The fitted (1921 – 2001) and projected (2002 – 2051) mortality experience using the Lee-Carter method, SVD and MLE with Poisson likelihood, female.



(a)



(b)

Fig. 3.9. The age pattern of mortality projected by Lee-Carter, SVD and MLE (Poisson), (a) male and (b) female, 2051.

3.3 EXTENSION OF LEE-CARTER (1): OUTLIER ANALYSIS

The objectives of the outlier analysis are (1) to verify the integrity of the mortality data by systematically examining the interruptive phenomena in the historical mortality series, particularly at time points where the coverage of the vital statistics were changed (see Section 2); (2) to discern how non-repetitive exogenous interventions, such as wars and pandemics, might have affected the mortality series, and to predict how the variability of the series might change if these or similar interruptive events recur; (3) to avoid possible erroneous model specifications due to the "masking effect" of the outliers (see Tsay, 1986).

To perform the outlier analysis, we firstly fit the Lee-Carter model to the historical data. The SVD method is used in the following illustration although the MLE method is also acceptable. Having fitted the model, we follow the Box-Jenkins approach to give the time-varying parameter, k_t, an appropriate ARIMA structure, which serves as the foundation of the outlier analysis. Let p, d and q be the order of the autoregressive polynomial, the number of differencing for stationary, and the order of the moving average polynomial respectively. Then, an ARIMA(p,d,q) model can be expressed as

$$\varphi(B)(1-B)^d k_t = \theta(B)a_t, \tag{3.30}$$

where k_t is the Lee-Carter time-varying parameter, assumed to be outlier free; B is the backshift operator such that $B^s k_t = k_{t-s}$,

$$\varphi(B) = 1 - \varphi_1 B - \ldots - \varphi_p B^p,$$
$$\theta(B) = 1 - \theta_1 B - \ldots - \theta_q B^q;$$

and $\{a_t\}$ is a sequence of white noise random variables, iid with mean 0 and constant variance σ^2.

An outlier-contaminated time-series k_t^* consists of an outlier-free time-series k_t and an exogenous intervention effect, denoted as $\Delta_t(T,\omega)$, i.e.,

$$k_t^* = k_t + \Delta_t(T,\omega), \tag{3.31}$$

where T is the location of the outlier and ω is the magnitude of the outlier.

We consider four common types of outliers, namely, additive outlier (AO), innovational outlier (IO), level shift (LS) and temporary change (TC).

- An AO affects only the level of a single observation, i.e.,

$$\Delta_t(T,\omega) = \omega D_t^{(T)}. \tag{3.32}$$

- An IO affects all observations beyond T through the memory of the underlying outlier-free process, i.e.,

$$\Delta_t(T,\omega) = \frac{\theta(B)}{\varphi(B)(1-B)^d}\omega D_t^{(T)}. \tag{3.33}$$

- A LS affects a series at a given time, and its effect is permanent, i.e.,

$$\Delta_t(T,\omega) = \frac{\omega}{1-B} D_t^{(T)}. \tag{3.34}$$

- A TC affects a series at a given time, and its effect decays exponentially according to a dampening factor, say δ, i.e.,

$$\Delta_t(T,\omega) = \frac{\omega}{1-\delta B} D_t^{(T)}. \tag{3.35}$$

  In practice, the value of δ often lies between 0.6 and 0.8 (Liu and Hudak, 1994, p.76). In this paper, we take $\delta = 0.7$ as recommended by Chen and Liu (1993).

In equations (3.32) – (3.35), $D_t^{(T)}$ is an indicator variable which equals 1 when $t = T$ and zero otherwise, representing the presence or absence of an outlier at time T. Figure 3.10 illustrates these outlier types graphically.

In general, the time-series of k_t may contain more than one, say m, outliers, and we have the following general time-series outlier model:

$$k_t^* = k_t + \sum_{i=1}^{m} \Delta_t(T_i, \omega_i). \tag{3.36}$$

Define a polynomial $\pi(B)$ as

$$\pi(B) = \frac{\varphi(B)(1-B)^d}{\theta(B)} = 1 - \pi_1 B - \pi_2 B^2 - \cdots. \tag{3.37}$$

Then equation (3.30) can be rewritten as

$$\pi(B)k_t = a_t, \tag{3.38}$$

which allows the fitted residuals $\hat{e}_t$ to be expressed in the form of

$$\hat{e_t} = \pi(B)Y_t. \tag{3.39}$$

By series expansion, we arrive at the following expressions of fitted residual for each outlier type.

$$\text{AO:} \qquad \hat{e}_t = \omega D_t^{(T)} + a_t, \tag{3.40}$$

$$\text{IO:} \qquad \hat{e}_t = \omega \pi(B) D_t^{(T)} + a_t, \tag{3.41}$$

$$\text{TC:} \qquad \hat{e}_t = \omega \frac{\pi(B)}{(1-\delta B)} D_t^{(T)} + a_t, \tag{3.42}$$

$$\text{LS:} \qquad \hat{e}_t = \omega \frac{\pi(B)}{1-B} D_t^{(T)} + a_t. \tag{3.43}$$

Alternatively, we can rewrite equations (3.40) – (3.43) as a general time-series regression, i.e.,

$$\hat{e}_t = \omega\, d(j,t) + a_t\,, \tag{3.44}$$

where $j \in J = \{\text{AO}, \text{IO}, \text{TC}, \text{LS}\}$; $d(j,t) = 0$ for all j and $t < T$; $d(j,T) = 1$ for all j; and for all $h \geq 1$,

$$d(\text{AO}, T+h) = 0,$$

$$d(\text{IO}, T+h) = -\pi_h\,,$$

$$d(\text{TC}, T+h) = \delta_h - \sum_{i=1}^{h-1} \delta^{h-j} \pi_j - \pi_h\,,$$

$$d(\text{LS}, T+h) = 1 - \sum_{i=1}^{h} \pi_i\,.$$

Hence, for a given T (suspected location of the outlier) and j (suspected type of outlier), the magnitude of the outlier effect ω and its corresponding standardized t-statistic, $\tau(j,T)$, can be readily computed using the principle of least squares. The final test statistic is the maximum value of this t-statistic over all possible T and j, i.e.,

$$T = \max_{1 \leq T \leq n} \max_{j \in J} \{\tau(j,T)\}\,. \tag{3.45}$$

For a given j, it follows approximately a normal distribution. An outlier of type j is detected if the final test statistic is greater than a critical value of C. We choose $C = 2.5$ as recommended by Liu and Hudak (1994) for a reasonable level of sensitivity.

For the detection, estimation, and adjustment of outliers we use the iterative procedures proposed by Chen and Liu (1993) and Li and Chan (2005a). Working together,

these procedures may avoid the problem of erroneous model specification. The key steps in the iterative cycles are as follows.

i. *Tentative model identification.* Use the Box-Jenkins approach to tentatively identify the order of the underlying outlier-free ARIMA(p,d,q) model.

ii. *Outlier detection.* Compute the test-statistic in equation (3.45), and record the type and location of the corresponding outlier. Then, compare the test statistic with the critical value C. If it is smaller than C, jump to (i), otherwise, proceed to (iii).

iii. *Outlier adjustment.* With the type and location of the identified outlier, re-estimate simultaneously the model parameters and the intervention model for the outlier effect. After the incorporation of the outlier effects, an outlier-adjusted data series is obtained.

iv. *Model Re-identification.* Using the Box-Jenkins approach, re-identify the ARIMA model underlying the outlier-adjusted data series obtained in (iii). If the re-identification makes a difference in p, d, and/or q, go back to (ii) using the original unadjusted data series under the re-identified ARIMA(p,d,q) model. Otherwise, terminate the iteration cycle, and the ultimate estimates of outliers and ARIMA model parameters are those obtained in the immediately previous (iii).

In this analysis, we found that the "masking effect" of outliers is minimal, since the ARIMA order of (0,1,0) remains unchanged throughout the above iterative procedures. Table 3.1 presents a synopsis of the outliers detected in k_t for both sexes. The positive outliers may be interpreted as unexpected mortality deteriorations, while the negative ones may be interpreted as abrupt mortality improvements. It is noteworthy that no outlier is detected at 1944, 1947, 1961, 1966 and 1971 when the coverage of the vital statistics was altered. Table 3.1 also suggests that interruptions in the mortality trend of Canadians from 1921 – 2001 are infrequent. In all cases, the magnitude and the t-values are small, indicating that the effect of the interruptions are relatively mild.

To learn more about how interruptive events in the early century might affect human mortality, we replicate the exercise using the US population mortality, which is available back to 1901. Table 3.2 shows all the outliers detected in the k_t's for the US population. The positive outlier in 1918 is a consequence of the Spanish flu epidemic, which infected about 28% of the US population and killed approximately 500,000 Americans. It is interesting to note that the additive nature of the outlier in 1918 might suggest that the Spanish flu, often regarded as the most deadly pandemic in the human history, affected only temporarily and could not halt the continual improvement of human mortality.

Readers interested in the explanation of other detected outliers may refer to Li and Chan (2005b).

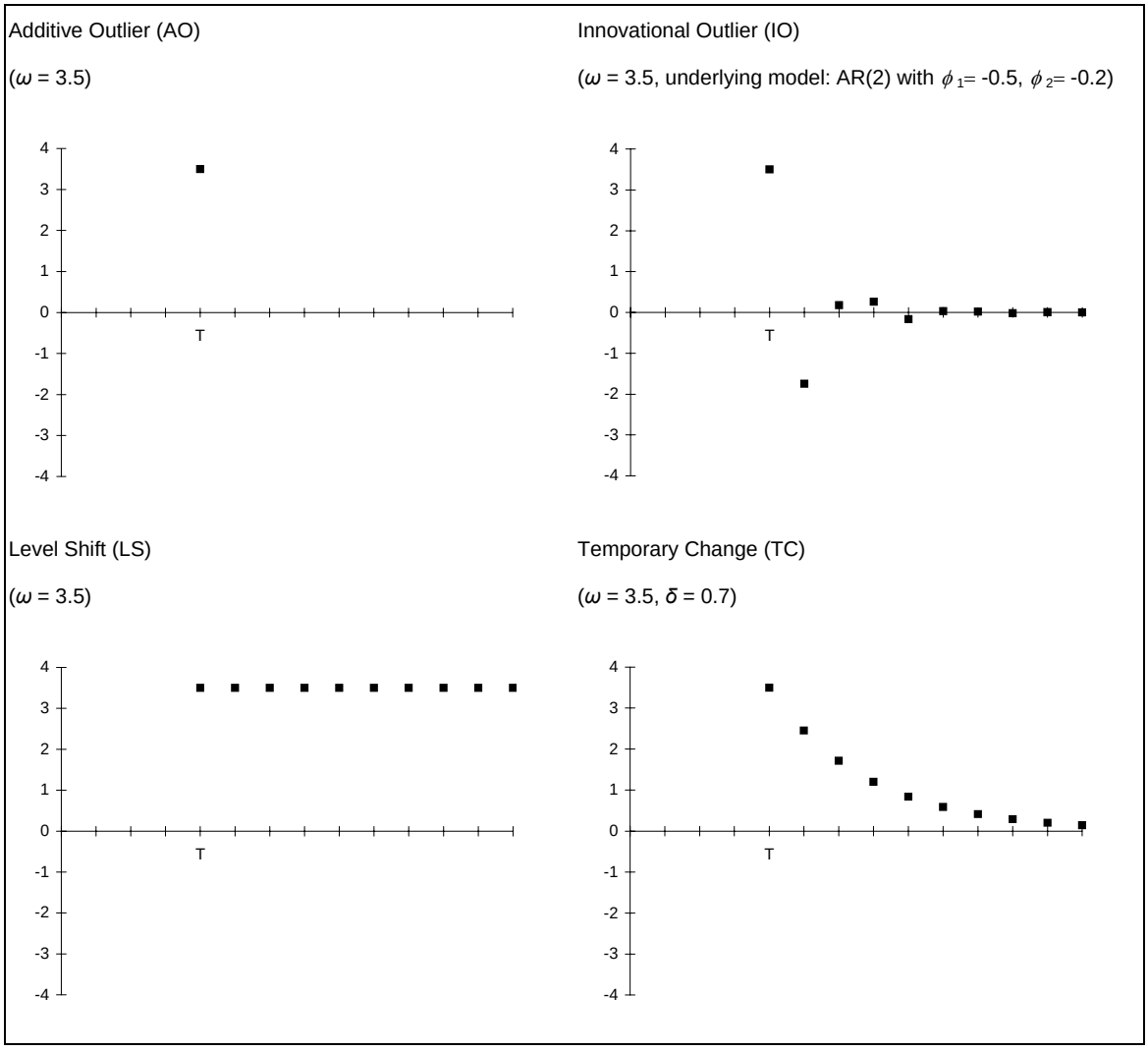


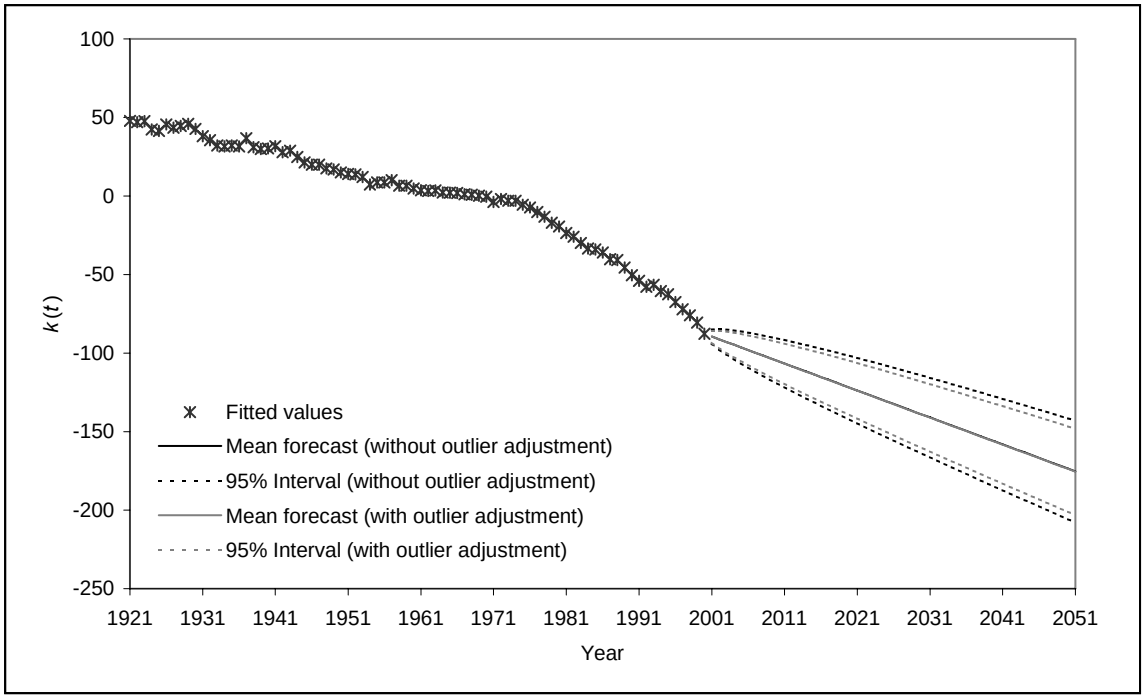
Fig. 3.10. Different types of time-series outlier.

Year	Magnitude	*t*-value	Type
Male			
1937	5.497	3.69	AO
Female			
1926	6.729	4.05	LS
1929	5.271	3.22	TC
1937	6.420	4.51	AO
1954	-4.940	-3.02	TC

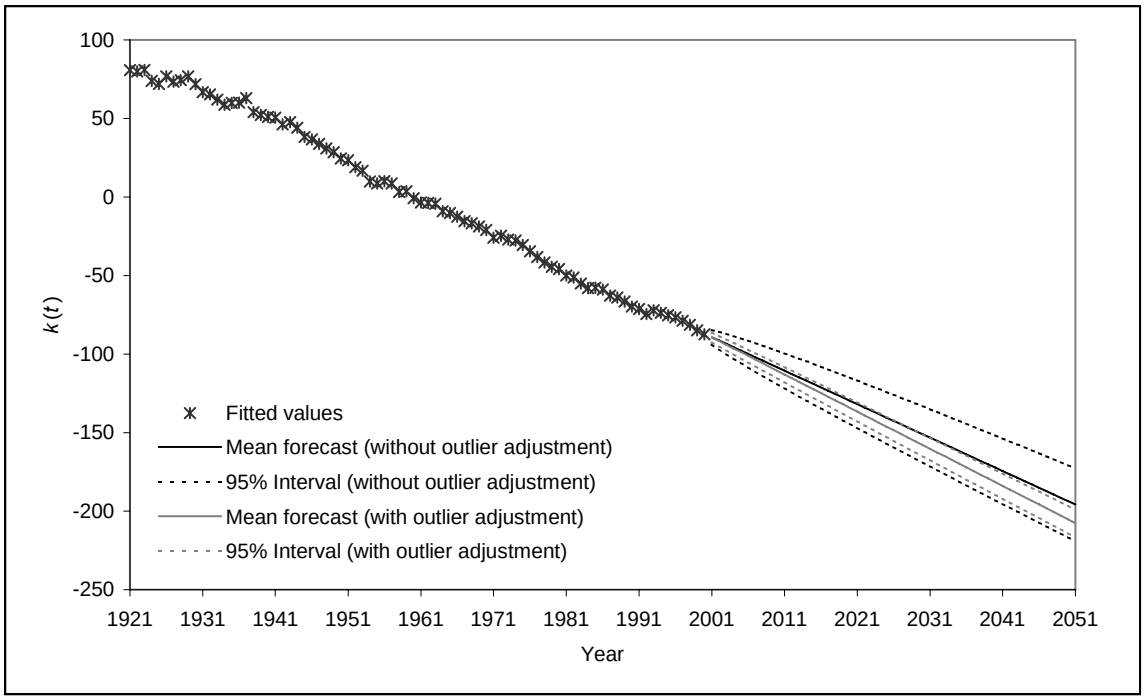
Table 3.1. Outliers detected in k_t, Canadian population, 1921-2001.

Year	Magnitude	*t*-value	Type
Male			
1918	29.444	11.67	AO
1921	-12.471	-3.88	TC
1927	-7.806	-3.09	AO
1936	9.815	3.05	TC
Female			
1916	10.117	3.48	LS
1918	30.374	11.87	AO
1921	-11.795	-4.07	TC
1926	7.610	2.94	AO
1927	10.630	3.62	TC
1936	11.531	4.00	TC
1954	-8.059	-2.79	TC
1975	-10.459	-3.61	LS

Table 3.2. Outliers detected in k_t, the US population, 1921-2001.



(a)



(b)

Fig. 3.11. Mean and 95% interval forecast of k_t, (a) male and (b) female. Confidence intervals are obtained by $\hat{k}_{t+s} \pm 1.96\sigma_{k,t+s}$.

Figure 3.11 plots the mean forecast and the 95% confidence interval of k_t's under both the original and the outlier adjusted model. The adjustment of outliers slightly deviates the mean forecasts, but substantially narrowed the width of the interval forecast. The reduction in width, however, could not imply higher precision. Li and Chan (2005a) suggested that the increased narrowness is in large part related to the optimistic nature in the outlier-adjusted forecast – The adjustment of outlier essentially presumes that the detected outliers are not to be recurred in the future. As a result, more detected outliers would mean more neglected variability in the interval forecast. This explains why the outlier adjusted interval for female is so narrow that its upper bound is even lower than the mean forecast under the original model.

The assumption of non-recurrence of outliers is undeniably not legitimate, since, as stated at the outset, the extreme stochastic fluctuations are important in the revision of mortality improvement scales, particularly in the context of insurance. Rather than using the outlier-adjusted forecast, we seek modifications so that the model is sympathetic to outliers in the data. Following Li and Chan (2005b), we explicitly allow the detected outliers to recur in the future by introducing the following routine in the parametric bootstrapping of the interval forecast.

> In each simulation, we sample with replacement from the pool of detected outliers. Then, we superimpose the sampled outliers to simulated sample path of future k_t.

The pool comprises all outliers in Table 3.1 and those before 1921 in Table 3.2. By allowing events like pandemics and wars to recur in the future, the confidence intervals could be wider and could include more stochastic uncertainty.

3.4 Extension of the Lee-Carter (2): MLE with Negative Binomial Likelihood

The confidence intervals plotted in some of the individual graphs in Figures 3.7 and 3.8 above appear too narrow compared with the historical experience. In Section 3.2, we pointed out that the rigid structure of the Lee-Carter model constrains the behavior of the future death rates so that a reasonable age pattern of mortality could be maintained far into the future. At the same time, however, the rigidity might too tightly bind the movements of future death rates, resulting an overly narrow interval forecast that might not reflect the true level of uncertainties. In other words, the narrow confidence intervals are more a result of the strong assumptions required of this model than a true feature of the data. The less parametric P-spline approach, while unsatisfactory in some ways, did

indicate huge forecast uncertainty.

In this section, we first explain how the assumptions made may understate the true confidence interval, and then weaken the assumptions slightly. This is done carefully so that a wider confidence interval can emerge, without too much distortion in the projected age mortality patterns.

The SVD method is a purely mathematical approximation. Values of k_t obtained from the first right singular vector often require further adjustments through an *ad hoc* procedure. The computation of the interval forecast is rather simple, but needs several very strong assumptions. First, even though an analytic formula for the variance of forecast error is available, for mathematical tractability, we have assumed independence between the model parameter estimates. This assumption could result in a significantly smaller variance than we would find if we allowed fully for dependence. Second, as the method specifies no probabilistic distribution during the fit, normality has to be assumed in the computation of the interval forecast. This is a very strong assumption; the normal distribution may well be too thin tailed, and also, of course, the assumption of normality completely rules out the possibility of an asymmetric interval forecast.

The original MLE method assumes that observed number of deaths is a realization of a Poisson distribution having the Lee-Carter expected number of deaths as its mean. In this setting, interval forecasts as well as parameter uncertainties can be evaluated by parametric bootstrapping, which is computationally intensive but allows us to model explicitly the statistical dependency between parameters. In addition, as the prediction intervals are given by the percentiles in the empirical distribution, they can be asymmetric.

While the MLE method has various competitive advantages, the assumption is very restrictive. In assuming Poisson models, we are assuming that the variance is equal to the mean; it cannot be separately estimated. If this assumption of mean variance equality is untrue (as we suspect), then we will effectively require the variance to be less than it really is, and so overestimate the precision of the forecasted future death rates. In the statistical literature, this situation is commonly referred to as overdispersion. McCullagh and Nelder (1989) pointed out that this is commonplace in the analysis of data using a single parameter family, such as the Poisson distribution. Cox (1983) stated the two major adverse effects of overdispersion. One is that the summary statistics have a larger variance than anticipated under the simpler model, and the other is a possible loss of efficiency in using statistics appropriate for the single parameter family.

To correct this undesirable phenomenon, we modify the original Lee-Carter by replacing the Poisson distribution by a Negative Binomial distribution as the stochastic structure of the number of deaths. This modification has two functions. First, as the two-parameter Negative Binomial distribution removes the equality restriction on the mean-variance relationship, so measures of uncertainties under this modification could capture a large part of the variation that is ignored in the original version. If so, we will obtain a wider interval forecast. Second, the assumption of Negative Binomial brings an additional parameter vector to the Lee-Carter model without altering the structure specified by equation (3.16). The additional parameters could give more flexibility to the model, and the unaltered structure could ensure that the nice properties in the original version, such as stability of age-pattern over time, are well kept.

We begin with the properties of the Negative Binomial distribution. Recall that the probability density function of a non-negative, integer-valued random variable, Y, taking on a Negative Binomial distribution is typically given as

$$\Pr[Y = y] = \binom{r+y-1}{y}\left(\frac{1}{1+\beta}\right)^{r}\left(\frac{\beta}{1+\beta}\right)^{y}, \tag{3.46}$$

for $r > 0$ and $\beta > 0$. Anscombe (1950) noted that there exists a finite probability of observing a data set from which the maximum likelihood estimate of r may not be calculated. A more useful parameterization may be obtained by letting $r = \alpha^{-1}$ and $\beta = \alpha\mu$, which gives

$$\Pr[Y = y] = \frac{\Gamma\left(y+\alpha^{-1}\right)}{y!\,\Gamma\left(\alpha^{-1}\right)}\left(\frac{\alpha\mu}{1+\alpha\mu}\right)^{y}\left(1+\alpha\mu\right)^{-1/\alpha}. \tag{3.47}$$

We denote this as $Y \sim \mathrm{NB}(\alpha, \mu)$. It can be easily shown that

$$\mathrm{E}(Y) = \mu, \tag{3.48}$$

$$\mathrm{Var}(Y) = \mu + \alpha\mu^2. \tag{3.49}$$

α is often known as the dispersion parameter. Notice that the limiting case $\alpha \to 0$ yields a Poisson distribution, and $\alpha > 0$ gives over-dispersion, i.e. $\mathrm{Var}(Y) \geq \mathrm{E}(Y)$.

In the application, we assume that the number of deaths follows a Negative Binomial distribution, and that the mean of the assumed distribution is the expected number of deaths under the Lee-Carter model. Mathematically,

$$D_{x,t} \sim NB\left(\alpha_x, E_{x,t} \exp\left(a_x + b_x k_t\right)\right), \tag{3.50}$$

where α_x denotes the dispersion parameter for age x.

Then, we may estimate the parameters maximizing the likelihood function, which is given by

$$L(\mathbf{a},\mathbf{b},\mathbf{k},\boldsymbol{\alpha}) = \prod_{x,t} \frac{\Gamma\left(D_{x,t} + \alpha_x^{-1}\right)\left(\alpha_x E_{x,t} \exp\left(a_x + b_x k_t\right)\right)^{D_{x,t}}}{D_{x,t}!\,\Gamma\left(\alpha_x^{-1}\right)\left(1 + \alpha_x E_{x,t} \exp\left(a_x + b_x k_t\right)\right)^{D_{x,t}+\alpha_x^{-1}}}, \tag{3.51}$$

where **a**, **b**, **k** and **α** are vectors of the parameters a_x, b_x, k_t and α_x; or equivalently, by the maximizing the log likelihood function, given by

$$l(\mathbf{a},\mathbf{b},\mathbf{k},\boldsymbol{\alpha}) = \sum_{x,t} \left\{ \begin{array}{l} \left[\sum_{i=0}^{D_{x,t}-1} \ln\left(\frac{1+\alpha_x i}{\alpha_x}\right)\right] + D_{x,t} \ln\left(\alpha_x E_{x,t} \exp\left(a_x + b_x k_t\right)\right) \\ -\left(D_{x,t} + \alpha_x^{-1}\right)\ln\left(1 + \alpha_x E_{x,t} \exp\left(a_x + b_x k_t\right)\right) \end{array} \right\} + c\,, \tag{3.52}$$

where c is a constant independent of a_x, b_x, k_t and α_x. Again, the maximization may be done by the Newton's method. At each update, we impose respectively the scaling constraint and the location constraint to $\hat{b}_x^{(v)}$ and $\hat{k}_t^{(v)}$ for parameter uniqueness. For faster convergence, the starting values $\hat{a}_x^{(0)}$, $\hat{b}_x^{(0)}$ and $\hat{k}_t^{(0)}$ may be taken as the estimates obtained from equations (3.24) – (3.26). The iteration stops when the change in equation (3.52) is sufficiently small, say 10^{-6}.

Figure 3.12 shows that the estimates of α_x are non-negative, justifying the presence of overdispersion. It is also noteworthy that values of α_x are especially high at $x > 85$. This agrees with our observation that the mortality rates at very old ages are highly volatile, and implies that previous interval estimations might have disregarded a substantial part of actual uncertainties.

Table 3.3 evaluates the model performance. We apply the Akaike Information

Criterion (AIC), the Schwarz Bayes Criterion (SBC) and the likelihood ratio test (LRT). Each of these is a slightly different way of assessing whether the improved fit of the negative binomial model is worth the added complexity, which is measured by the number of parameters. For the AIC and SBC, we are looking for a higher value of the penalized log-likelihood; for the likelihood ratio test, a small p-value (say, less than 1%) indicates that the more complicated model is preferred. The formulae for the criteria are as follows, where l is the maximum log likelihood, k is the number of parameters, n is the size of the data set, d is the difference between the number of parameters of the negative binomial model and of the Poisson model.

$$AIC = l - k$$
$$SBC = l - (k/2)\log n$$
$$LRT \text{ p - value} = \Pr[\chi_d^2 > l_{NB} - l_P]$$

Figures 3.13 and 3.14 reflect the increased goodness of fit on the "best estimate" of future death rates. Being more flexible, the Negative Binomial extension seems capable of correcting the under-fit manifested in the original version, particularly at the very young ages. The improved flexibility has however no discernable harm to the desired age-pattern of mortality, as shown in Figure 3.15. Having computed the "best estimate", we obtain the confidence intervals by the usual parametric bootstrap and the additional routine that allows recurrence of the detected outliers in the future. The effect of the extension on the confidence intervals varies by age – The average increase in width at younger ages (0 – 40) is around 6 percent and that at higher ages (75 and over) is more than 100 percent. This agrees with our previous assertion that the mean-variance equality restriction in the Poisson version has lead us to understating the variations, mostly at the higher ages. However, it should be emphasized that the model structure after this relaxation is still strong, and hence that the confidence intervals might still be too narrow to cater for the possibility of future structural changes.

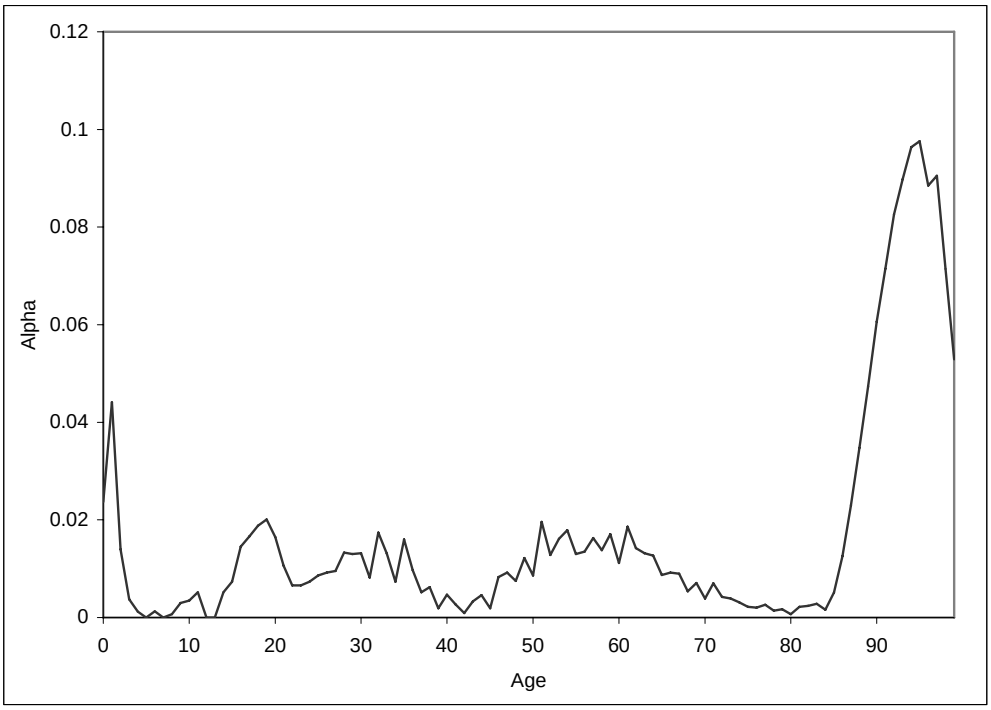
Male

	Model	
	Poisson	NB
Number of Parameters	279	379
Log-likelihood	-66021	-43517
AIC	-66300	-43896
SBC	-67276	-45222
Likelihood Ratio Test		
p-value	0.000000	

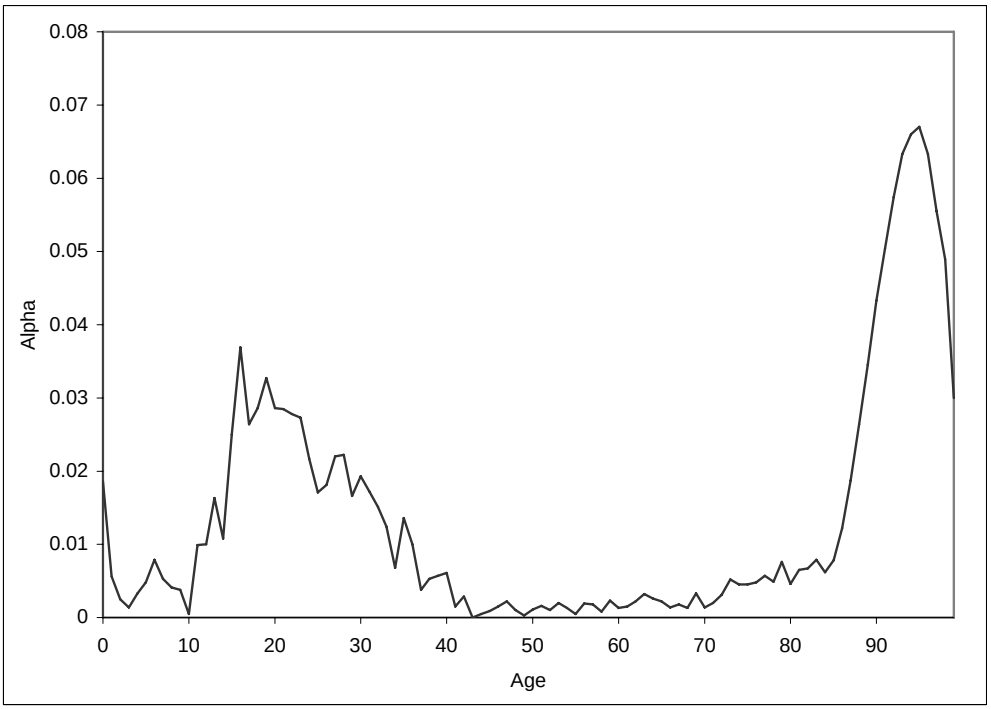
Female

	Model	
	Poisson	NB
Number of Parameters	279	379
Log-likelihood	-52643	-40748
AIC	-52922	-41127
SBC	-53898	-42453
Likelihood Ratio Test		
p-value	0.000000	

Table 3.3. MSE and AIC of the Lee-Carter fit to $\ln(m_{x,t})$, assuming the number of deaths follows a Poisson / Negative Binomial distribution, Canadian population.

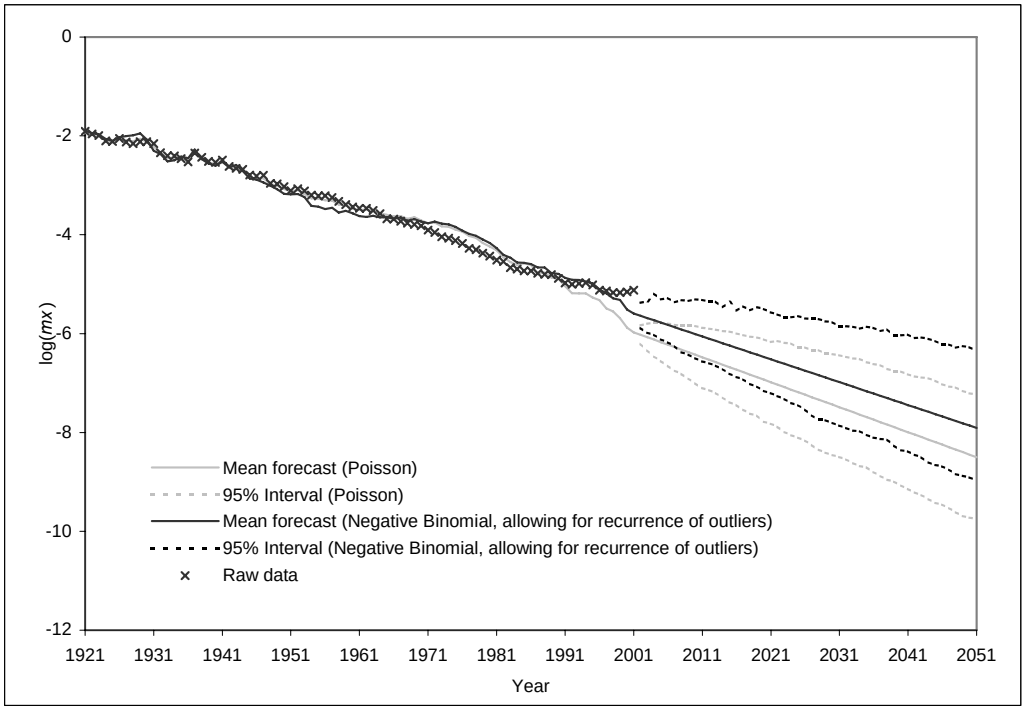


(a)

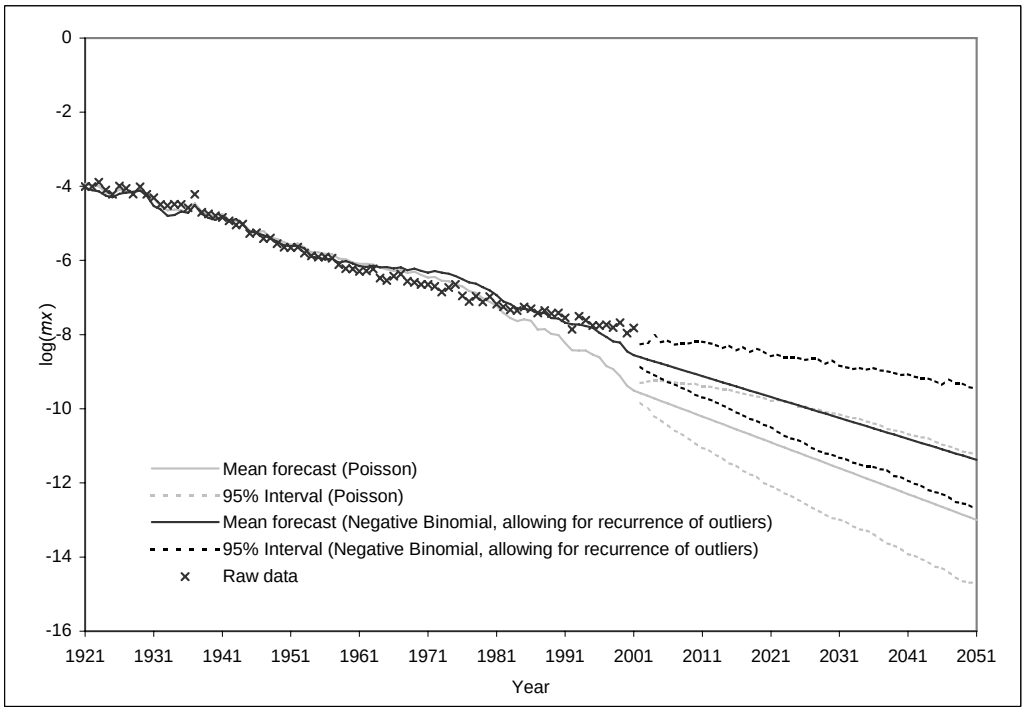


(b)

Fig. 3.12. Fitted values of α_x, (a) male and (b) female.

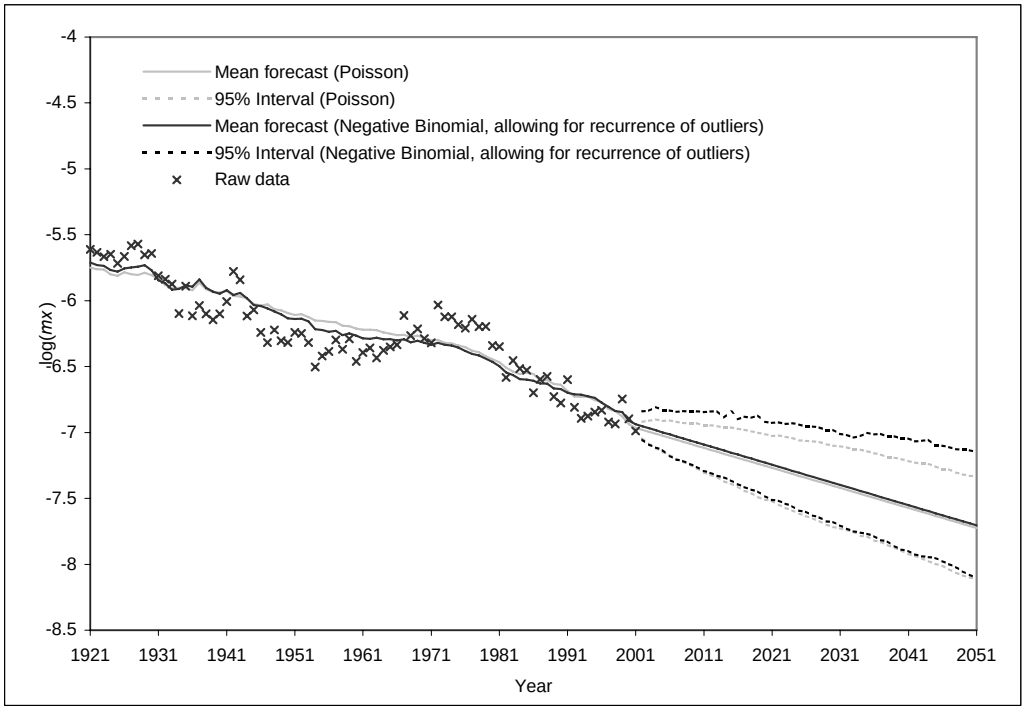


Age 0

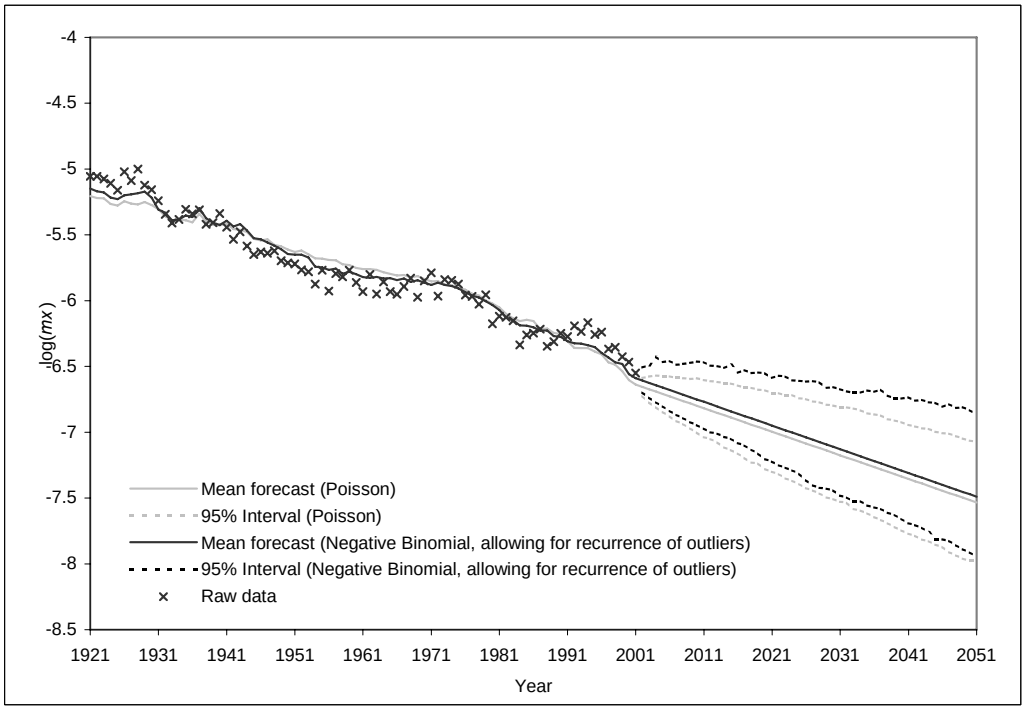


Age 1

Fig 3.13. The Lee-Carter fit (1921 – 2001) and projection (2002 – 2051), MLE (Poisson) and MLE (Negative Binomial) allowing for recurrence of outliers, male.

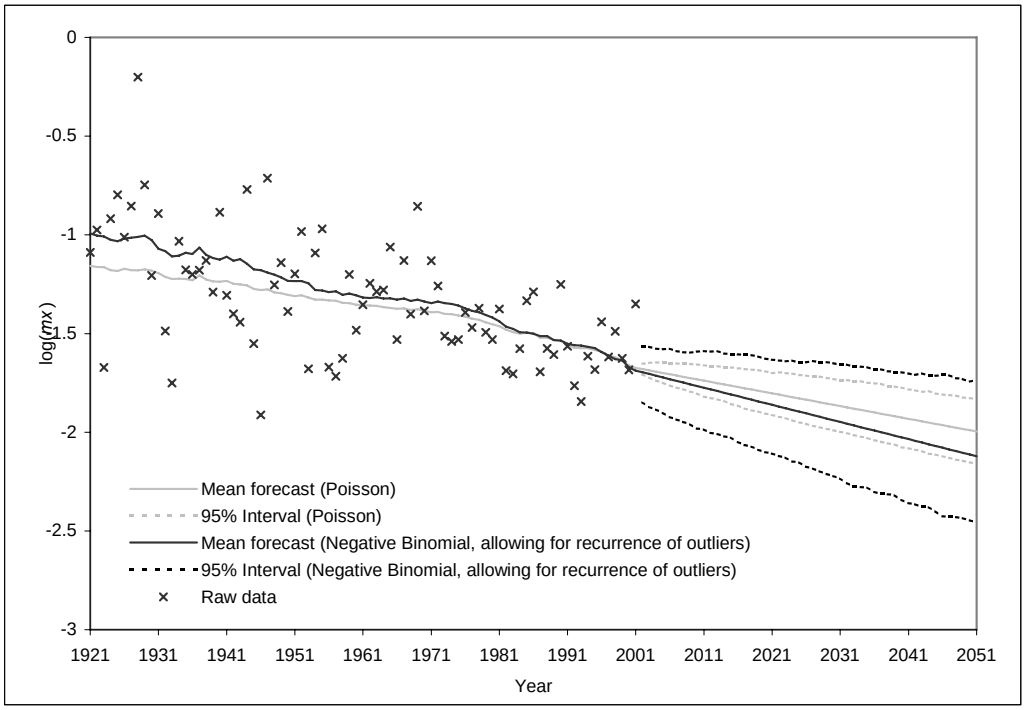


Age 20

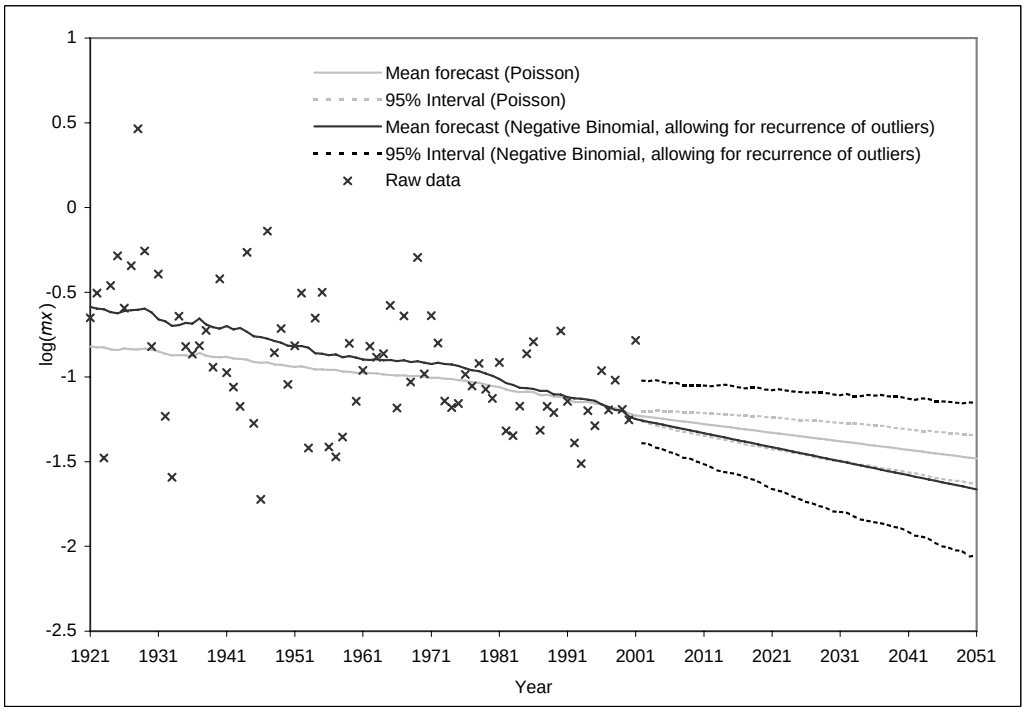


Age 40

Fig 3.13 (cont'd). The Lee-Carter fit (1921 – 2001) and projection (2002 – 2051), MLE (Poisson) and MLE (Negative Binomial) allowing for recurrence of outliers, male.

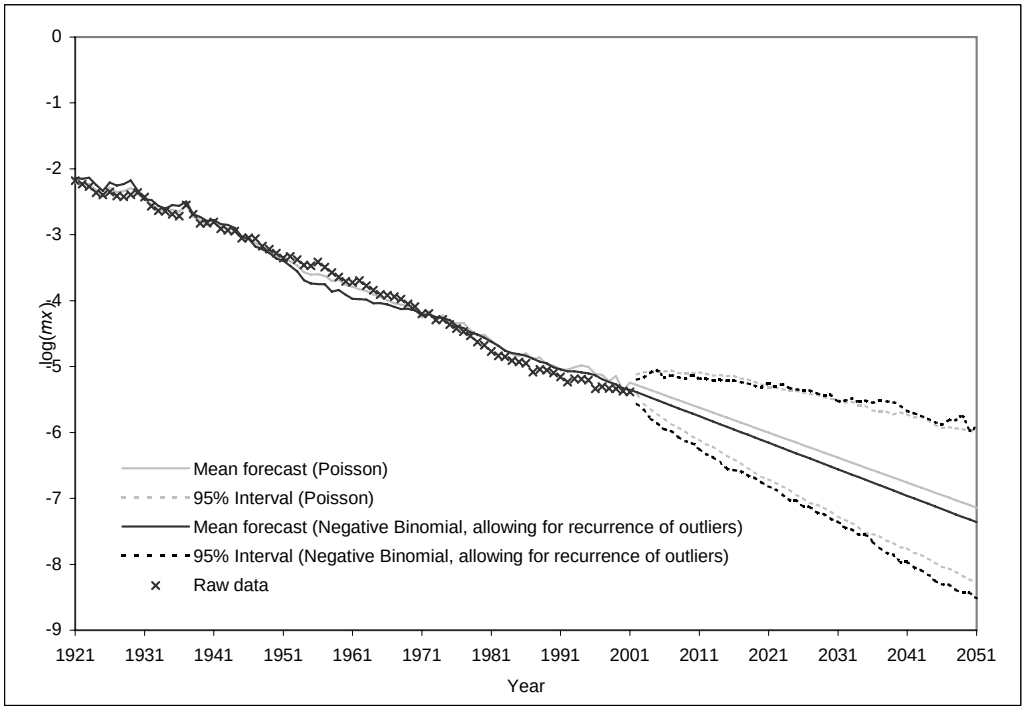


Age 90

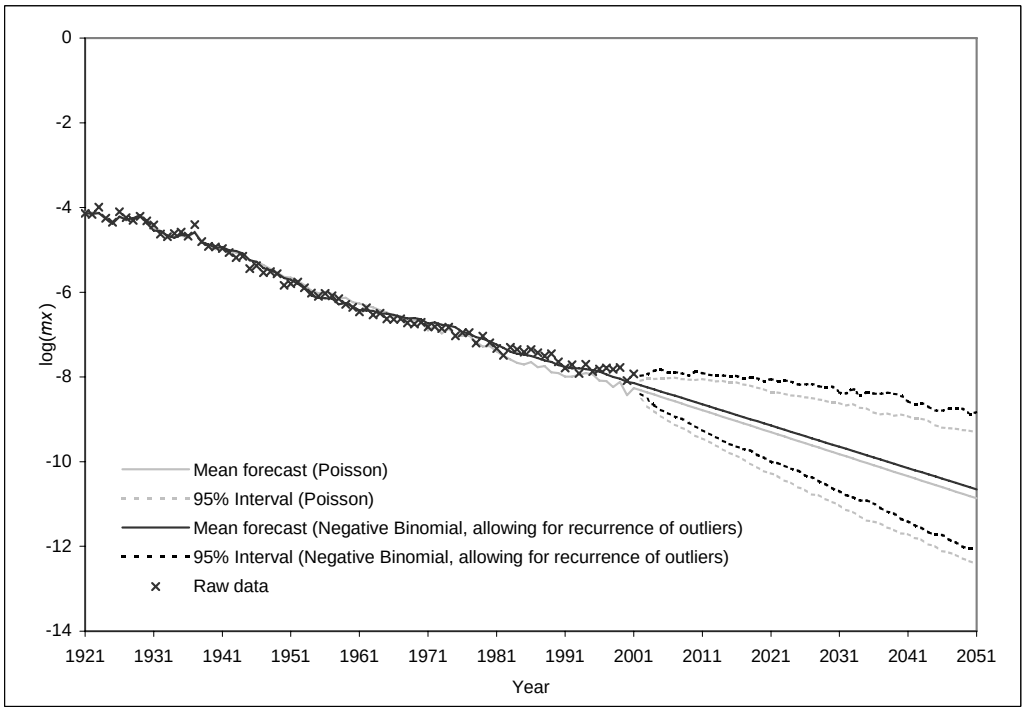


Age 95

Fig 3.13 (cont'd). The Lee-Carter fit (1921 – 2001) and projection (2002 – 2051), MLE (Poisson) and MLE (Negative Binomial) allowing for recurrence of outliers, male.

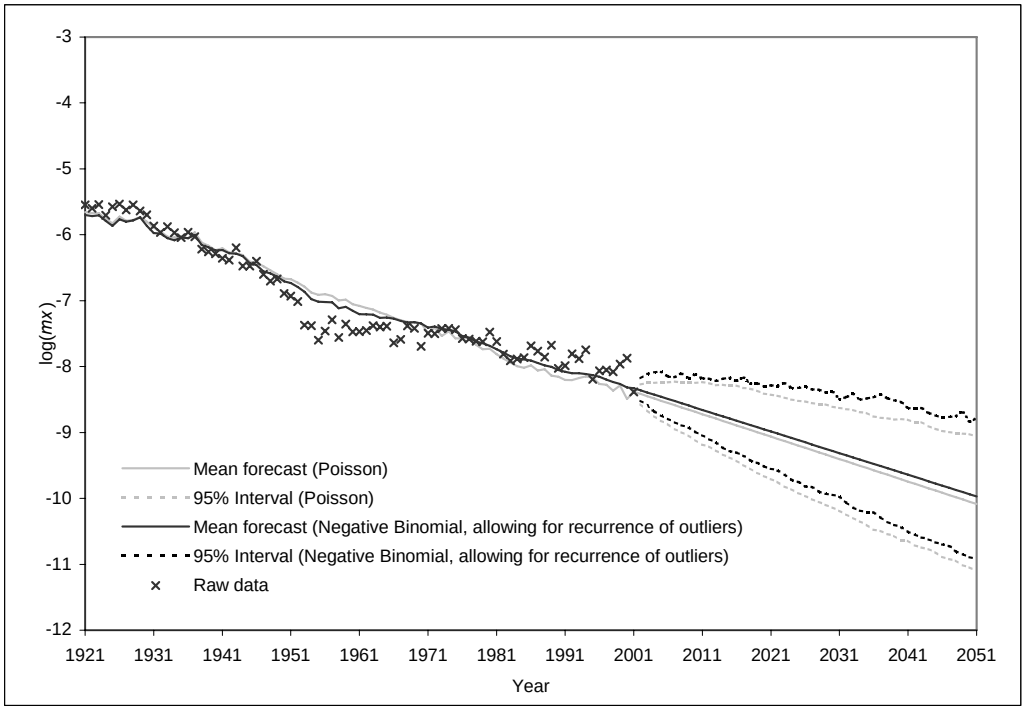


Age 0

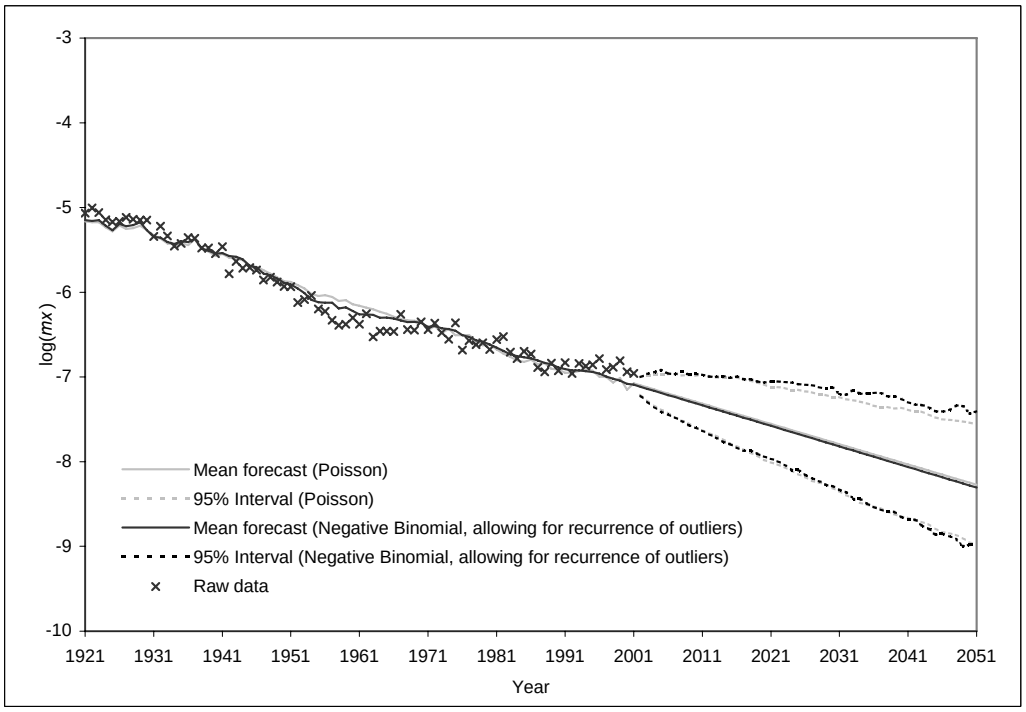


Age 1

Fig 3.14. The Lee-Carter fit (1921 – 2001) and projection (2002 – 2051), MLE (Poisson) and MLE (Negative Binomial) allowing for recurrence of outliers, female.

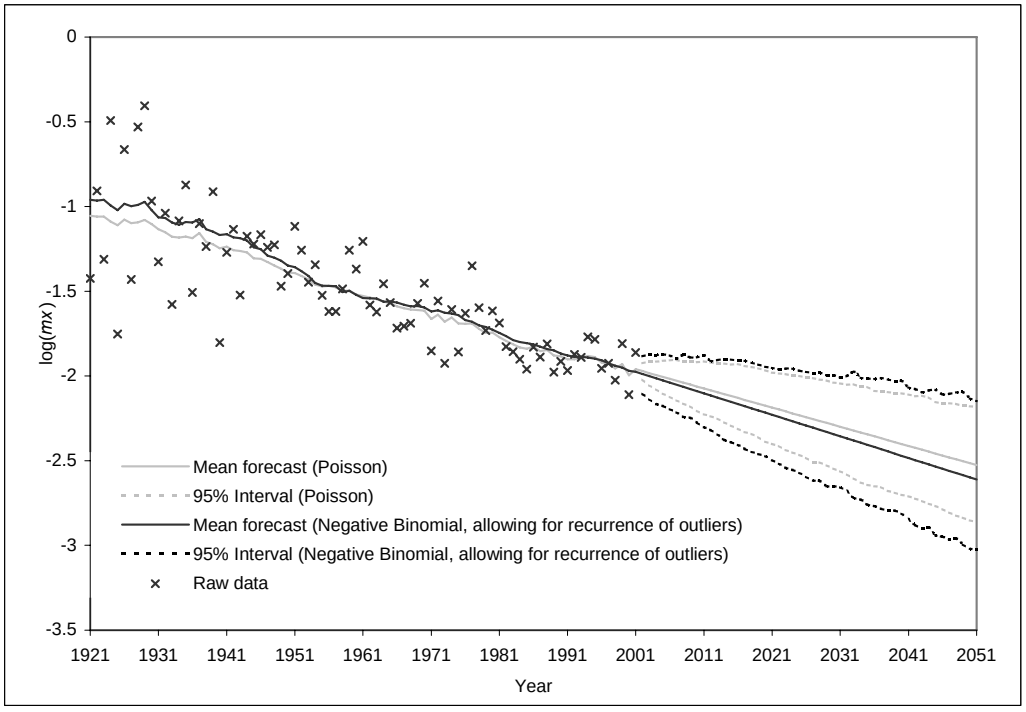


Age 20

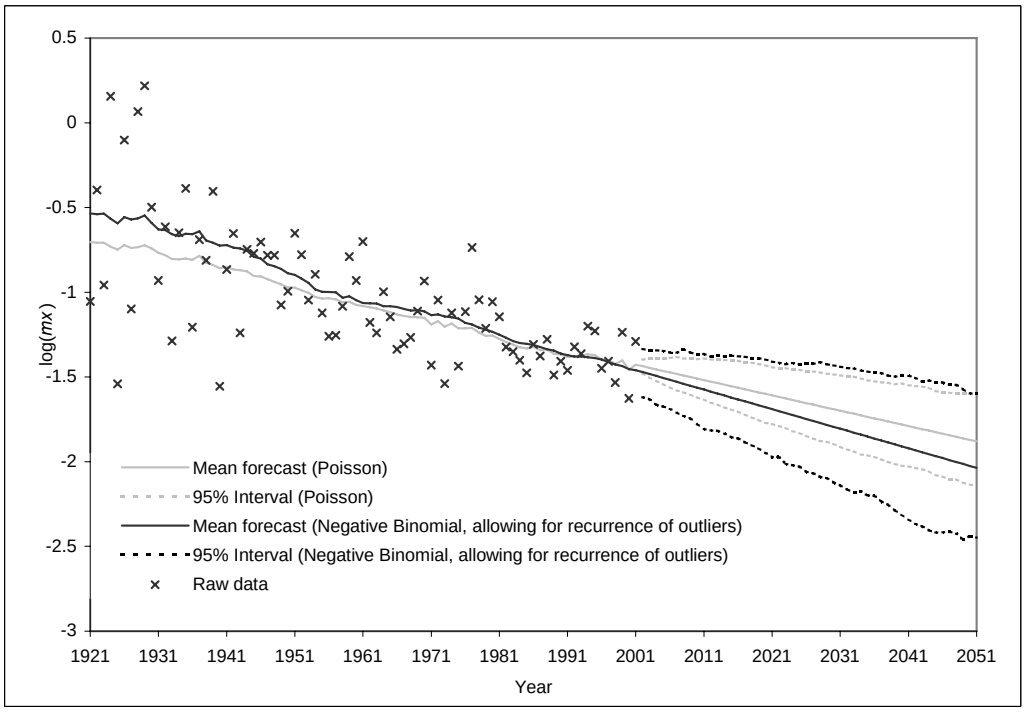


Age 40

Fig 3.14 (cont'd). The Lee-Carter fit (1921 – 2001) and projection (2002 – 2051), MLE (Poisson) and MLE (Negative Binomial) allowing for recurrence of outliers, female.

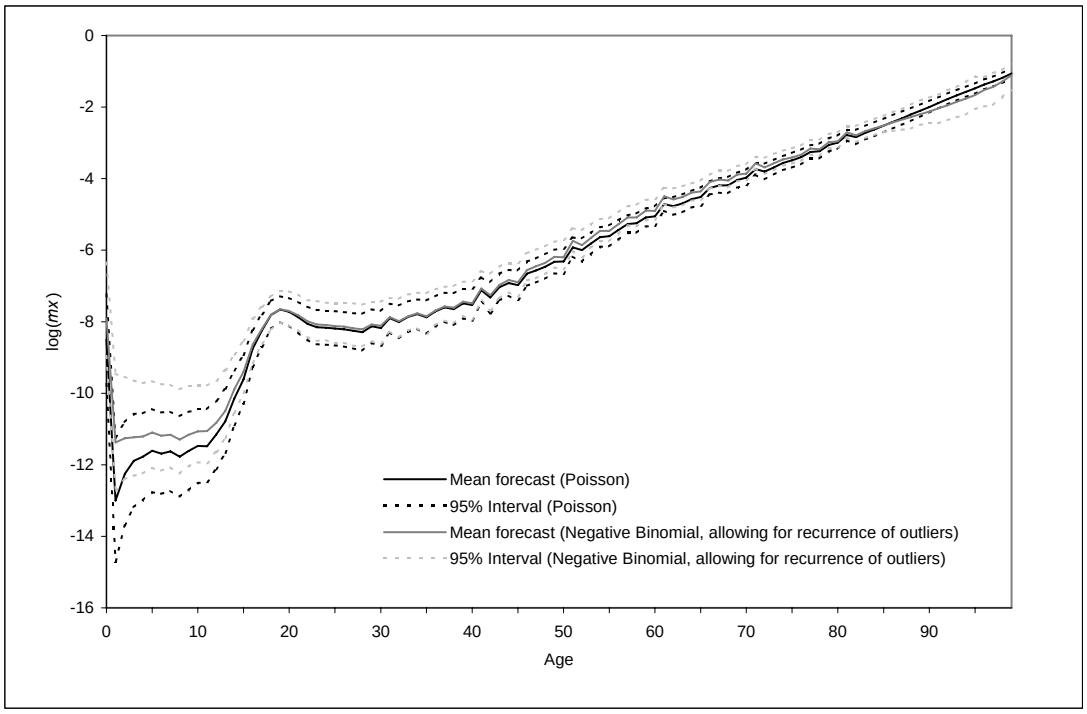


Age 90

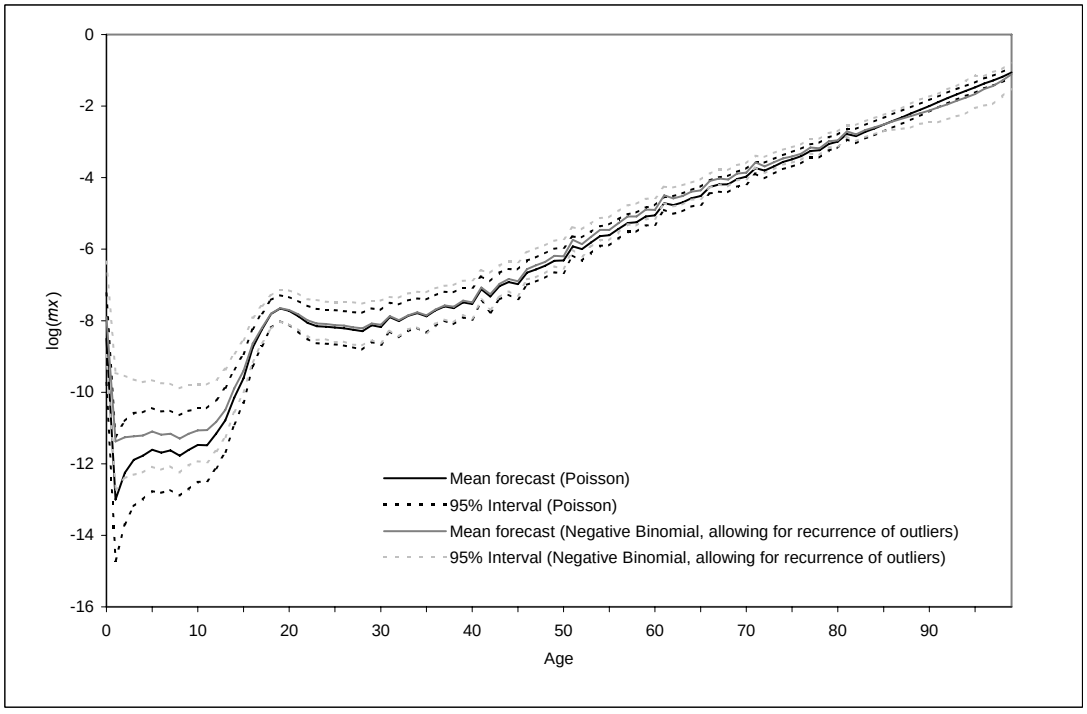


Age 95

Fig 3.14 (cont'd). The Lee-Carter fit (1921 – 2001) and projection (2002 – 2051), MLE (Poisson) and MLE (Negative Binomial) allowing for recurrence of outliers, female.



(a)



(b)

Fig. 3.15. The projection of age pattern of mortality by Lee-Carter, MLE (Poisson) and MLE Negative Binomial) allowing for recurrence of outliers, (a) male and (b) female, 2051.

3.5 The Improvement Scales

In previous sections, we have introduced several stochastic mortality models and applied them to the mortality data of the Canadian population. However, the forecasts are in the form of vast arrays, which makes retrieval difficult. In this section, we attempt to summarize both the "best estimate" and the interval forecast of future death rates by some tractable formulae, or mortality improvement scales.

The improvement scales shall consist of two parts. One is a current period life table, also known as "base table", to which the mathematical formulae are applied. To avoid the adverse effect of random fluctuations, we graduate the historical life tables by the two-dimensional *P*-splines regression and take the graduated 2001 experiences as the illustrative "base tables". These "base tables" are included in Appendix A for the reader's reference. The other part is a set of multiplicative factors that specifies the percentage of reduction in age-specific death rates during a particular time period. These multiplicative factors, along with their associated measure of uncertainties, are to be derived from the Lee-Carter model with the two suggested extensions.

Recall that under the Lee-Carter model, the "best estimate" of future death rates can be written as

$$\hat{\ln}\left(m_{x,T+s}\right) = \hat{a}_x + \hat{b}_x \hat{k}_{T+s}. \tag{3.53}$$

Here, we denote T as the "base table" year, which is 2001 in this application; and denote s as the number of years after T. Rearranging equation (3.53) gives

$$\begin{aligned} \hat{\ln}\left(m_{x,T+s}\right) &= \hat{\ln}\left(m_{x,T}\right) + \hat{b}_x\left(\hat{k}_{T+s} - \hat{k}_T\right) \\ &\approx \ln\left(m_{x,T}\right) + \hat{b}_x\left(\hat{k}_{T+s} - \hat{k}_T\right). \end{aligned} \tag{3.54}$$

Furthermore, if k_t follows an ARIMA(0,1,0) process,

$$\hat{k}_{T+s} - \hat{k}_T = \hat{c}\, s, \tag{3.55}$$

where $\hat{c}$ is the estimated drift term in the ARIMA(0,1,0) time-series equation. Define the mortality improvement scale as the estimated percentage reduction of death rate at age x in a time period of s years, i.e.,

$$IS_{2001}^{\text{population}}(x,s) = \frac{m_{x,2001+s}}{m_{x,2001}}, \tag{3.56}$$

where the subscript and superscript respectively refers to the "base table" year 2001 and the application to the population data. The combination of equations (3.54) – (3.56) gives

$$\begin{aligned} \hat{IS}_{2001}^{\text{population}}(x,s) &= \exp\left(\hat{b}_x \hat{c}\, s\right) \\ &= \exp\left(\hat{w}_x\, s\right), \end{aligned} \tag{3.57}$$

which means the representation of the forecast can be reduced from a two dimensional array of values to a one dimensional vector of parameters (w_x). Crude estimates of w_x are not necessarily smooth, since the Lee-Carter model does not guarantee smoothness age by age. We remove the raggedness by a one-dimensional *B*-splines regression, illustrated in Figure 3.16. Note that w_0 is not included in the *B*-splines regression to avoid any unwanted inflation / deflation of other w_x's. The graduated values of w_x, separately for each sex, are included in Appendix B.

The parametric bootstrapping method described in Section 3.2 could generate an empirical distribution of a particular improvement scale, and the percentiles of this distribution could give an estimate of the required confidence interval in terms of numerical values. However, to obtain algebraic expressions for the confidence interval, we need an additional approximation. Noting that the ARIMA(0,1,0) structure gives uncertainties that are increasing linearly with time, and that the interval forecast should be close to zero at the origin, we may approximate the variance of the logarithm of the improvement scale by a straight line without intercept. Figure 3.17 indicates that the linear model gives a good fit. Let v_x be the slope parameter of this straight line. Then, the approximate 95% point-wise confidence interval of the improvement scale can be expressed as

$$\left[\exp\left(\hat{w}_x\, s - 1.96\sqrt{\hat{v}_x\, s}\right), \exp\left(\hat{w}_x\, s + 1.96\sqrt{\hat{v}_x\, s}\right)\right]. \tag{3.58}$$

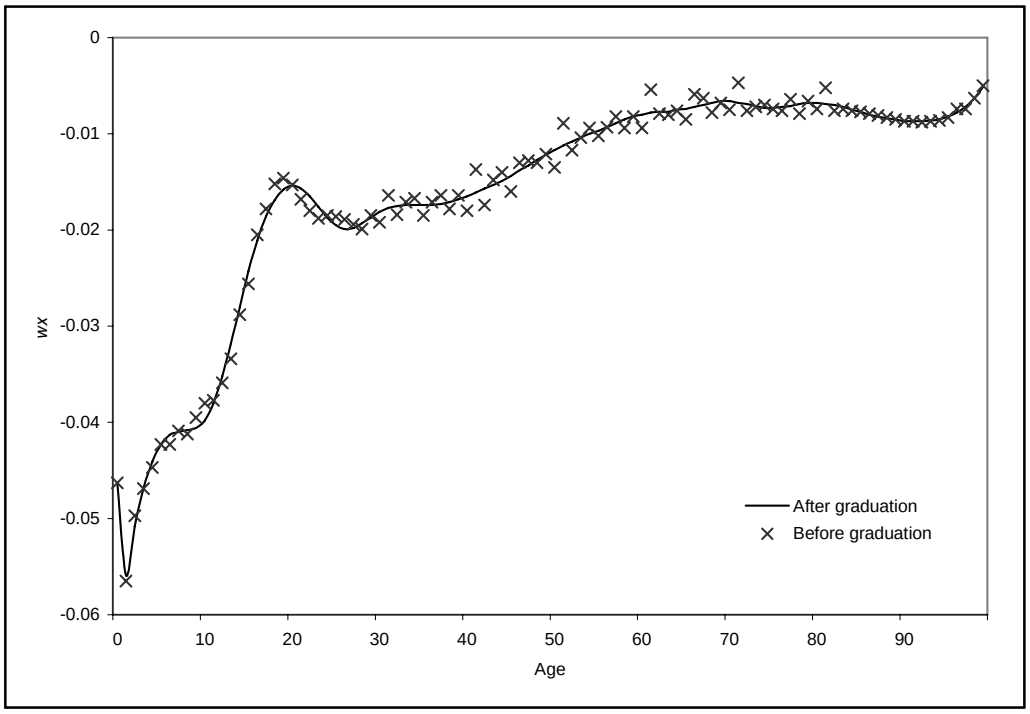
Note that the variability measure v_x has incorporated both parameter uncertainties and stochastic uncertainties (forecast error in the time-series component). The roughness in v_x is again graduated by a one-dimensional *B*-splines regression, illustrated in Figure 3.18. The graduated values of v_x are included in Appendix B.

The improvement scale allows a simultaneous retrieval of the "best estimate" and interval forecast of future death rates. To illustrate, suppose that we want to forecast m_{90} (male) in 2021. Appendix A gives $m_{90} = 0.232874$ in 2001 and Appendix B gives $w_{90} = -0.008655$ and $v_{90} = 0.000368$. Hence, the "best estimate" and confidence interval of m_{90} in 2021 is $0.232874 \times \exp(-0.008653 \times 20) = 0.195867$, and $0.232874 \times [0.710910, 0.995101] = [0.165566, 0.231733]$.

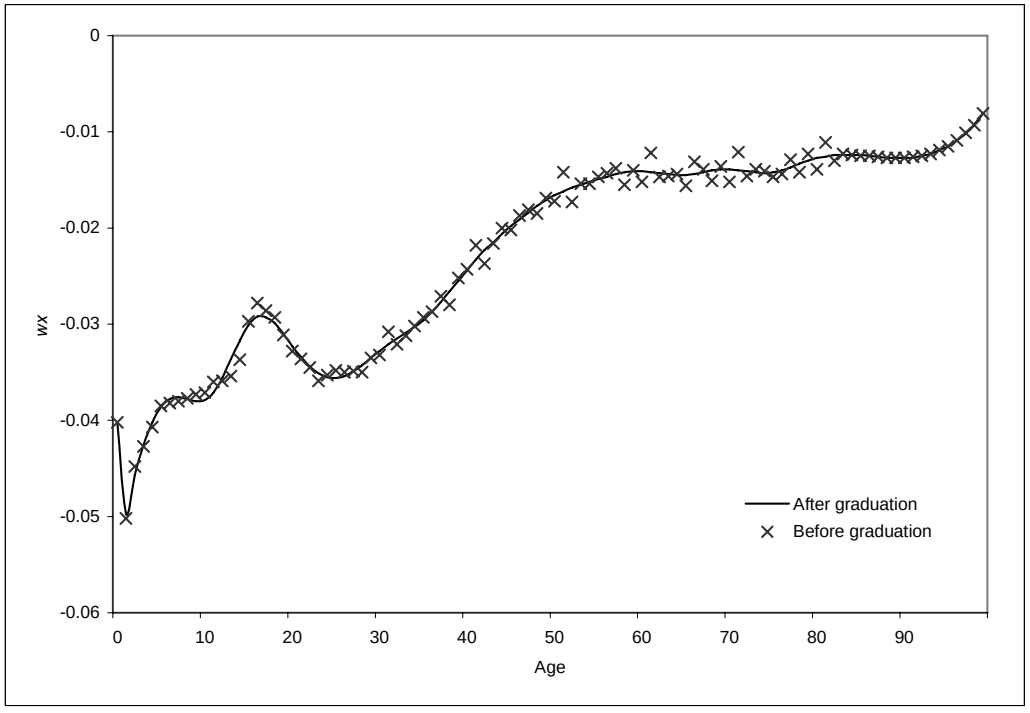
If a base table of p_x is given, we may express the forecast of p_x by

$$p_{x,2001+s} = \left(p_{x,2001}\right)^{\hat{IS}_{2001}^{\text{population}}(x,s)} \tag{3.59}$$

assuming constant force of mortality over each year of age. The corresponding interval forecast can be obtained by replacing $\hat{IS}_{2001}^{\text{population}}(x,s)$ in equation (3.59) by the upper and lower limits in expression (3.58).

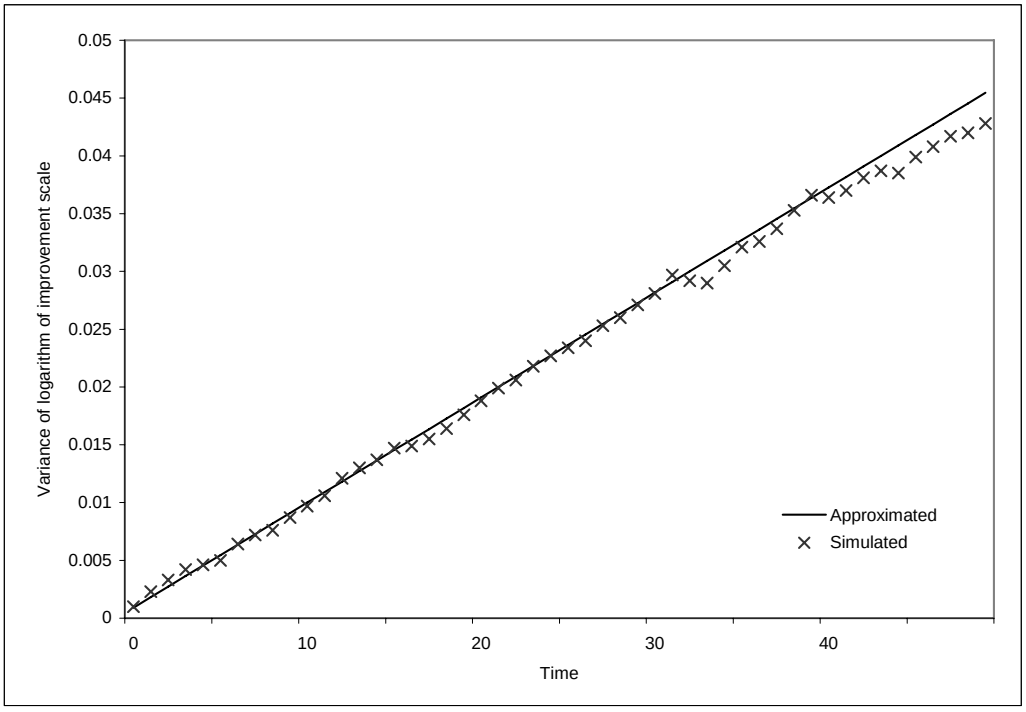


(a)

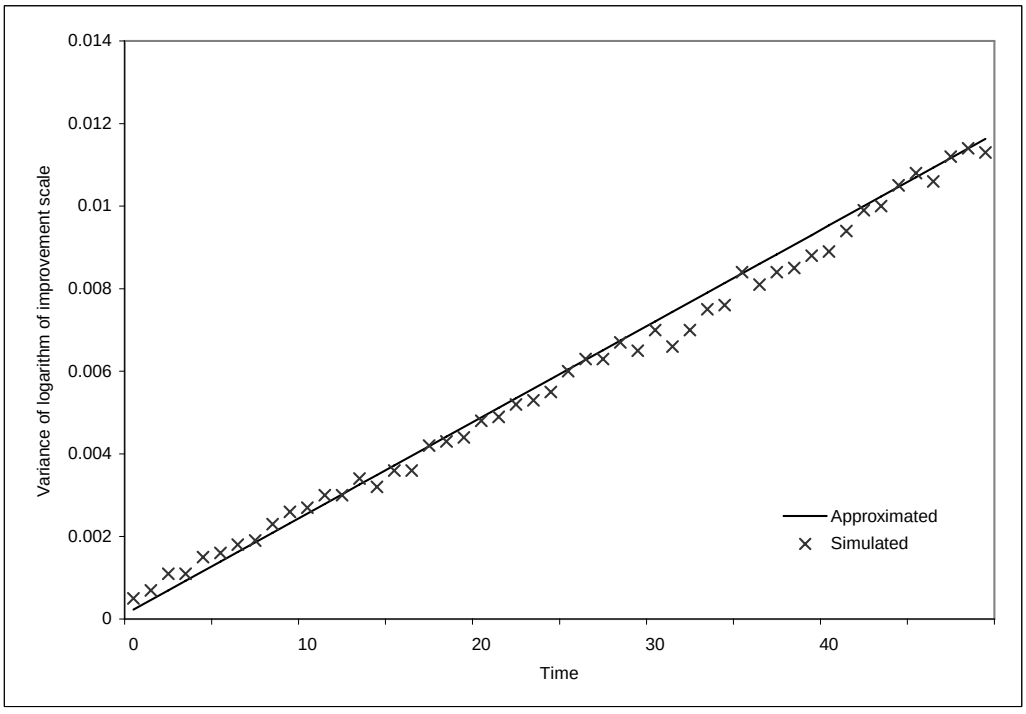


(b)

Fig. 3.16. Parameter w_x before and after graduation, (a) male and (b) female.

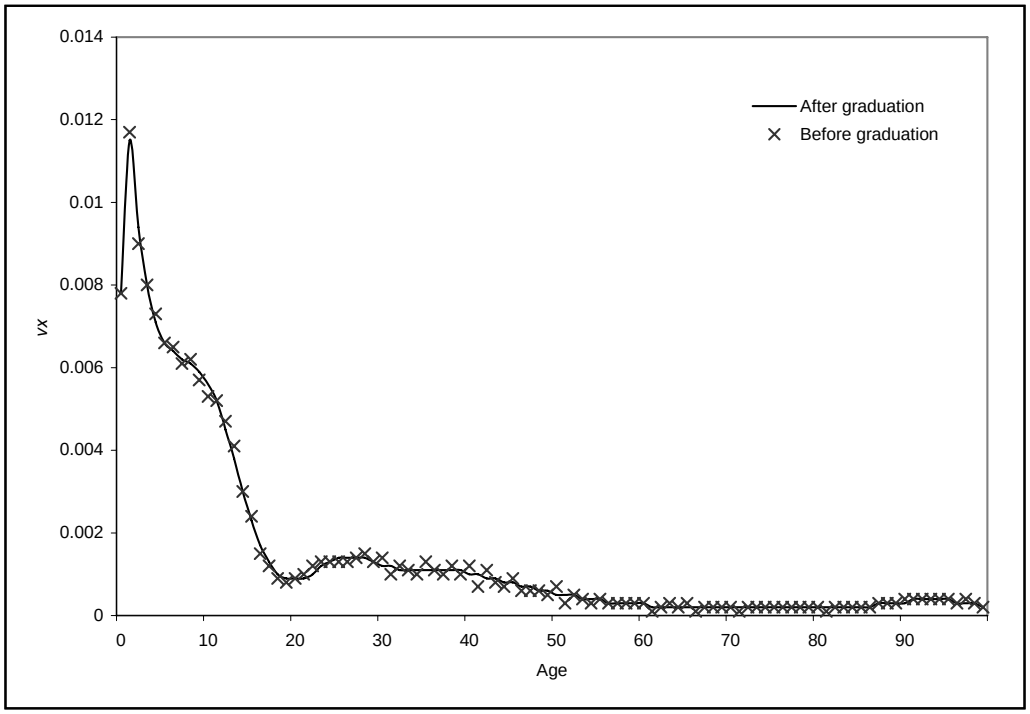


(a)

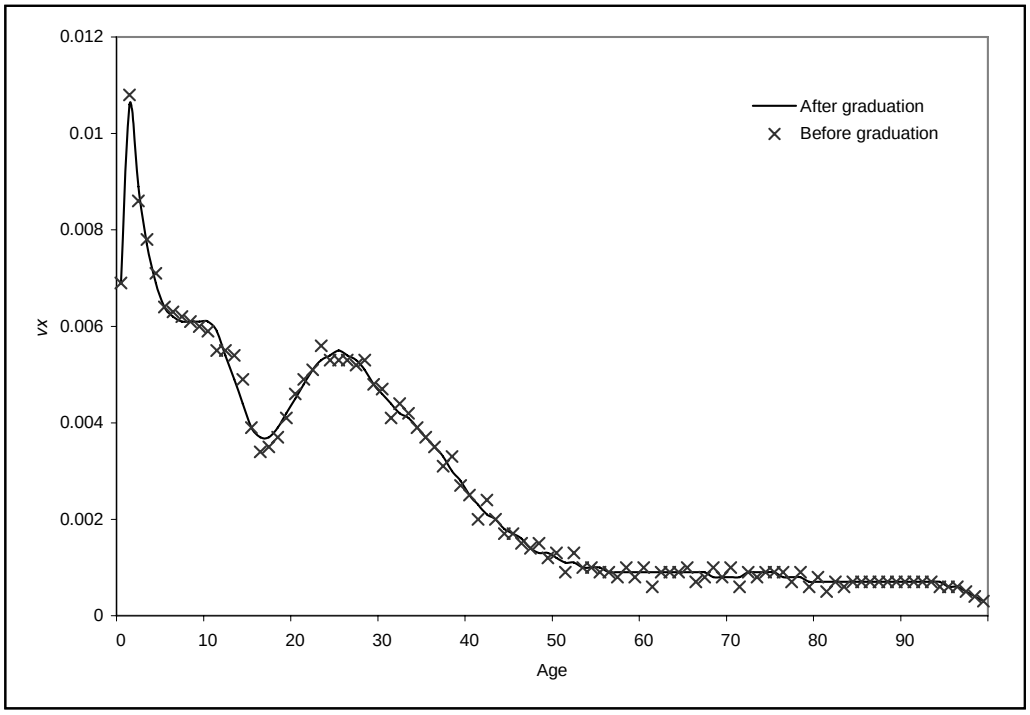


(b)

Fig. 3.17. Variance of logarithm of improvement scale obtained by parametric bootstrapping and a linear approximation, age 65, (a) male and (b) female.



(a)



(b)

Fig. 3.18. Parameter v_x before and after graduation, (a) male and (b) female.

3.6 Additional Comments

The implementation of all models requires a crucial subjective judgment on the period over which the models parameters are estimated. Some researchers prefer using the maximum period for which data is available (e.g. Lee and Nault, 1993; Lee and Carter, 1992a, 1992b), while others prefer a restricted fitting period, say 50 years, when the age pattern of mortality is the most stable (e.g. Renshaw and Haberman, 2003a and 2003b). In the statistical literature, this decision is usually made according to the range of the forecast. If the range is short, recent trends should be given more emphasis and consequently a restricted fitting period may be used; and on the other hand, if the range is long, the forecaster should not ignore older data to prevent the long term trend from being biased by short term phenomenon. As the improvement scales are to be used in projecting mortality far into the future, we prefer not to discard any long-term experience which is likely to be informative and relevant. Nevertheless, for the purpose of comparison, we replicate the model fitting exercise using data from 1952 to 2001. Figures 3.19 and 3.20 show that the restricted fitting period yields rather dissimilar projections, but the dissimilarities are unsystematic and are captured by the confidence intervals in the original fit.

In the case of the females at age 30, the original fit is much steeper, and possibly, this is because the original fit has taken into account the remarkable improvement of maternal risks occurred in the early century[2]. If this argument is justifiable, then the improvement scales we provide could lead us to overestimating future mortality improvements, as such an early century phenomenon is unlikely to be repeated. The legitimacy, however, is negated in a closer scrutiny of female mortality improvement at childbearing ages. Figure 3.21 shows that the peak of female mortality improvement occurred in around 1950, which is not the time when the reduction of maternal risks was the most remarkable. This suggests that the reduction of risks related to child birth might not be the major driving force of female mortality improvement in the early century, and that it might not be appropriate to assume heuristically a cessation of female mortality improvement at childbearing ages without sufficient knowledge on the genuine driving forces, and the possible interactions among them.

It is also noteworthy that the stability during the restricted fitting period has led to

[2] The reported maternal mortality ratio in Canada has declined from approximately 500 maternal deaths per 100,000 live births in the early 1920s to less than 100 per 100,000 live births in the 1950s (Statistics Canada, 1994)

substantially narrower confidence intervals. These intervals are of course not to be relied on since they contain little information on the possible changes in mortality rates.

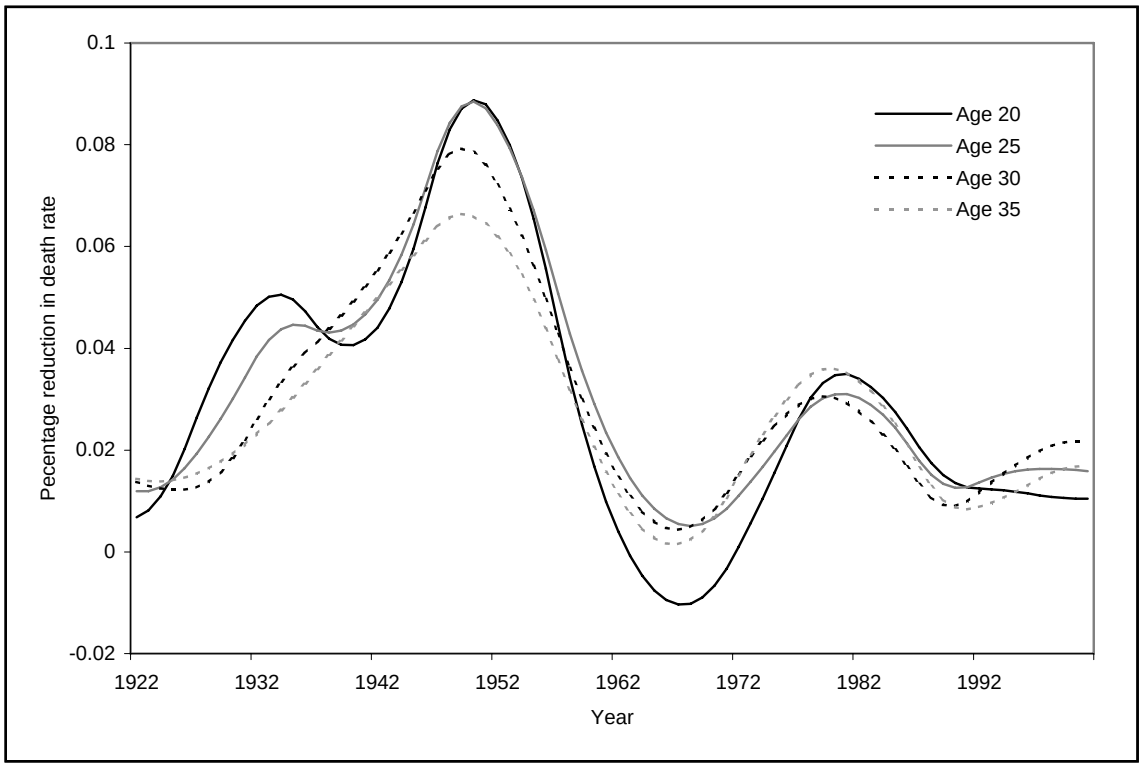
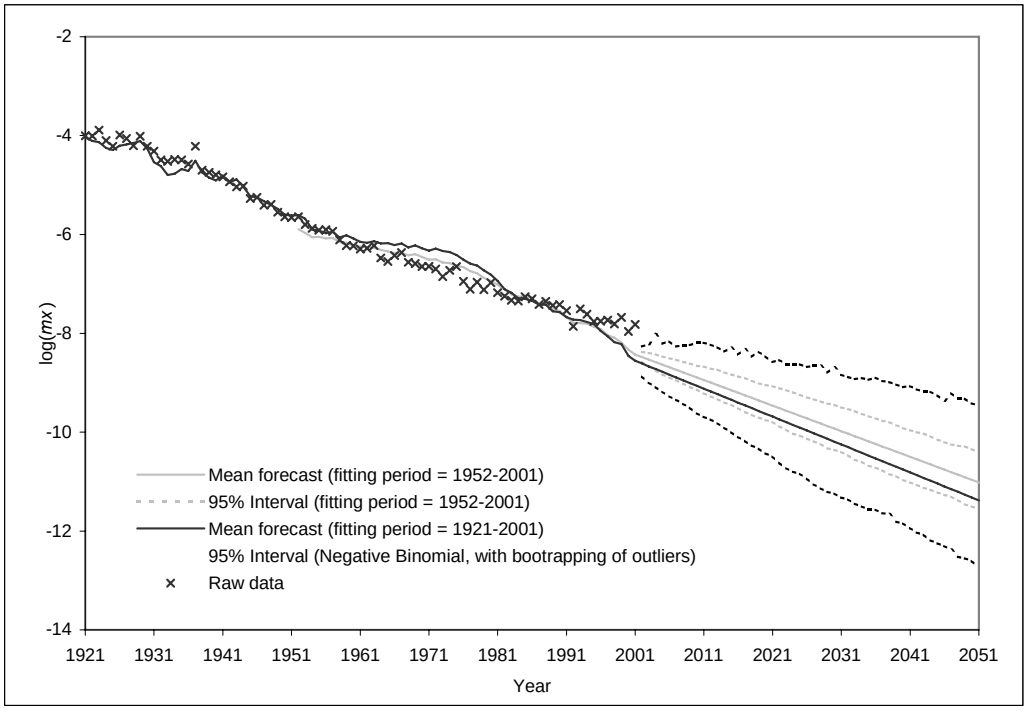
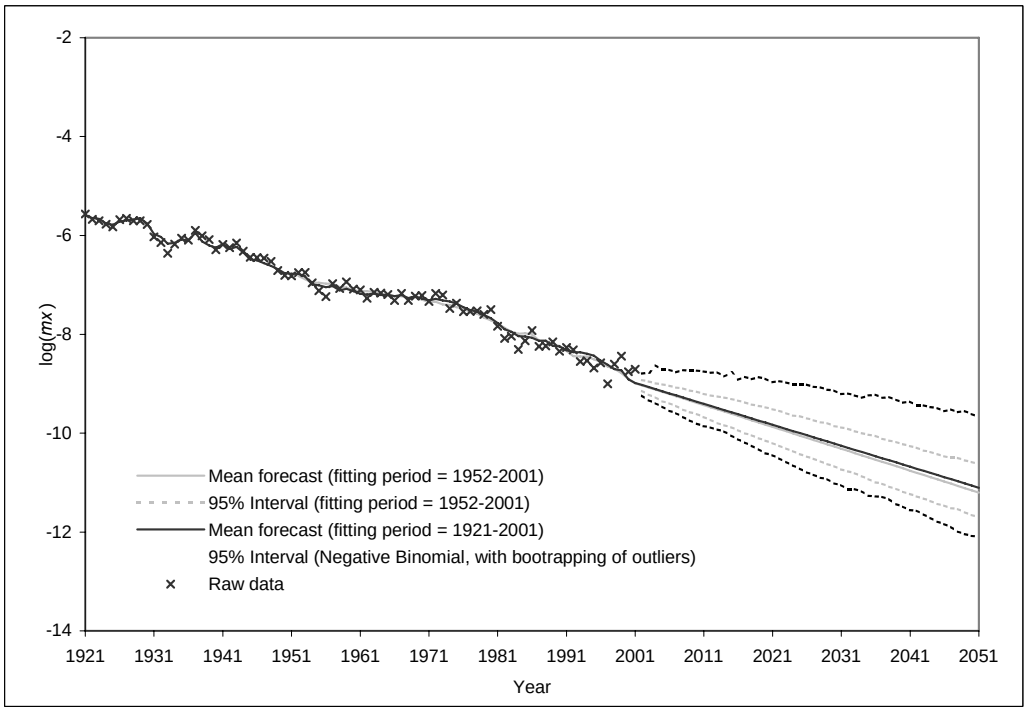


Fig. 3.21. Reduction in female mortality at childbearing ages.

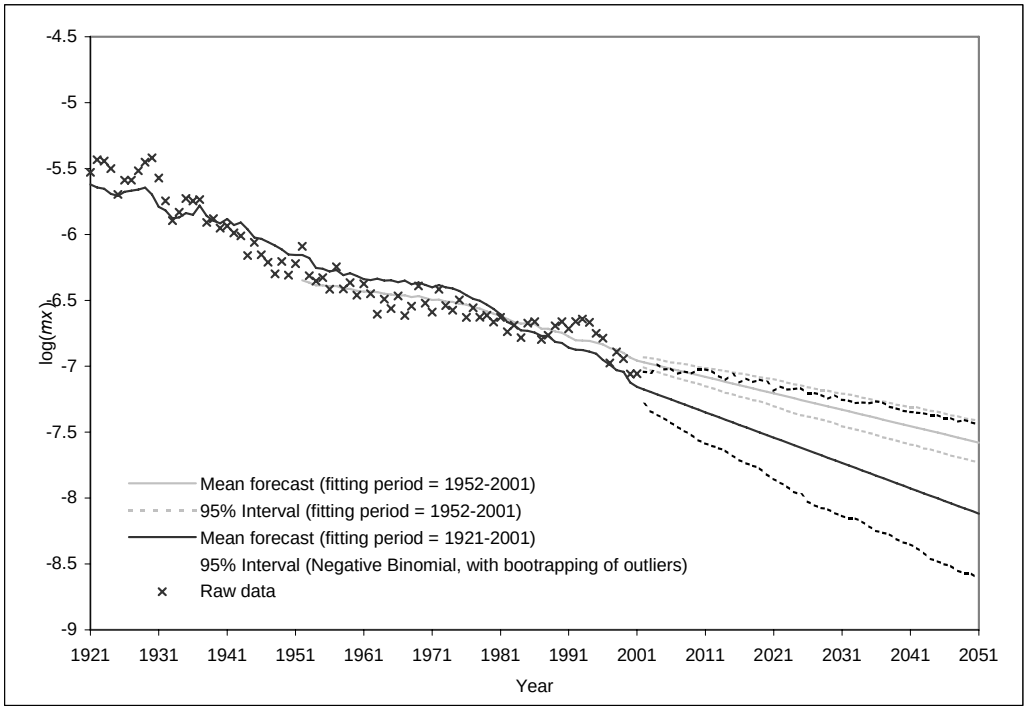


Age 1

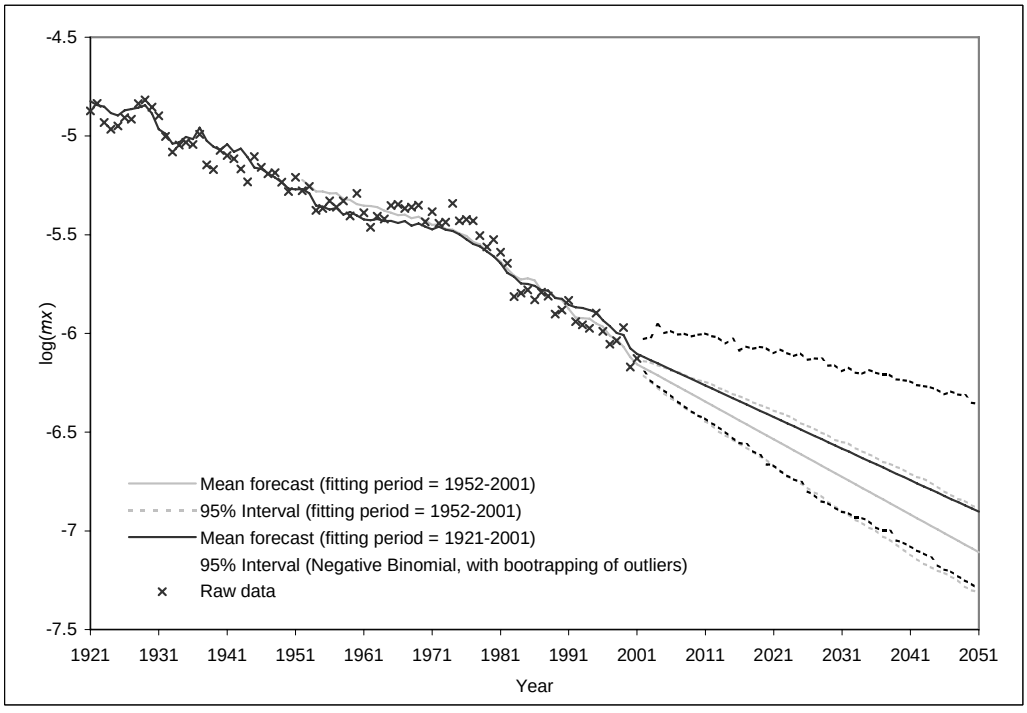


Age 5

Fig. 3.19. The Lee-Carter projection under different fitting periods, male.

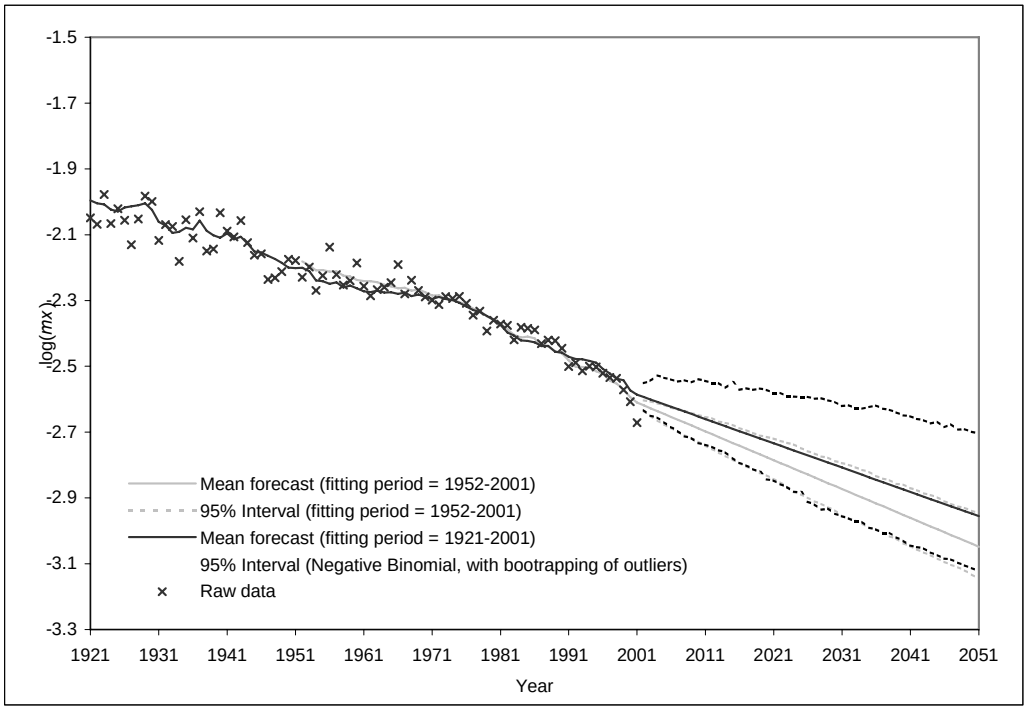


Age 30

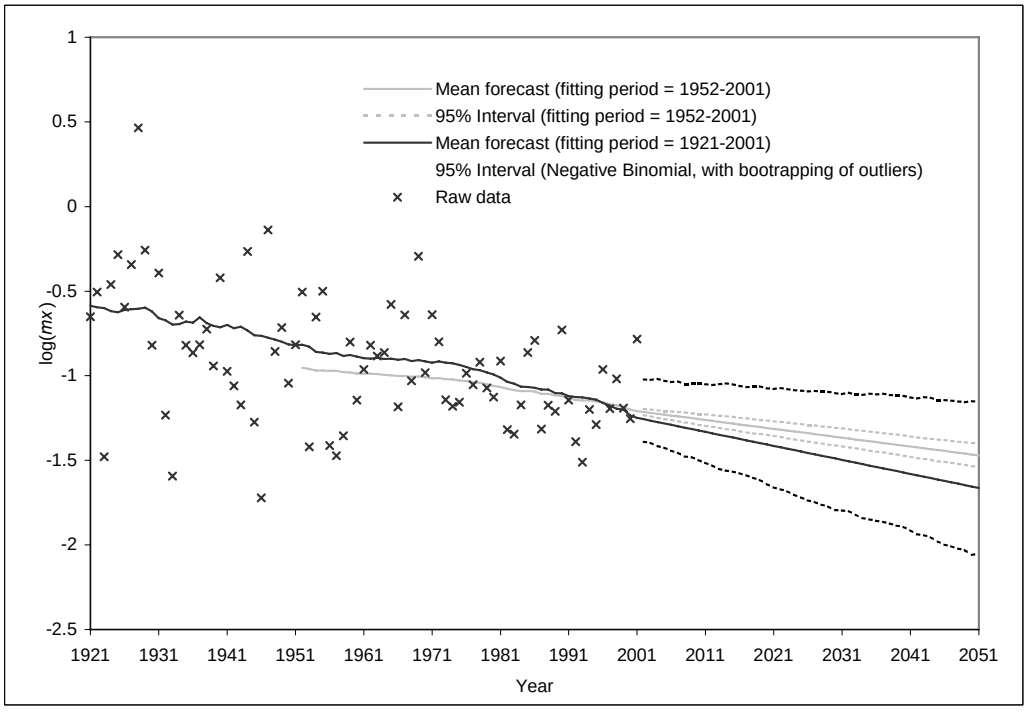


Age 45

Fig. 3.19 (cont'd). The Lee-Carter projection under different fitting periods, male.

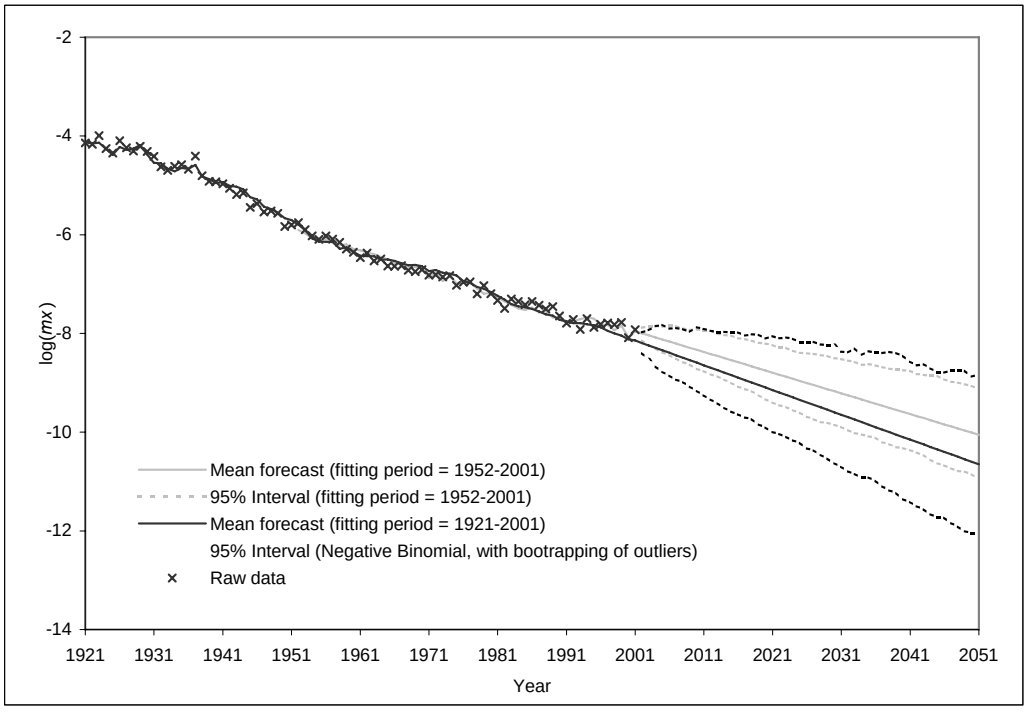


Age 80

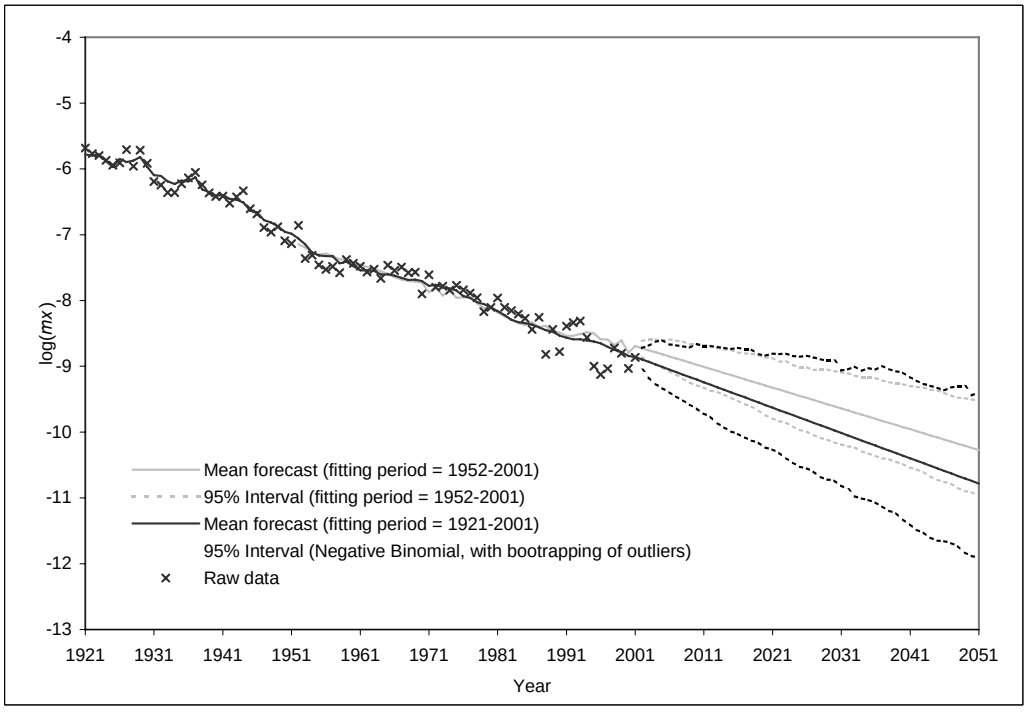


Age 95

Fig. 3.19 (cont'd). The Lee-Carter projection under different fitting periods, male.

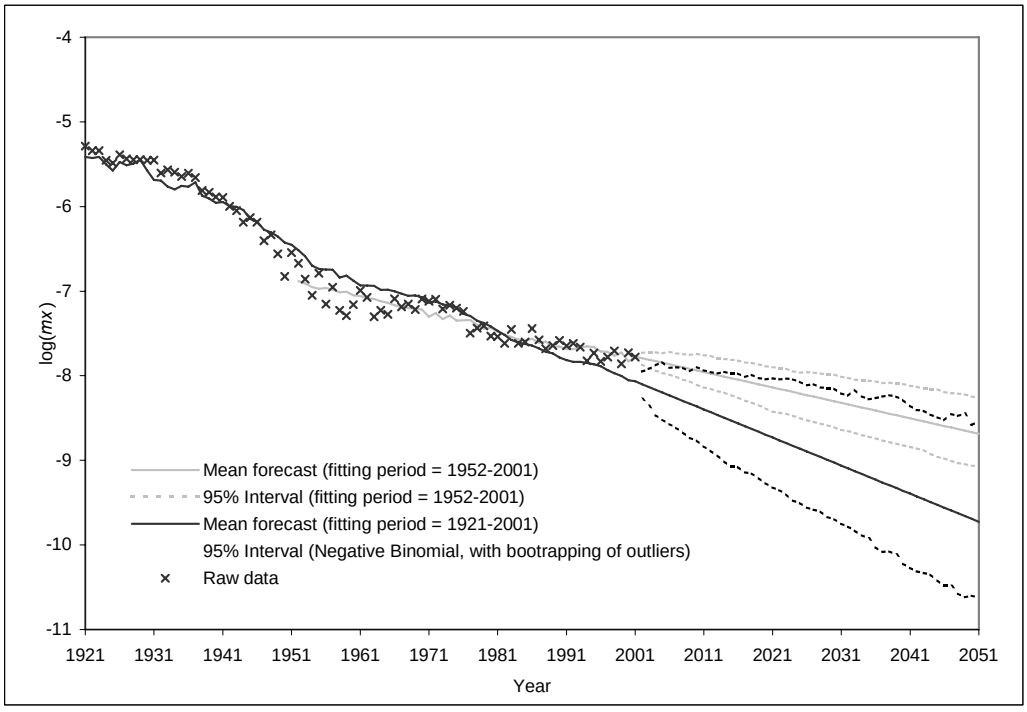


Age 1

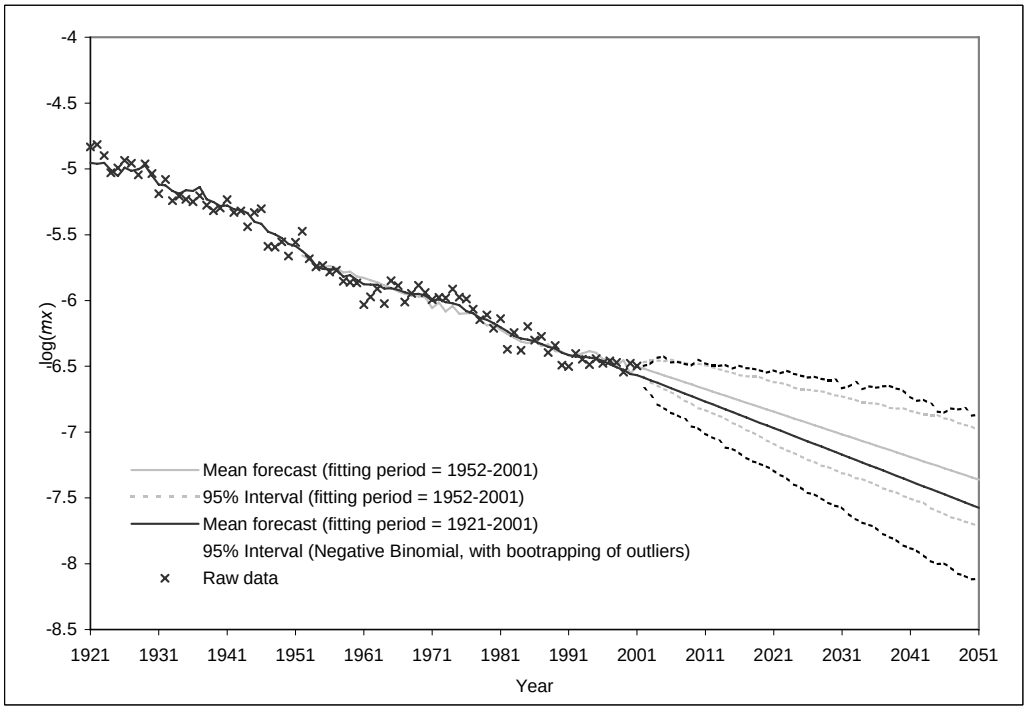


Age 5

Fig. 3.20. The Lee-Carter projection under different fitting periods, female.

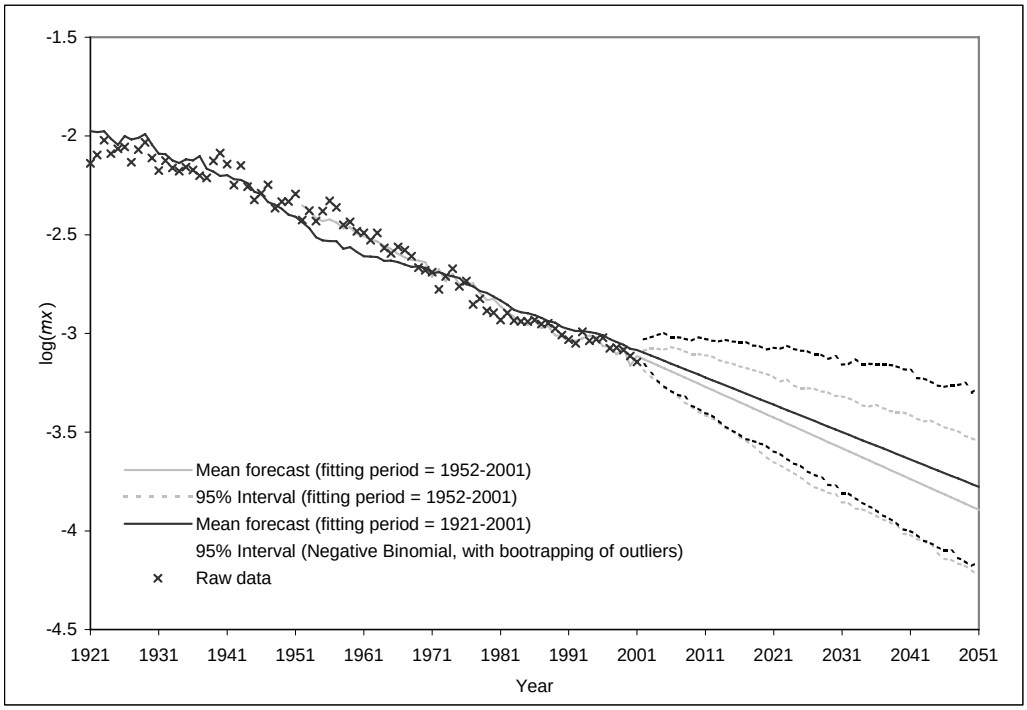


Age 30

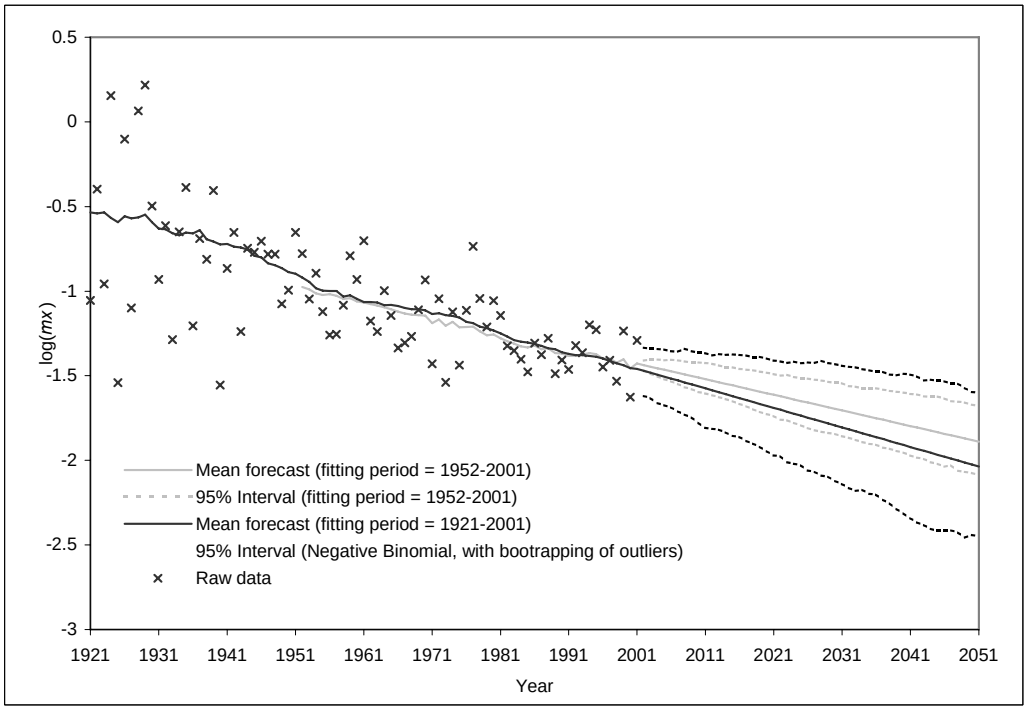


Age 45

Fig. 3.20 (cont'd). The Lee-Carter projection under different fitting periods, female.



Age 80



Age 95

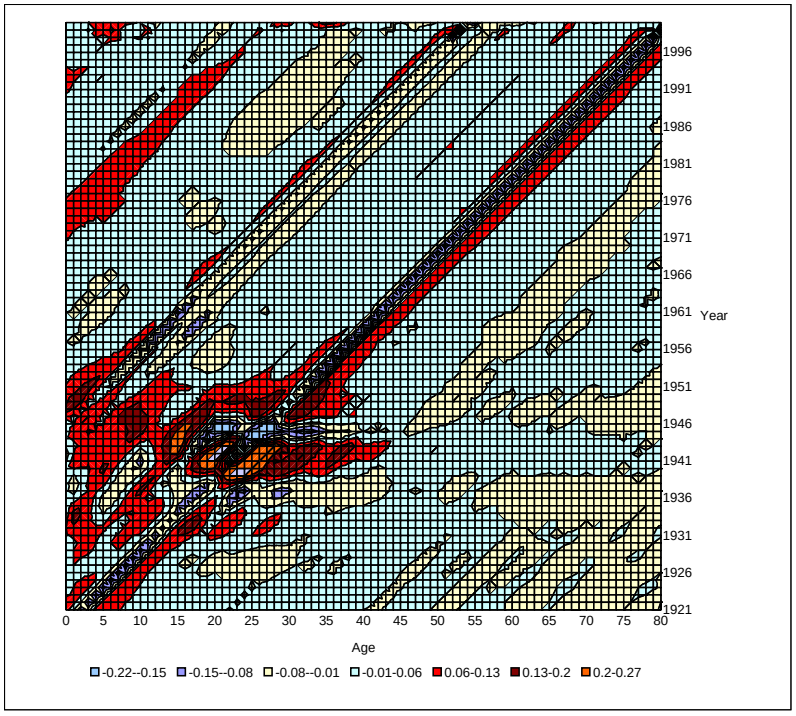
Fig. 3.20 (cont'd). The Lee-Carter projection under different fitting periods, female.

3.7 COHORT EFFECTS

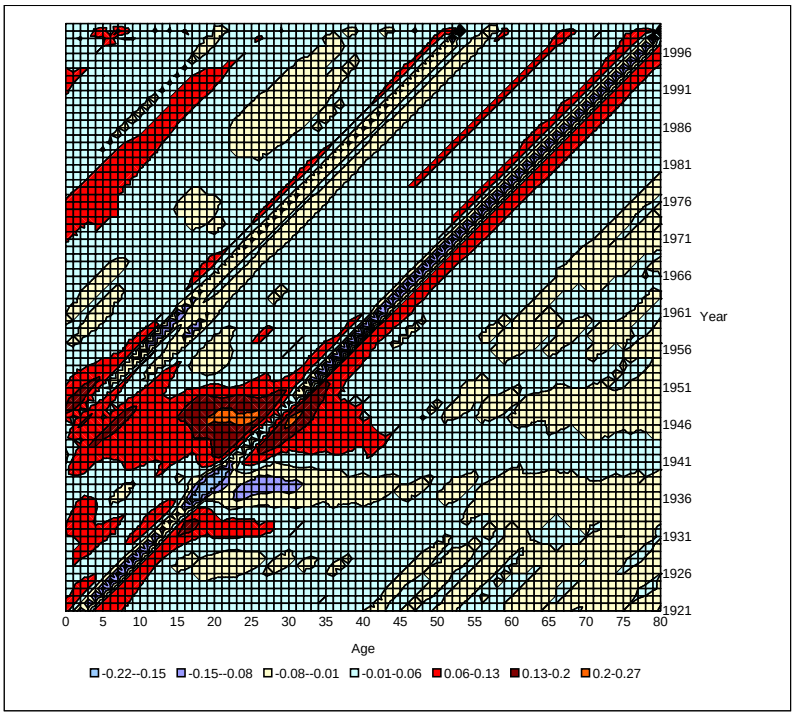
Recall that in all versions of Lee-Carter, future death rates are assumed to be driven by a linear stochastic time-series; and in P-splines regression, the penalty order n = 2 gives a linear forecast. In other words, these models assume that the percentage reduction of age-specific death rate is constant over time, and thus in a three dimensional data analysis (x: age, y: time, z: percentage reduction of m_x), we should expect the pattern to be highly vertical. Figure 3.22 shows that, however, the pattern based on the English and Welsh mortality is rather diagonal. These diagonals, from bottom left to top right, show cells for succeeding ages for the same year of birth, and therefore are clear evidence of the cohort, or year of birth, effects in the reduction of mortality rates. CMIB (2002) noticed that the pensioner experience also experienced similar cohort effects, which might be the reason that male mortality has improved more rapidly than the improvement contained in the projected "92" Series tables.

Dealing with cohort effects is generally difficult. Noting that the cohort effects are strongest in the cohort centered on births in 1926, CMIB (2002) revised the projected "92" Series tables with three sets of deterministic projections – The "Short Cohort" one extends the 1926 cohort period to 2010 while the "Medium Cohort" and "Long Cohort" projections extend it to 2020 and 2040 respectively. All of this was done by an *ad hoc* adjustment of annual improvement rates. CMIB (2005) and Richards et al. (2005) pointed out that we could choose age and year of birth (cohort) as the dimensions in the two dimensional *P*-splines regression, but fitting in this way might lead us to ignoring the period (time) effect, which could be equally, if not more, important.

Figure 3.23 suggests that cohort effects may also exist in the Canadian population mortality, but that the effects are comparatively mild. The only concern might be the particularly rapid improvements in the cohorts born in the few years either side of 1967. Even so, as these cohorts have been observed for no more than 40 years, any conclusion on the persistence of their year of birth effects would be premature. Furthermore, given that similar patterns in the English and Welsh mortality ceased in around 1991, we opt against introducing any *ad hoc* adjustment. However, we believe that in the ideal case, *both* period and cohort effect should be modeled simultaneously, and this requires further research on modifications of current stochastic mortality models.

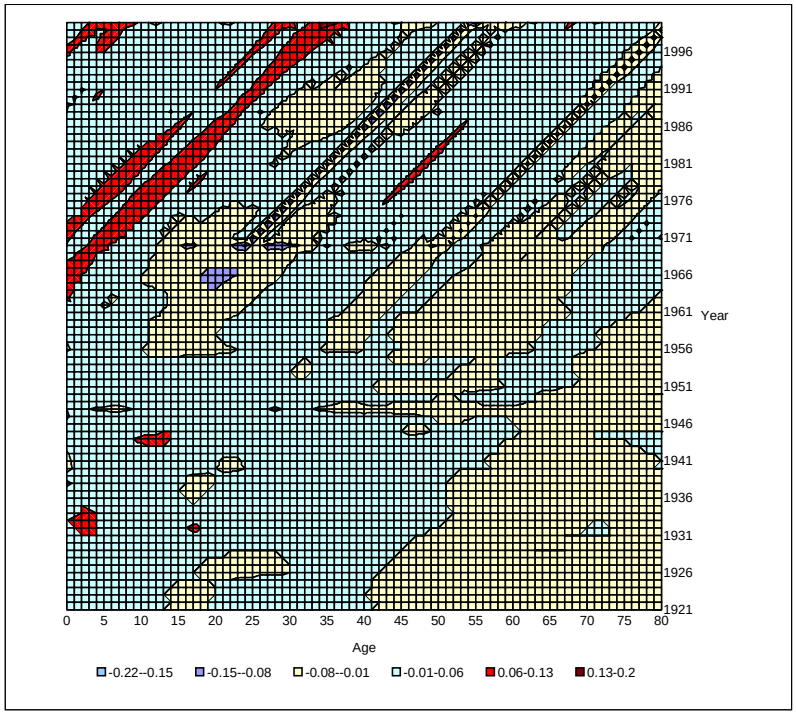


(a)

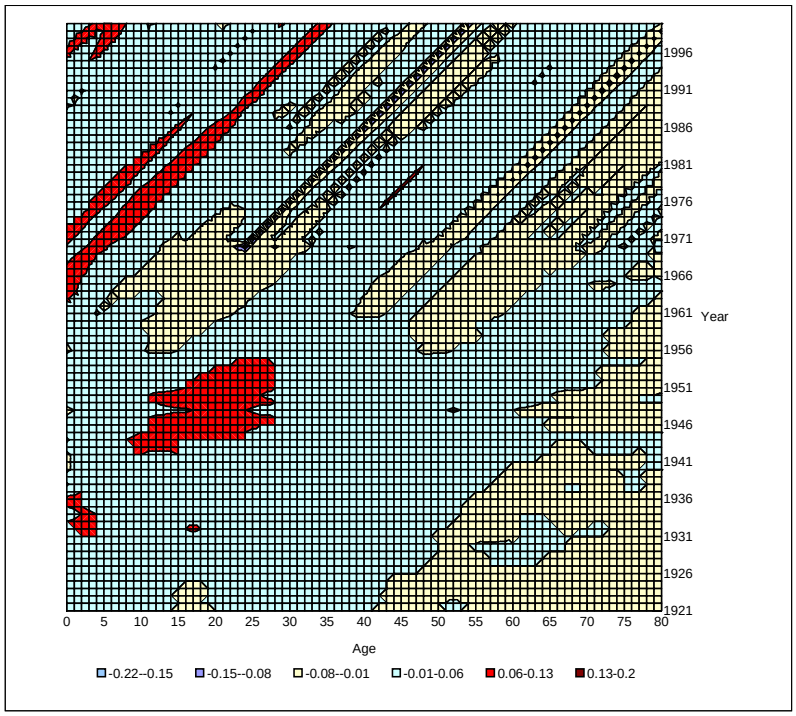


(b)

Fig. 3.22. Percentage improvement in age-specific death rates, (a) males and (b) females, English and Welsh population.



(a)



(b)

Fig. 3.23. Percentage improvement in age-specific death rates, (a) males and (b) females, Canadian population.

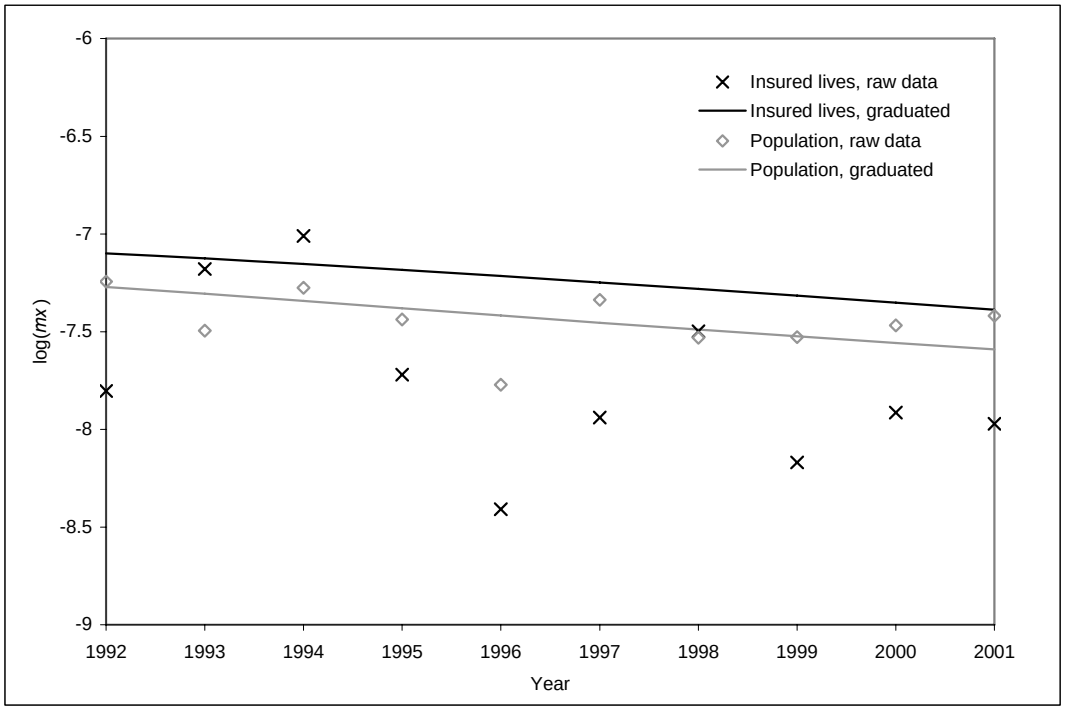
4. FORECASTING MORTALITY FOR CANADIAN INSURED LIVES

Having projected future values of mortality rates at the population level, we now proceed to the third fitting stage of the joint model. In this stage, we relate the experience of the population to that of the insured lives by means of a parametric equation. As this parametric equation has to be reasonably stable over time, it will be mainly based on the ultimate experience (smoker status combined) which has the largest exposure base and thus should form a more steady fit than any other duration. We believe that this approach makes more efficient use of the available data as it allows us to infer the relationship at ages and durations where data are sparse by borrowing strengths from the part of data where the number of exposure-to-risk is the most abundant. We can then impose further structure to reflect the select period information. Although, at this point we do not have sufficient data on the smoker/non-smoker data breakdown, as data becomes available adjustments can be superimposed on the fundamental parametric relationship for this also.

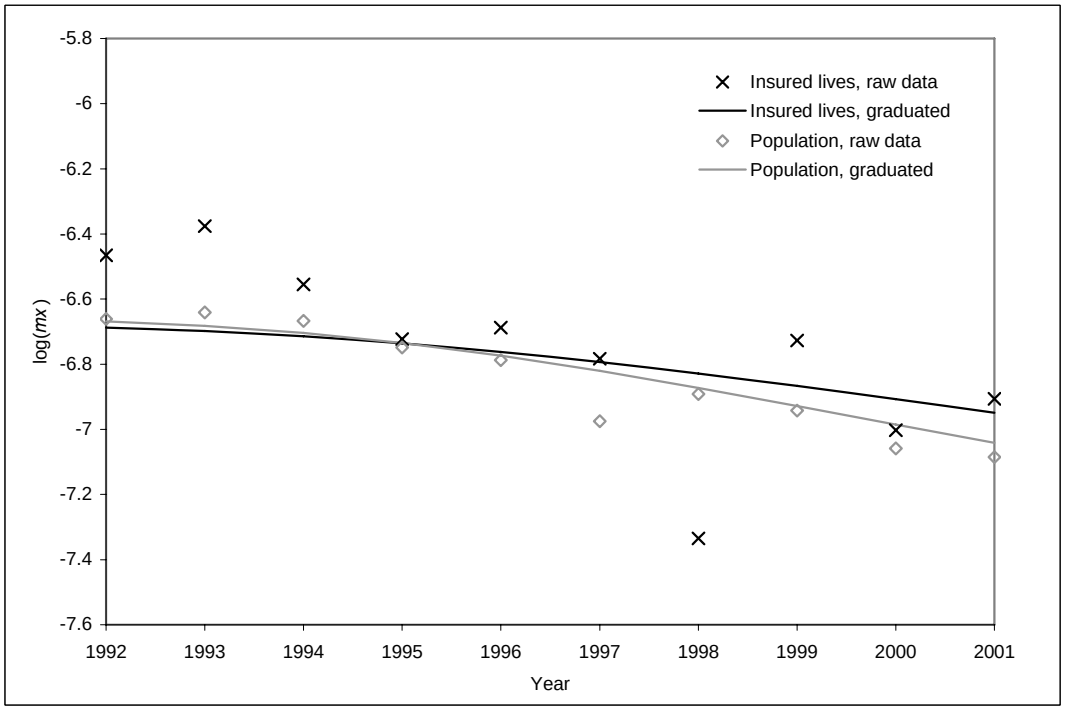
4.1 GRADUATION OF INSURED LIVES EXPERIENCES

We apply the two dimensional *P*-splines regression to the ultimate experience with combined smoker status. The graduation allows us to depict the improvements in the insured lives experiences, and gives us a smooth insured lives "base table" (shown in Appendix C) on which the ultimate improvement scales are to be applied.

For some insights in the specification of the parametric relationship, we compare graphically the graduated experiences of the insured lives and the population. Figures 4.1 and 4.2 show that both experiences follow closely to each other, although in some rare cases, e.g. the males at age 95, the relationship is not definite. It is interesting to note that the experience of the insured lives is not always lighter than that of the population. It is also noteworthy that the relationships are not necessarily in the form of level shifts -- see the males at age 30 for example. This suggests that the mortality of insured lives at different ages might be improving at a different speed to the population, and that this possibility should be allowed for in the specification of the relational model.

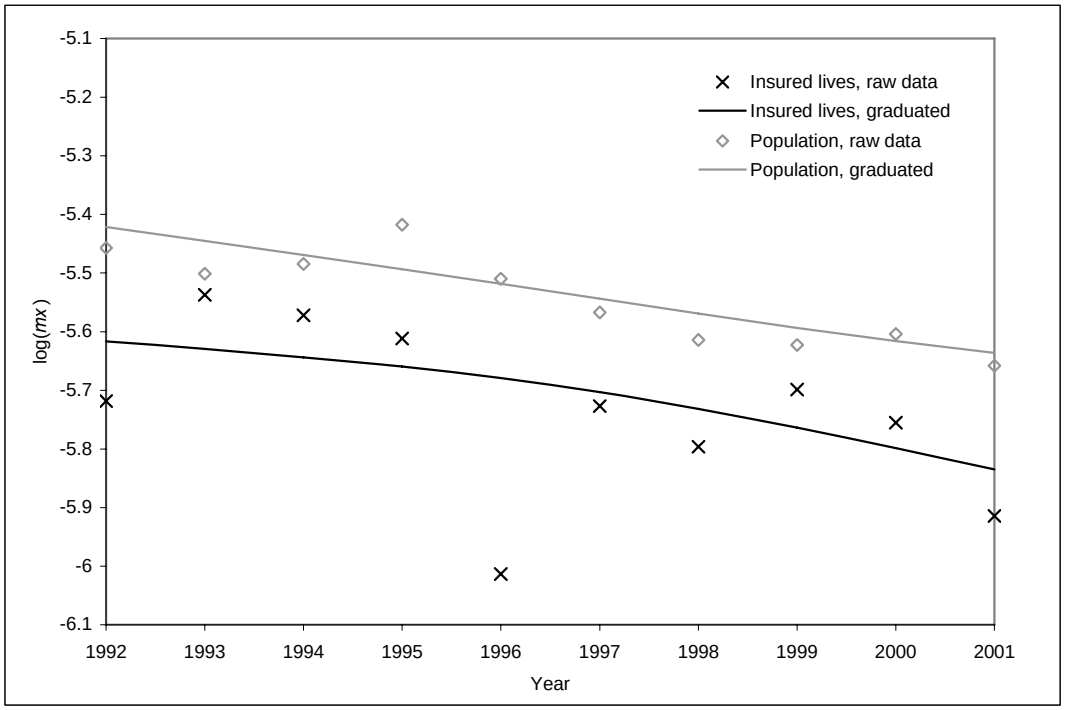


Age 16

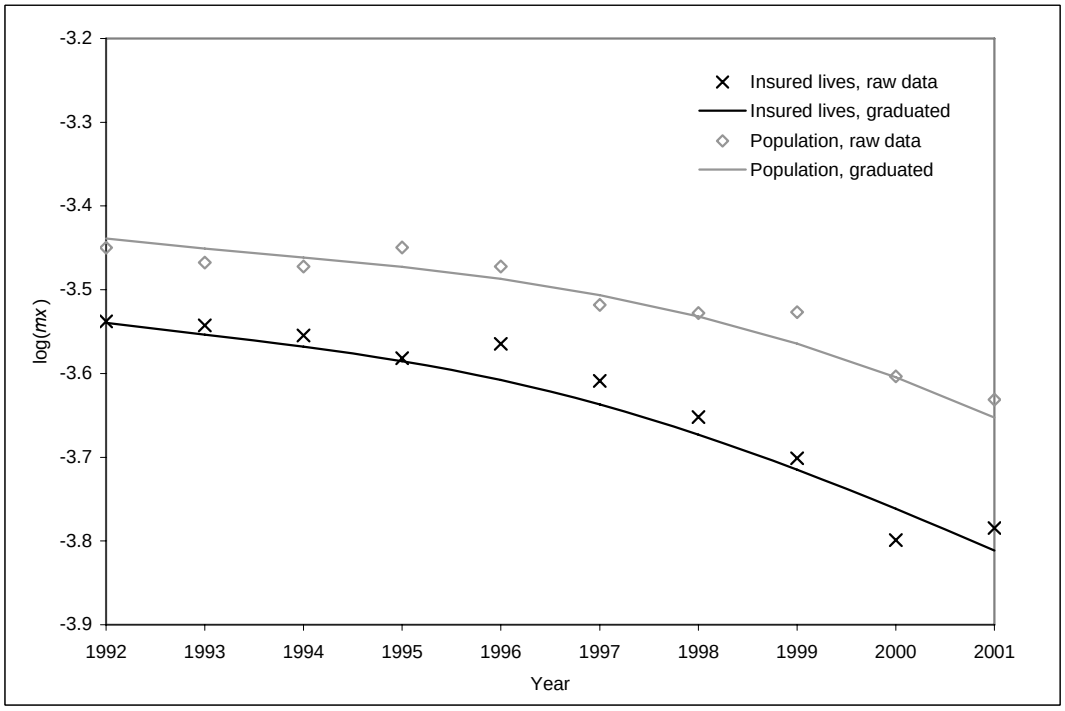


Age 30

Fig. 4.1. Comparison of insured lives and population experience, male.

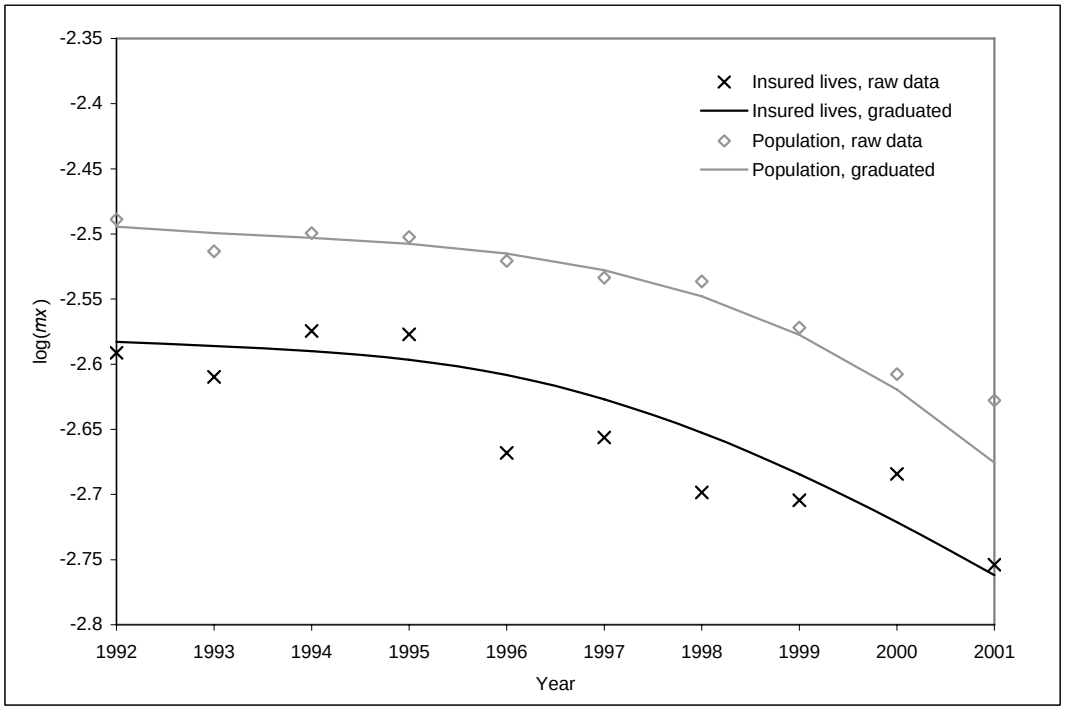


Age 50

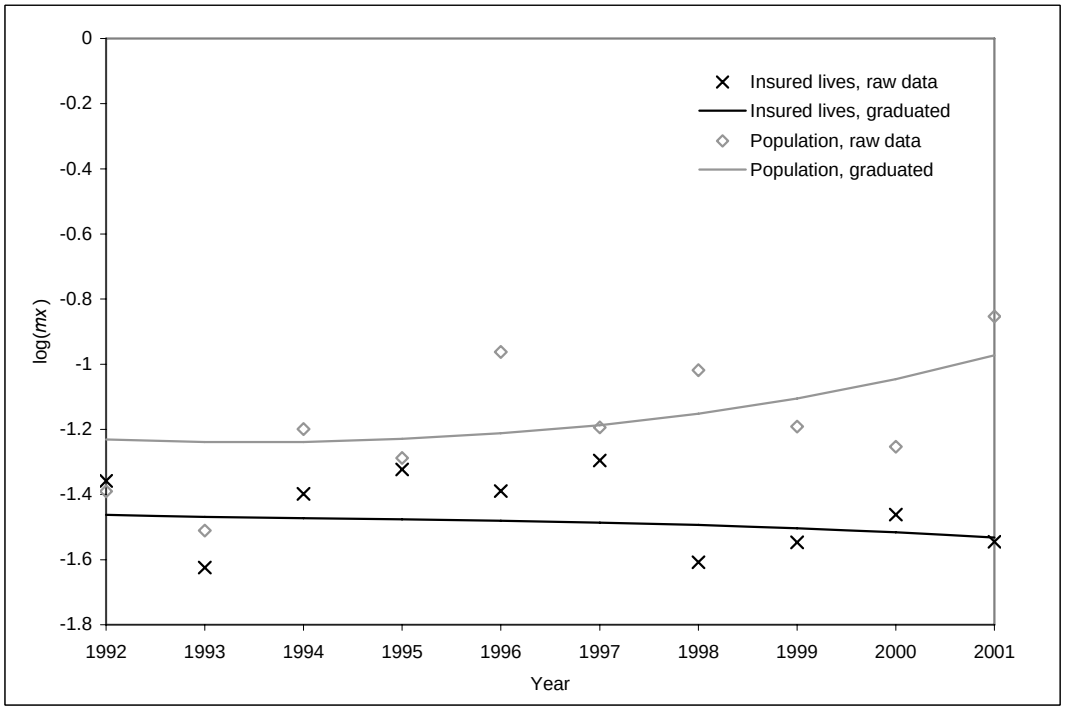


Age 70

Fig. 4.1 (cont'd). Comparison of insured lives and population experience, male.

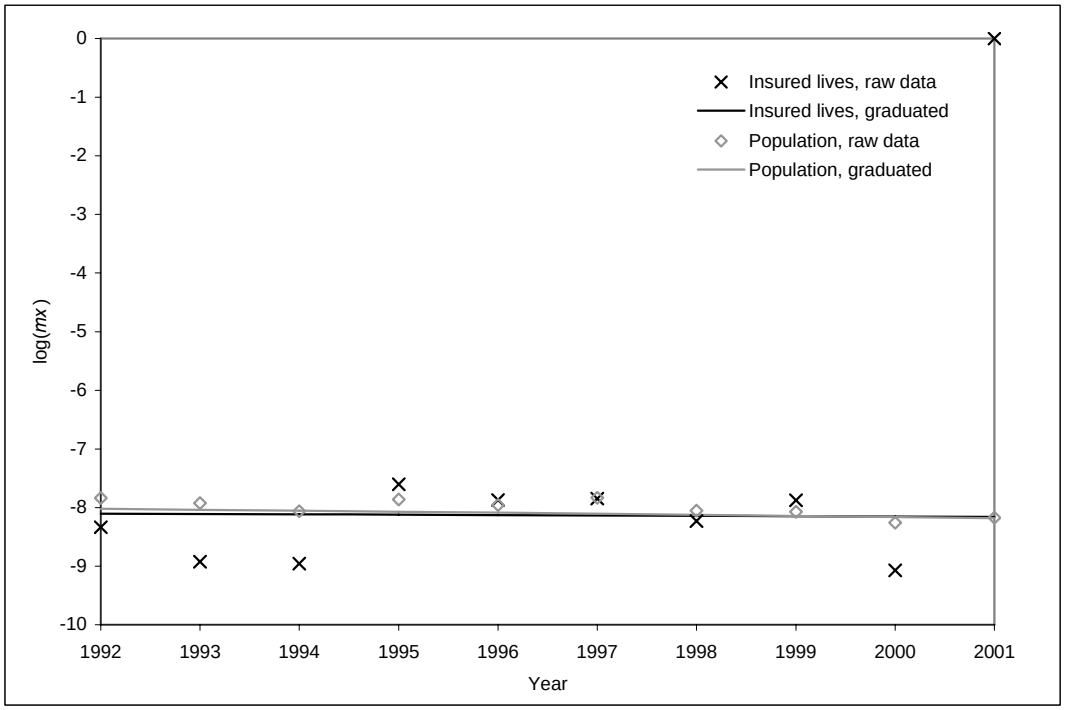


Age 80

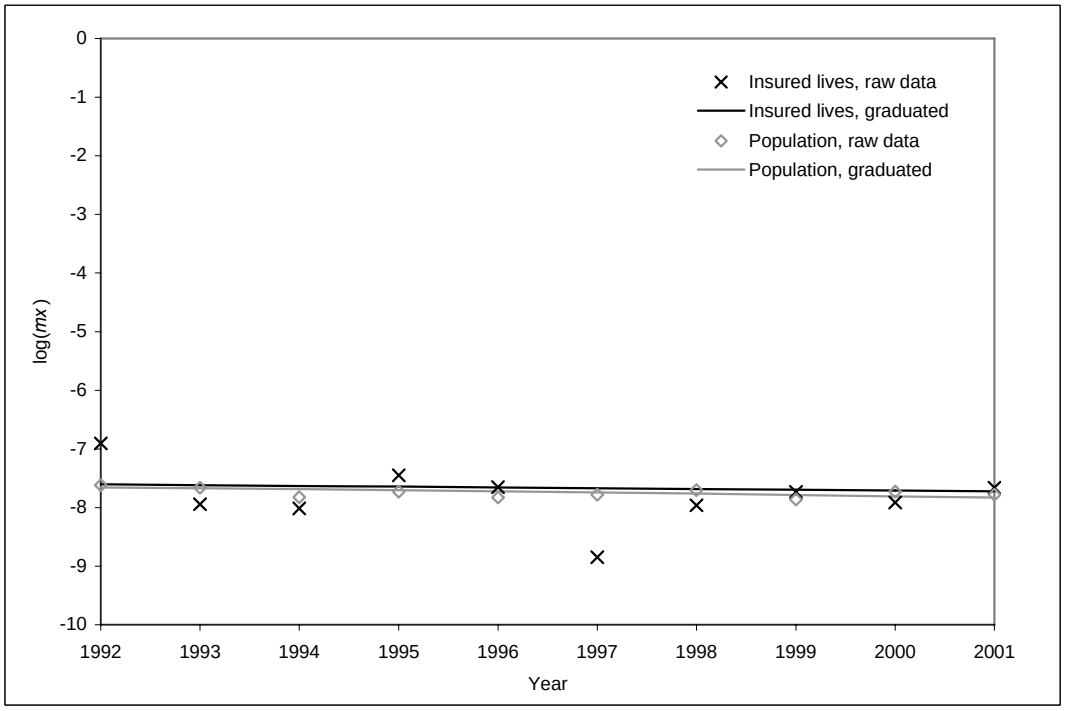


Age 95

Fig. 4.1 (cont'd). Comparison of insured lives and population experience, male.

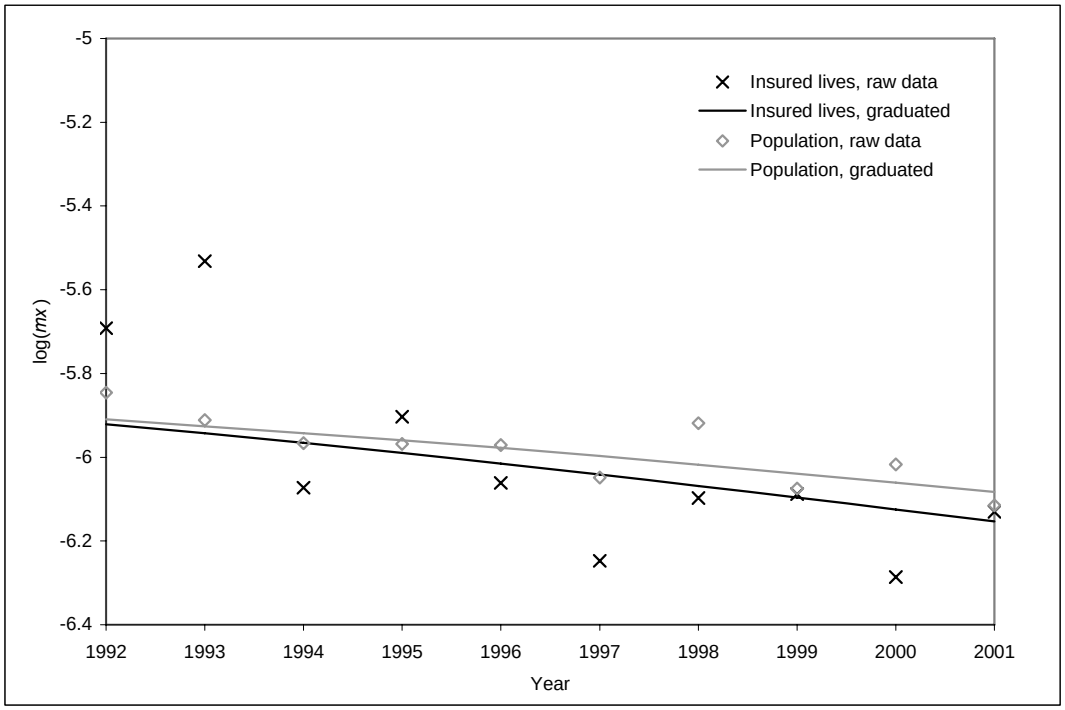


Age 16

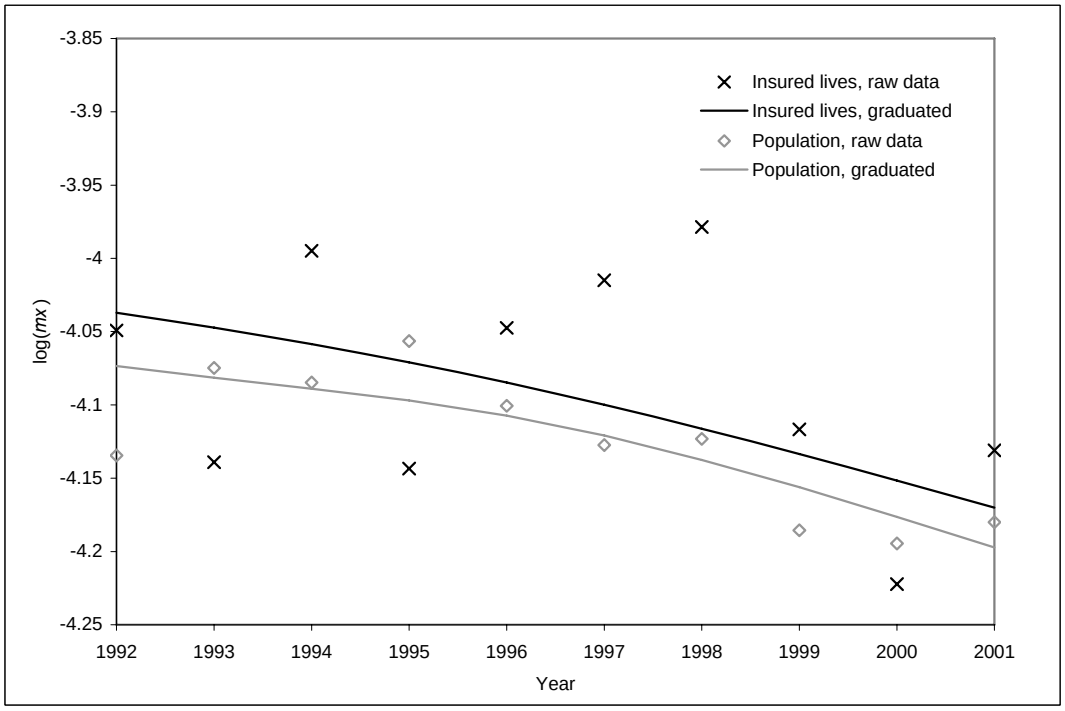


Age 30

Fig. 4.2. Comparison of insured lives and population experience, female.

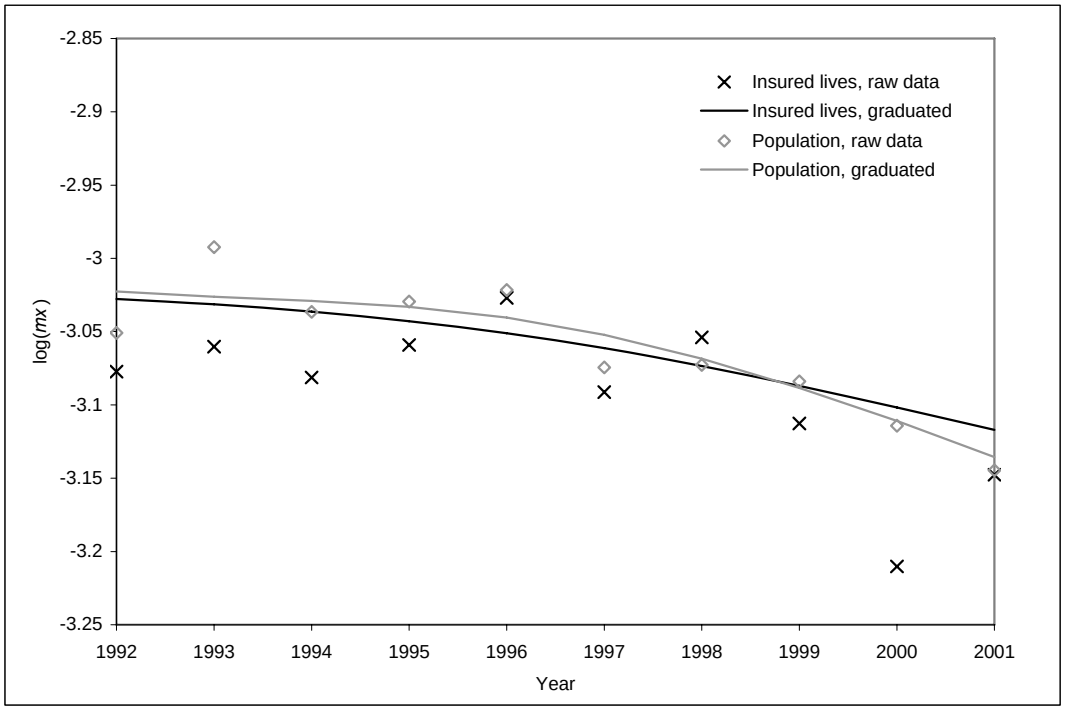


Age 50

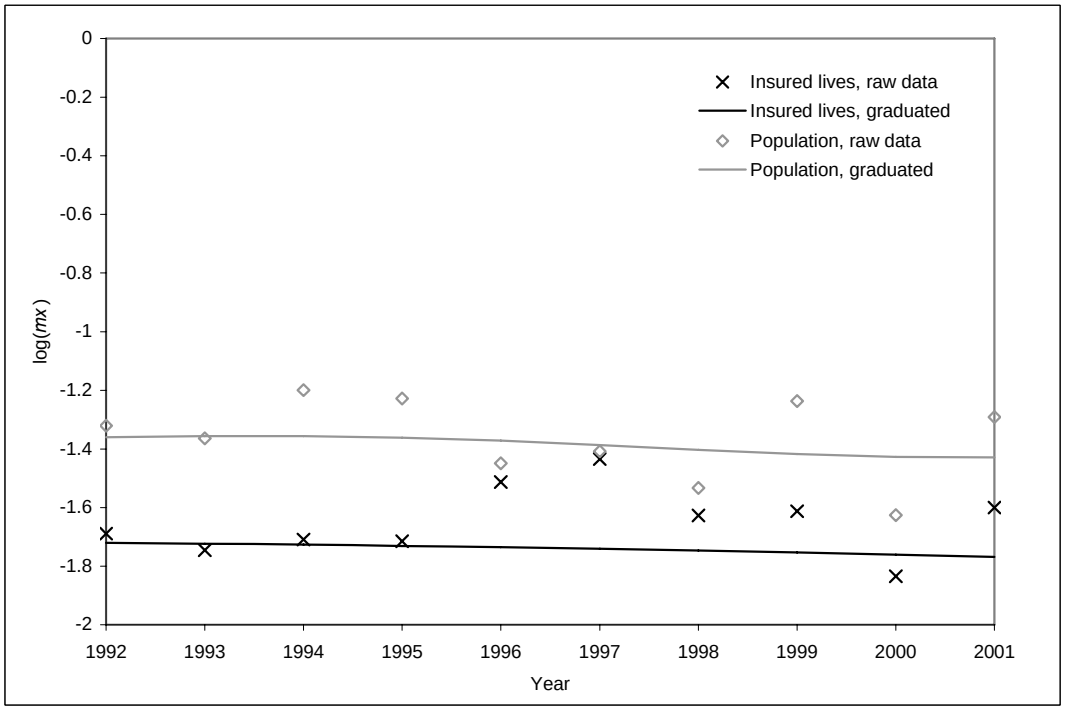


Age 70

Fig. 4.2 (cont'd). Comparison of insured lives and population experience, female.



Age 80



Age 95

Fig. 4.2 (cont'd). Comparison of insured lives and population experience, female.

4.2 IMPROVEMENT SCALES FOR THE INSURED LIVES

Having noted the properties of the relationships, we resort here to the Brass-type model (Brass, 1975), which has been an important tool for gaining an understanding of mortality conditions when the required data is limited, deficient or even non-existent. For example, the United Nations (1997) recommended the use of this model in relating death rates of a single population to a standard mortality schedule constructed by examining mortality rates from various low-mortality countries. Brouhns et al. (2002) also employed this model in relating the mortality experience of the Belgian annuitants to that of the whole Belgian population.

Mathematically, the model can be expressed as

$$f\left(m^*_{x,t}\right) = h_{1,x} + h_{2,x} f\left(m_{x,t}\right) + \varepsilon_{x,t}\,, \tag{4.1}$$

where $m^*_{x,t}$ denotes the death rate of the population under study (the ultimate insured lives experience); $m_{x,t}$ denotes the death rate of the reference population (the general population); $h_{1,x}$ and $h_{2,x}$ are age-specific parameters, assumed to be invariant over time; $\varepsilon_{x,t}$ is the error term, assumed to be normally distributed with mean zero and constant variance $\sigma^2_{\varepsilon,x}$; and $f(\cdot)$ is a function, typically specified by either ln(·) or logit(·). For mathematical tractability, we set $f(\cdot) = \ln(\cdot)$, which gives

$$\ln\left(m^*_{x,t}\right) = h_{1,x} + h_{2,x} \ln\left(m_{x,t}\right) + \varepsilon_{x,t}\,. \tag{4.2}$$

The model allows $m^*_{x,t}$ and $m_{x,t}$ to vary at a different speed and / or direction, specified by parameter $h_{2,x}$, which is a more general approach than that in Currie *et al* (2004). More specifically, the effect of parameter $h_{2,x}$ can be segregated into five cases.

i. $h_{2,x} > 1$
   The experience of the insured lives is improving faster than that of the population.

ii. $h_{2,x} = 1$
   The experiences of the population and the insured lives are moving at the same speed.

iii. $-1 > h_{2,x} > 0$

The experience of the insured lives is improving slower than that of the population.

iv. $h_{2,x} = 0$

The insured lives experience shows no improvement / deterioration relative to the population experience.

v. $h_{2,x} < 0$

The insured lives experience is deteriorating, relative to the population.

Parameters in equation (4.2) can be readily estimated by the OLS method. Figure 4.3 shows the estimates of parameter $h_{2,x}$. For males, the estimate of $h_{2,x}$ is less than zero for $x > 93$, indicating that the experience of the population and the insured lives are moving in opposite directions in that particular age range. To avoid any counter-intuitive projections at these ages, where the data is sparse, we replace all of the negative values by zero.

Using equation (4.2), we estimate $\ln\left(m^*_{x,2001+s}\right)$ and $\ln\left(m^*_{x,2001}\right)$, and by differencing the estimates, we obtain

$$\hat{\ln}\left(m^*_{x,2001+s}\right) - \hat{\ln}\left(m^*_{x,2001}\right) = \hat{h}_{2,x}\left[\hat{\ln}\left(m_{x,2001+s}\right) - \ln\left(m_{x,2001}\right)\right], \tag{4.3}$$

which gives

$$\hat{IS}^{\text{insured}}_{2001}(x,s) = \exp\left(\hat{h}_{2,x}\right)\hat{IS}^{\text{population}}_{2001}(x,s), \tag{4.4}$$

where $\hat{IS}^{\text{insured}}_{2001}(x,s)$ denotes the estimate of the improvement scale for the insured lives, using the graduated 2001 insured lives experience as the "base table". Note that parameter $h_{1,x}$, which measures the magnitude of parallel shifts, is irrelevant in the improvement scales.

Finally, recall that under Lee-Carter,

$$\hat{IS}^{\text{population}}_{2001}(x,s) = \exp\left(\hat{w}_x\, s\right). \tag{4.5}$$

The combination of equations (4.4) and (4.5) yields

$$\begin{aligned} \hat{IS}^{\text{insured}}_{2001}(x,s) &= \exp\left(\hat{h}_{2,x}\,\hat{w}_x\, s\right) \\ &= \exp\left(\hat{z}_x\, s\right). \end{aligned} \tag{4.6}$$

In other words, the improvement of the insured lives mortality can again be summarized by a single parameter vector. Having smoothed out the bumps in the crude estimates of z_x by a *B*-splines regression, we obtain the improvements scales applicable to the insured lives experience. The graduation of z_x is illustrated in Figure 4.4 and the graduated values of z_x are provided in Appendix D.

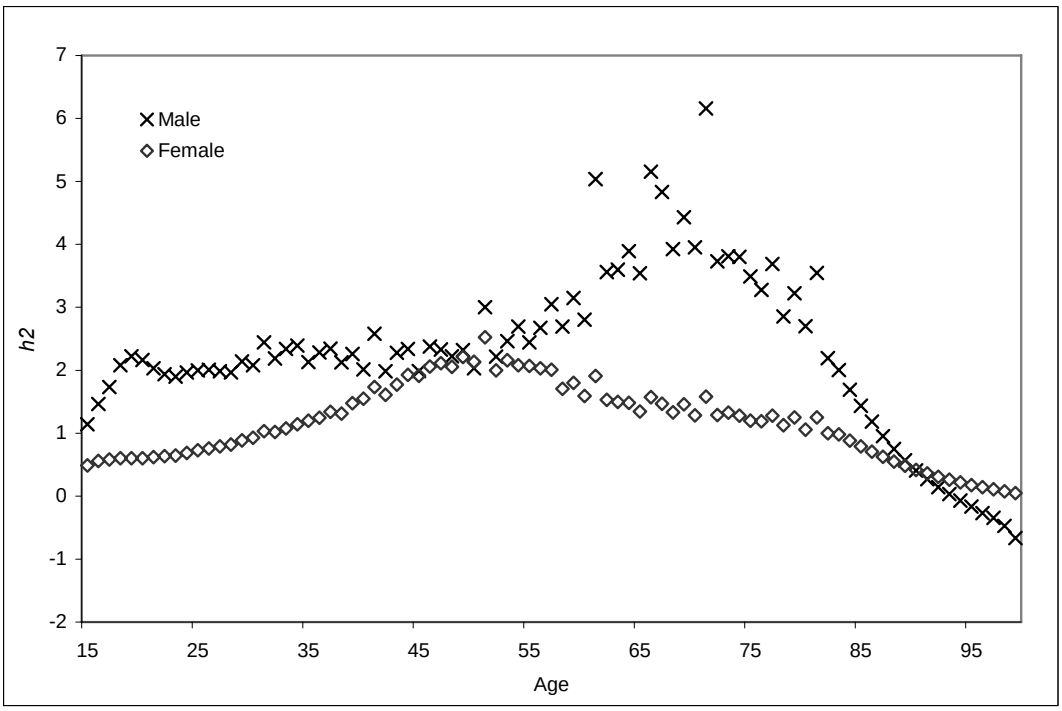


Fig. 4.3. Estimates of parameter $h_{2,x}$.

We now turn to the measure of variability. Assuming that the model specified by equation (4.2) is correct, the true value of $IS_{2001}^{\text{insured}}(x,s)$ can be written as

$$\ln\left[IS_{2001}^{\text{insured}}(x,s)\right]=\left(\hat{h}_{2,x}+\eta_{2,x}\right)\left\{\hat{\ln}\left[IS_{2001}^{\text{population}}(x,s)\right]+\iota_{x,s}\right\}+\varepsilon_{x,2001+s}-\varepsilon_{x,2001}\,,$$

(4.7)

where $\eta_{2,x}$ and $\iota_{x,s}$ denote the error in estimating $h_{2,x}$ and the logarithm of $IS_{2001}^{\text{population}}(x,s)$ respectively. Hence, the forecast error of the logarithm of $IS_{2001}^{\text{insured}}(x,s)$ is given by

$$E_{x,s}=\eta_{2,x}\,\hat{\ln}\left[IS_{2001}^{\text{population}}(x,s)\right]+\iota_{x,s}\,\hat{h}_{2,x}+\eta_{2,x}\iota_{x,s}+\varepsilon_{x,2001+s}-\varepsilon_{x,2001}\,,\qquad(4.8)$$

Assuming independence between $\eta_{2,x}$ and $\iota_{x,s}$, and using the results in Section 3.5, the variance of the forecast error can be expressed as

$$\sigma^2_{E,x,s} \approx \sigma^2_{\eta,2,x}\left(\hat{w}_x\, s\right)^2 + \left(\hat{h}^2_{2,x} + \sigma^2_{\eta,2,x}\right)\hat{v}_x\, s + 2\sigma^2_{\varepsilon,x}\,, \tag{4.9}$$

where $\sigma^2_{\eta,2,x}$ denoted the error variance of $\hat{h}_{2,x}$. The estimate of $\sigma^2_{\eta,2,x}$ can be readily obtained by the OLS method. The first term in equation (4.9) has negligible effect on the overall error variance and hence the equation can be further simplified as

$$\sigma^2_{E,x,s} = u_{1,x} + u_{2,x}s\,, \tag{4.10}$$

where

$$u_{1,x} = 2\sigma^2_{\varepsilon,x}\,, \tag{4.11}$$

and

$$u_{2,x} = \left(\hat{h}^2_{2,x} + \sigma^2_{\eta,2,x}\right)\hat{v}_x\, s\,. \tag{4.12}$$

The error variance now consists of two parts. The first part, specified by parameter $u_{1,x}$, measures the uncertainties involved in establishing the relationship between the experiences of the insured lives and the population. This part is independent of s as the relationship is assumed to be fixed over time. The second part, specified by parameter $u_{2,x}$, measures the variability associated to the forecast of population improvement scales. This part tends to be large when the insured live experience is sensitive to the population mortality, i.e., when the value of $h_{2,x}$ is high. Following the stochastic structure of the population improvement scale, this part is proportional to s. The estimates of $u_{1,x}$ and $u_{2,x}$ are smoothed by the B-splines regression, illustrated in Figures 4.5 and 4.6, and the graduated values are included in Appendix D.

Based on the above results, the approximate 95% point-wise confidence interval of the insured lives improvement scales is given by

$$\left[\exp\left(\hat{z}_x\, s - 1.96\sqrt{\hat{u}_{1,x} + \hat{u}_{2,x}s}\right), \exp\left(\hat{z}_x\, s + 1.96\sqrt{\hat{u}_{1,x} + \hat{u}_{2,x}s}\right)\right]. \tag{4.13}$$

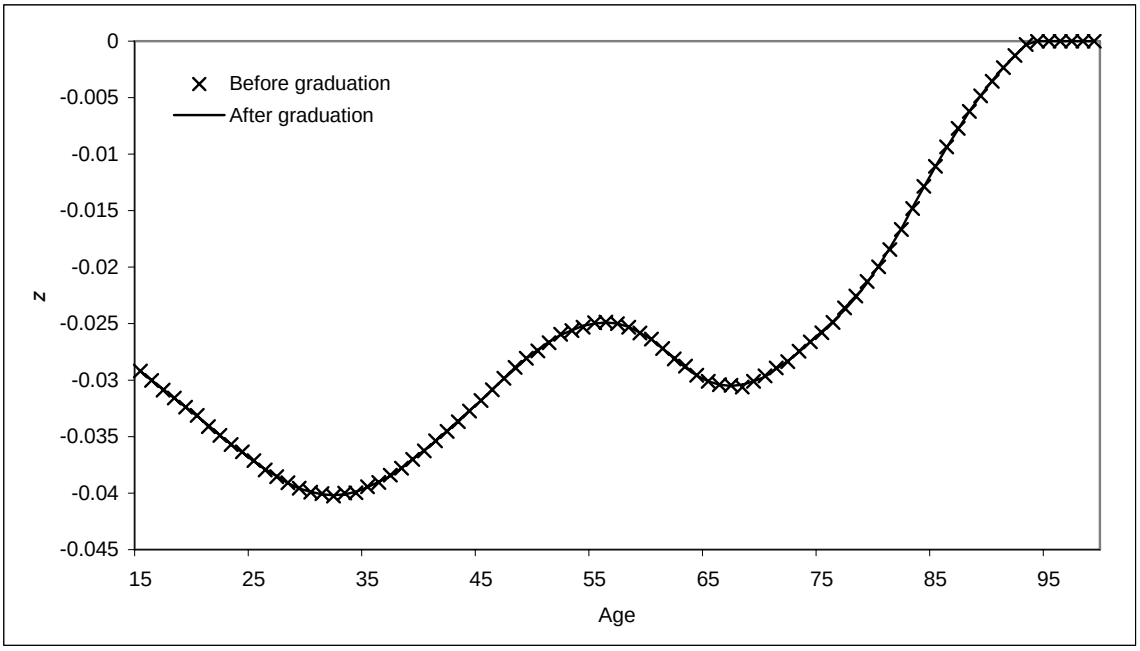
As the ultimate experience consists of no mortality information for age 0 to 14, we are unable to derive precise parametric relationships for these ages. For completeness, we

estimate the improvement scales for age 0 to 14 based on the population improvement scales for these ages as well as the trend of parameter $h_{2,x}$. An additional margin is included in the parameter $u_{2,x}$ for $x = 0, 1, \ldots, 14$, to reflect the subjectivity involved in estimating the scales.

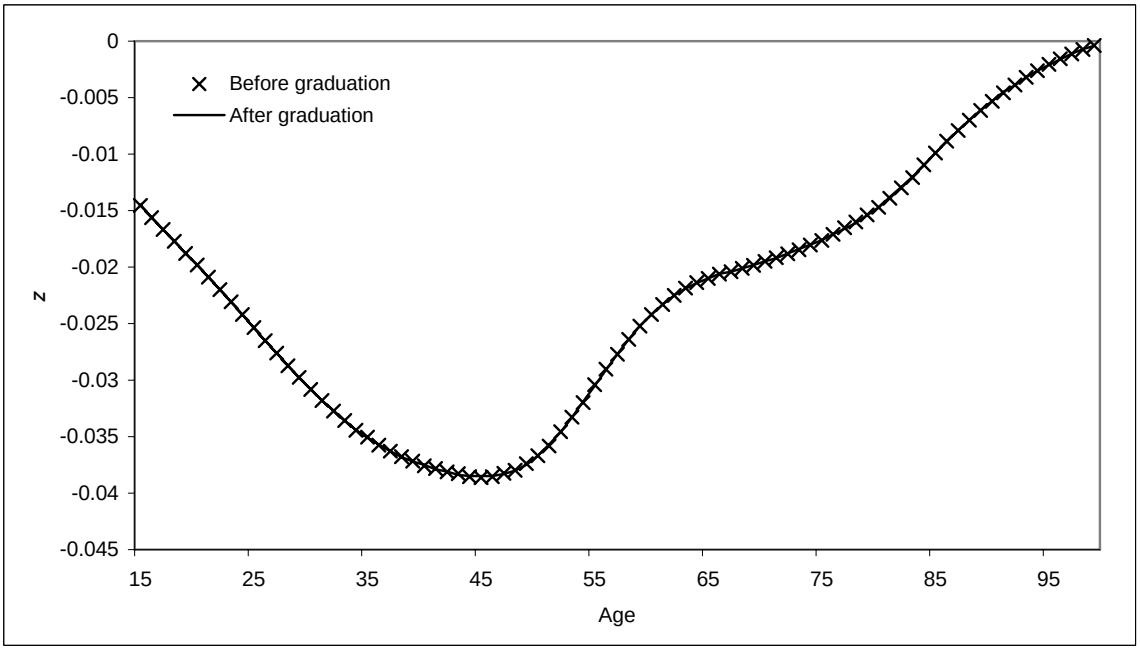
To illustrate the use of the insured lives improvement scales, let us consider the forecast of m_{80} (male, ultimate, composite smoker/non-smoker) in 2021. Appendix C gives $m_{80} = 0.063156$ in 2001 and Appendix D gives $z_{80} = -0.019931$, $u_{1,80} = 0.000924$ and $u_{2,80}=0.001597$. Hence, the "best estimate" and the confidence interval of m_{80} in 2021 is $0.063156 \times \exp(-0.019931 \times 20) = 0.042393$, and $0.063156 \times [0.470510, 0.957621] = [0.029716, 0.060480]$ respectively. Given a base table of p_x, we may use equation (3.59) to obtain the forecast of p_x.

To access the performance of the insured lives improvement scales, we conduct an ex-post analysis by fitting only to experiences prior to year 1993. Using the graduated 1993 insured lives experience as the "base table", we obtain forecasts of future insured lives mortality rates up to year 2001. These forecasts are compared with the actual experience during the ex post period. The comparison is shown in Figures 4.7 and 4.8.

In addition, we compare our scales with some previous deterministic improvement scales, namely, the AA Scale in the Society of Actuaries 1994 Group Annuity Mortality Table, the Improvement Factors in the Society of Actuaries 2001 Valuation Basic Experience Table, and the Reduction Factors in the Institute of Actuaries "92" Series Base Table. Mathematical formulae for these scales are reviewed in Appendix E. Once more, we apply these scales to the 1993 "base table", and this gives different forecasts of future rates, which are shown simultaneously in Figures 4.7 and 4.8.

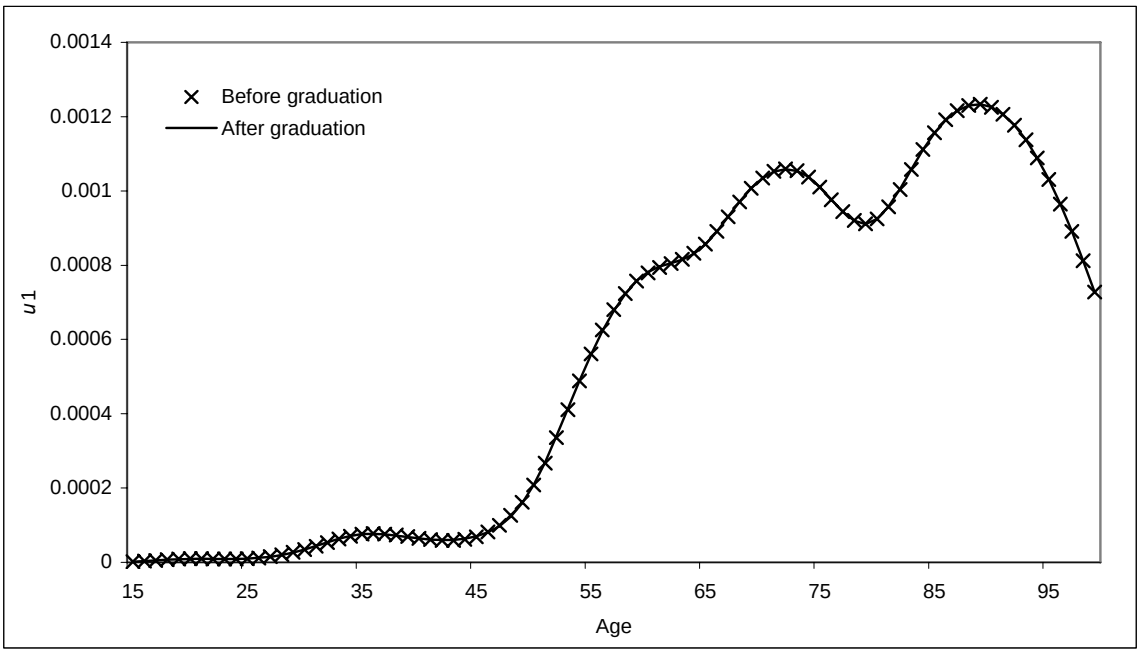


(a)

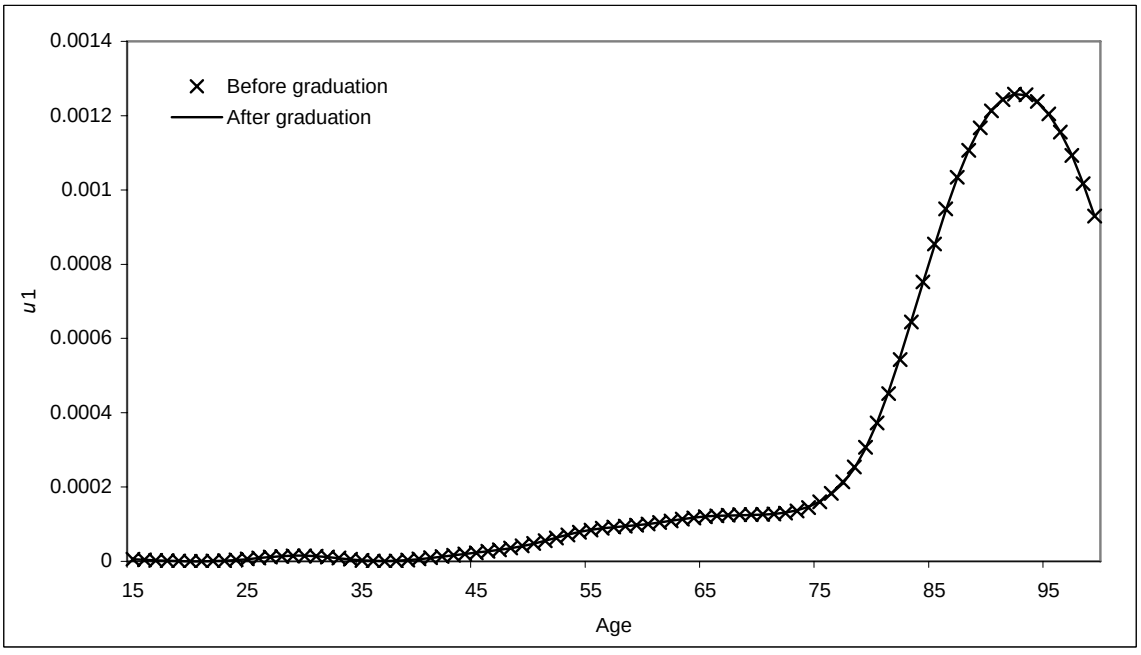


(b)

Fig. 4.4. Parameter z_x before and after graduation (a) male and (b) female.

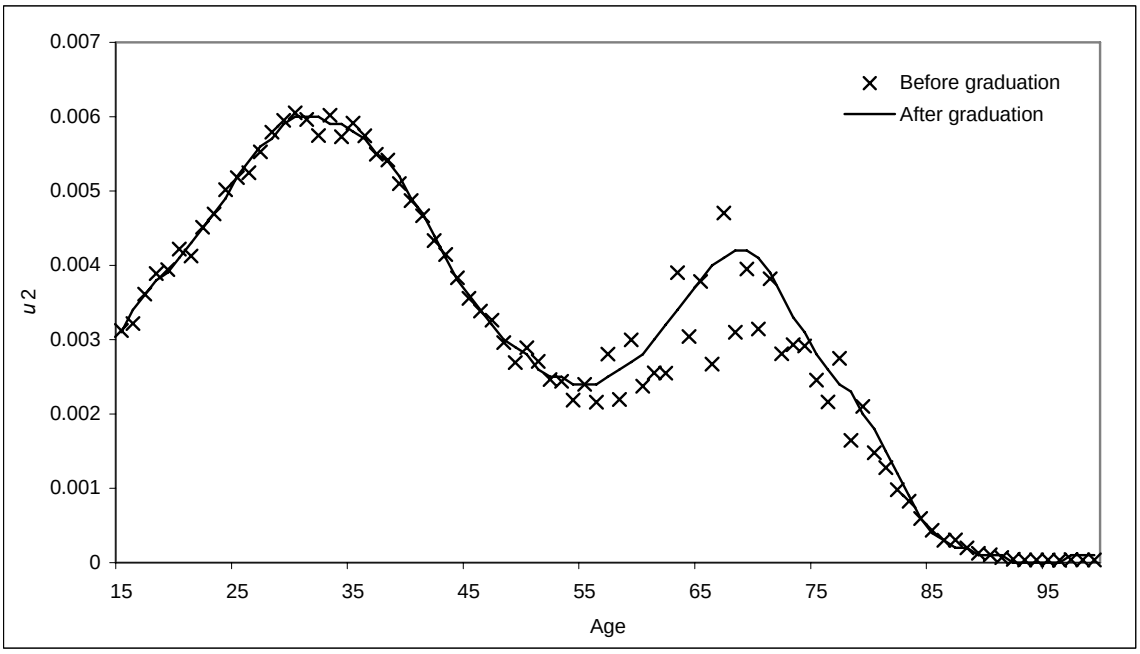


(a)

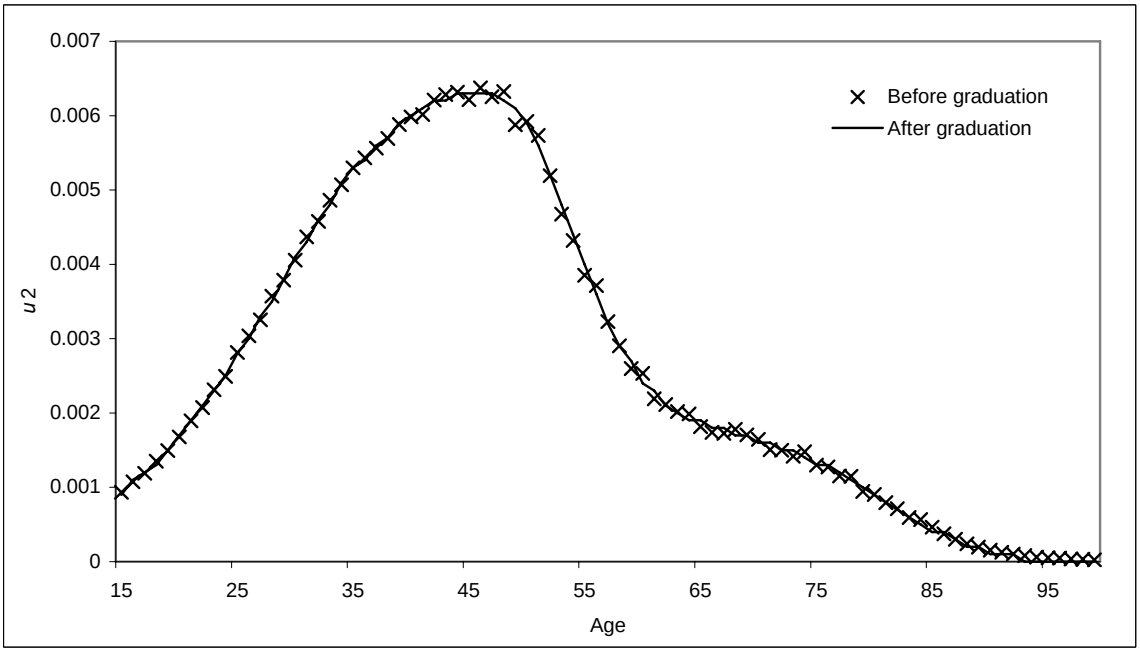


(b)

Fig. 4.5. Parameter $u_{1,x}$ before and after graduation (a) male and (b) female.

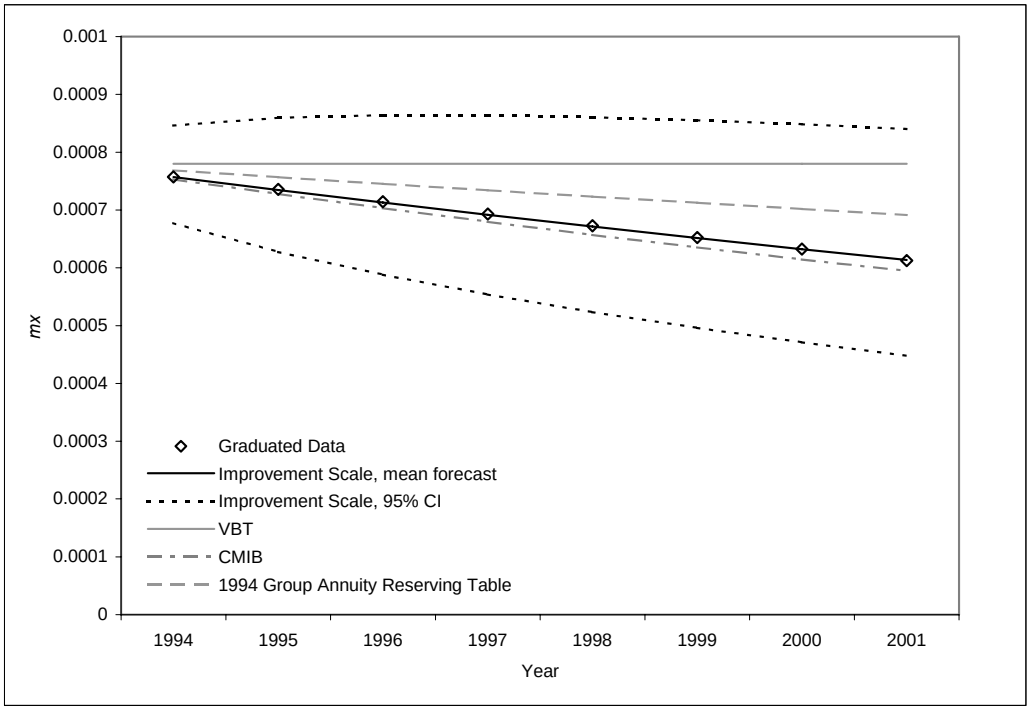


(a)

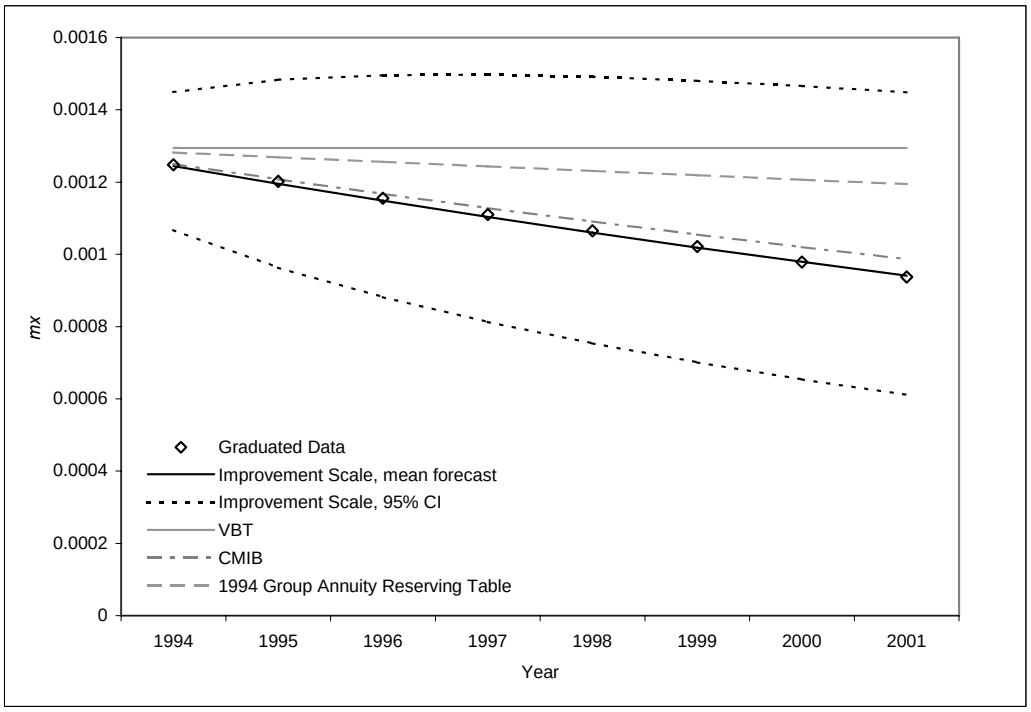


(b)

Fig. 4.6. Parameter $u_{2,x}$ before and after graduation (a) male and (b) female.

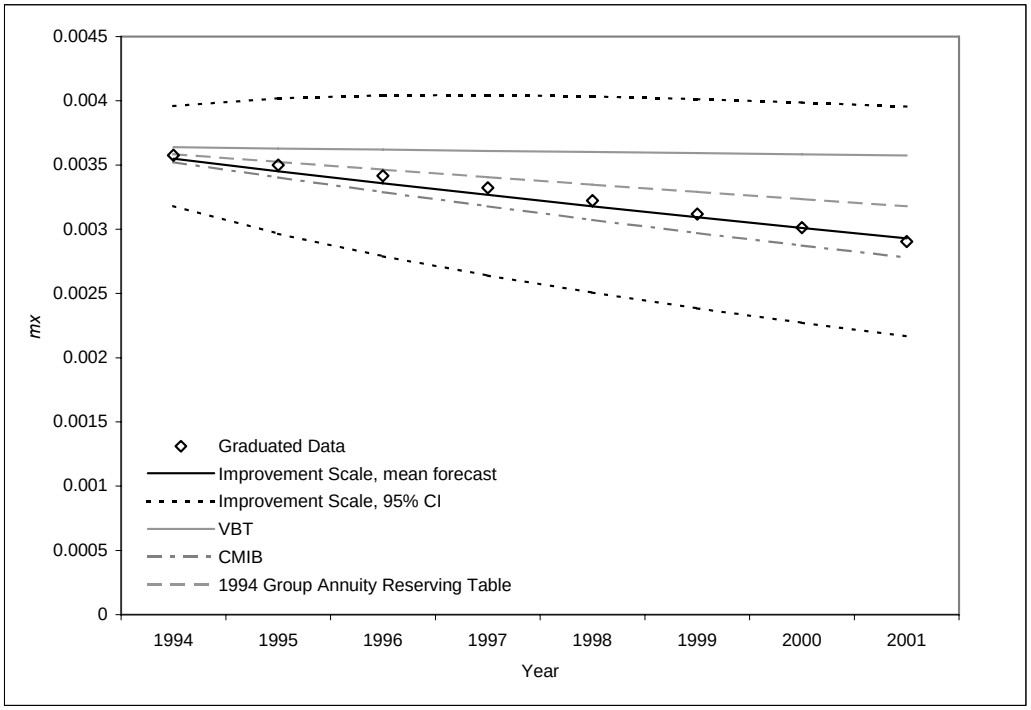


Age 16

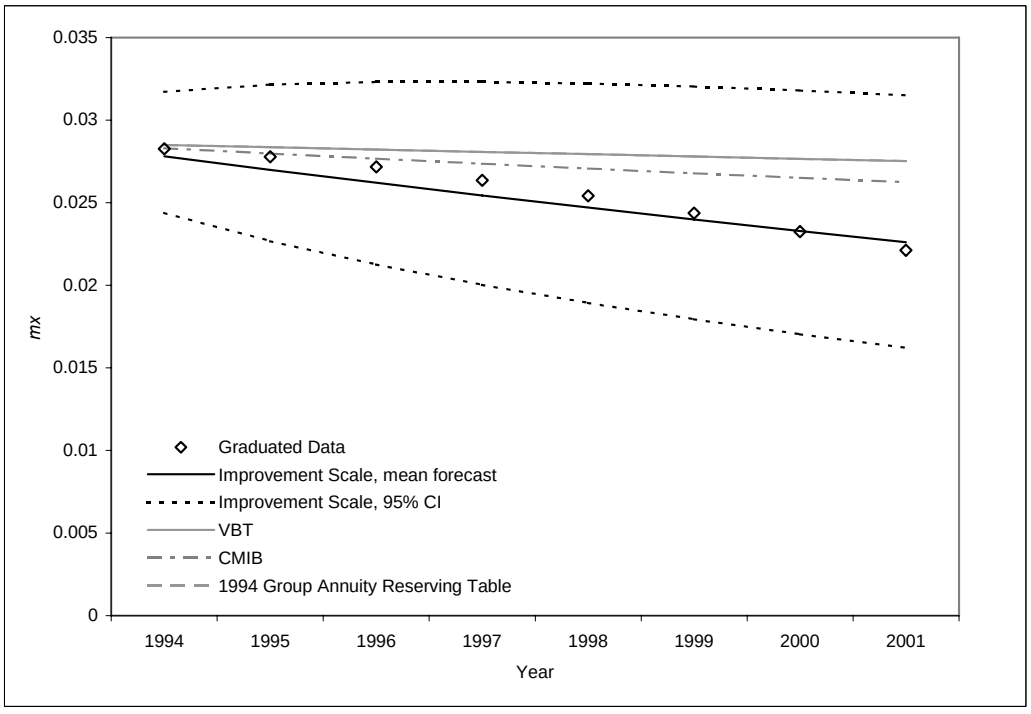


Age 30

Fig 4.7. Ex post analysis of insured lives improvement scales, male.

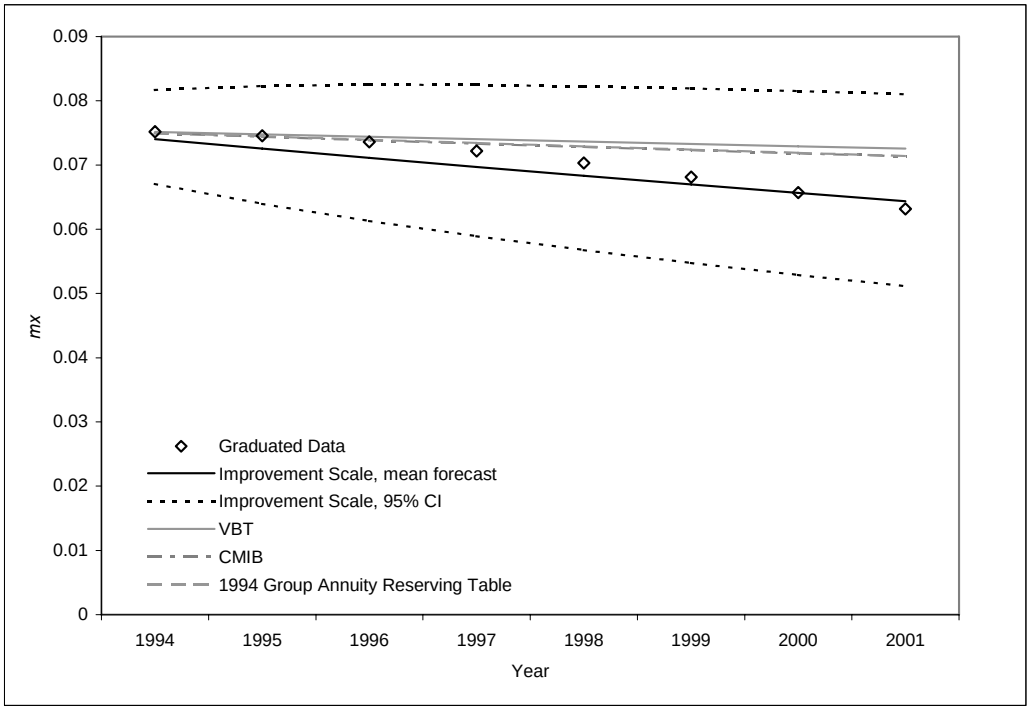


Age 50

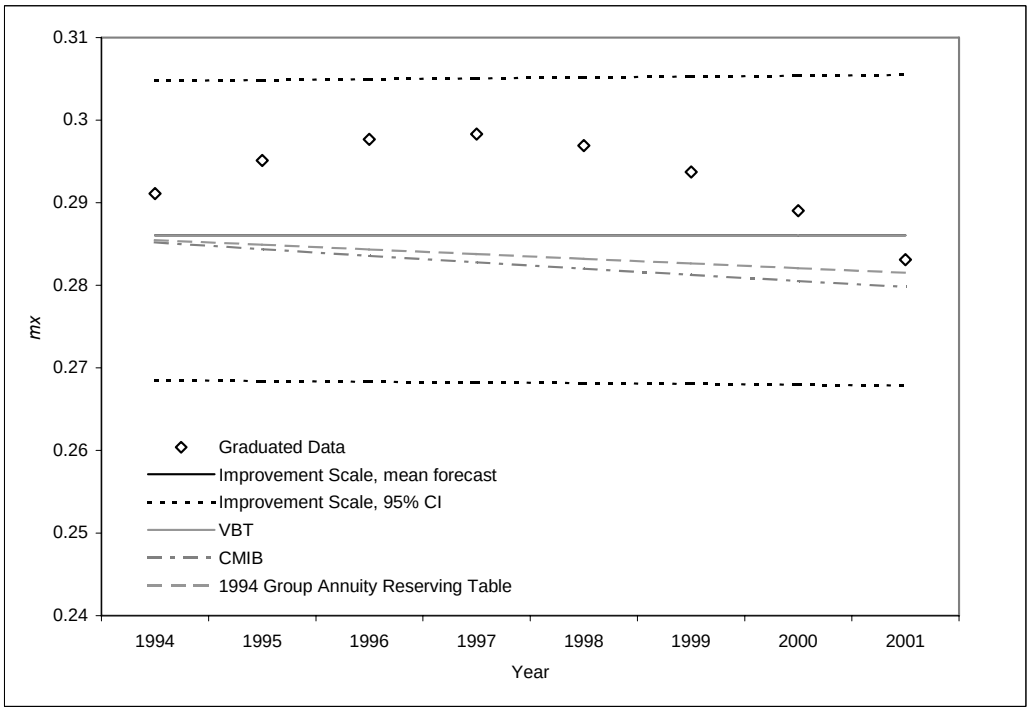


Age 70

Fig 4.7 (cont'd). Ex post analysis of insured lives improvement scales, male.

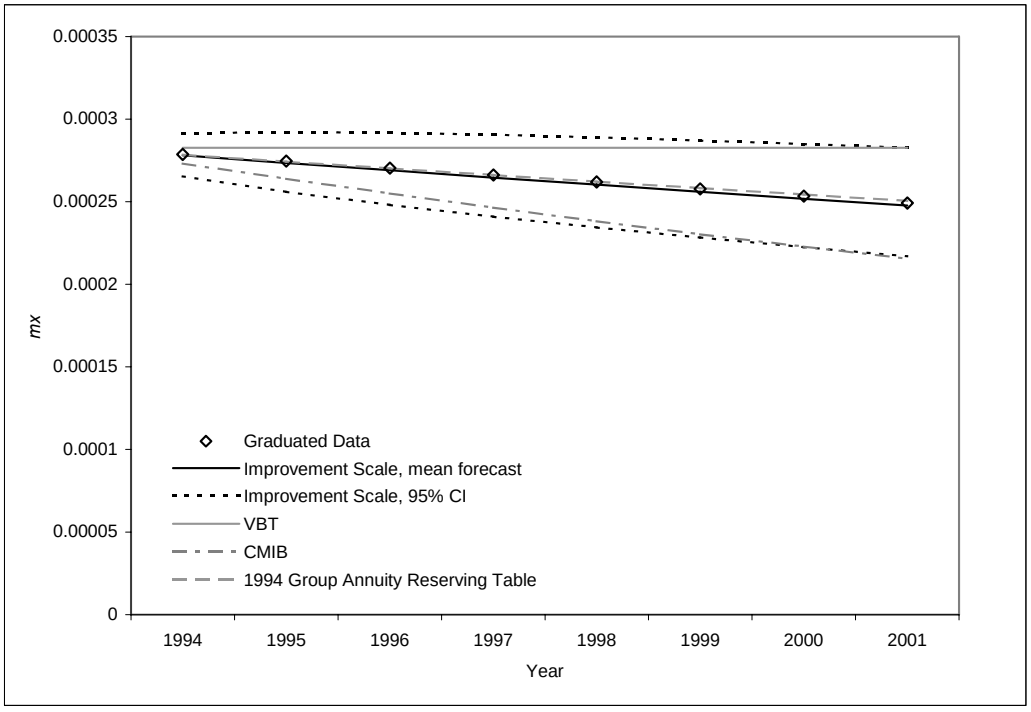


Age 80

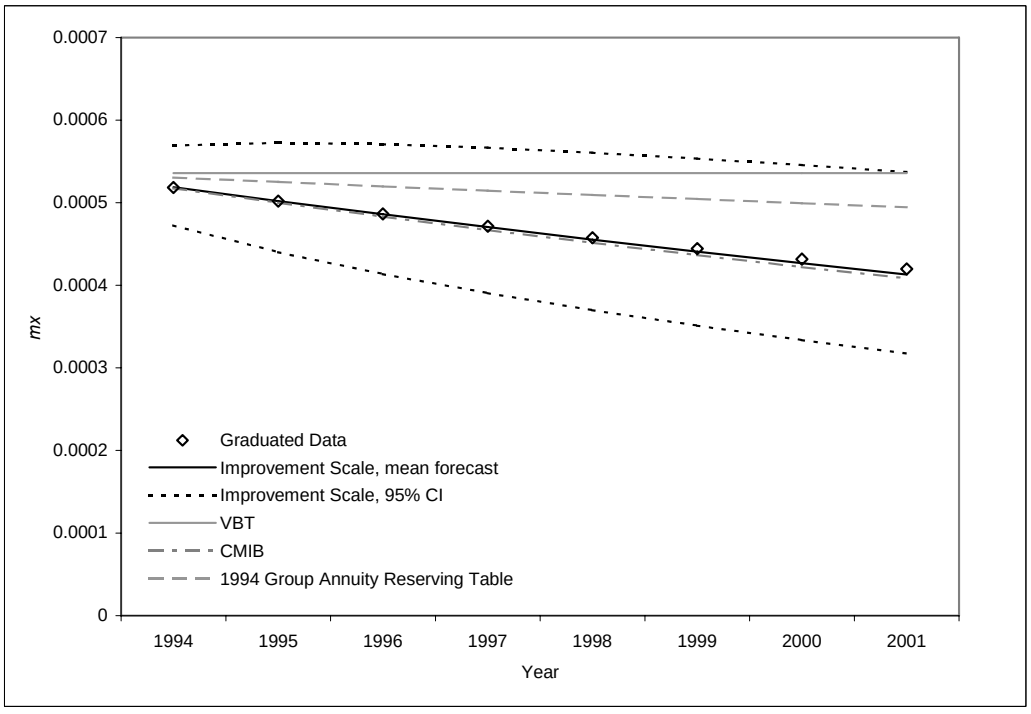


Age 95

Fig 4.7 (cont'd). Ex post analysis of insured lives improvement scales, male.

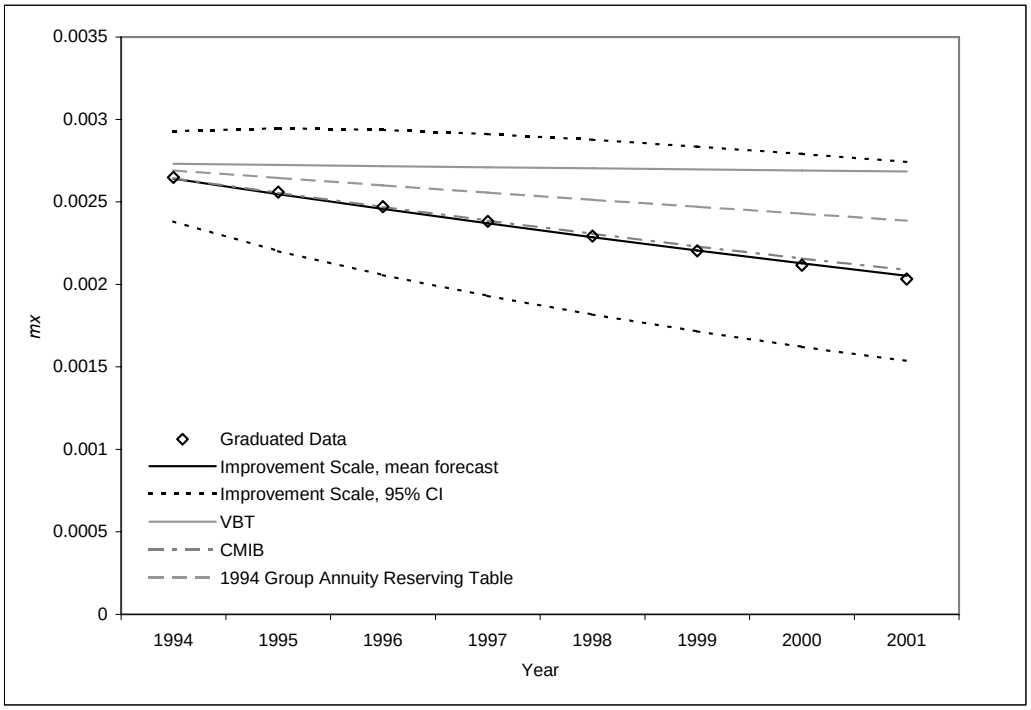


Age 16

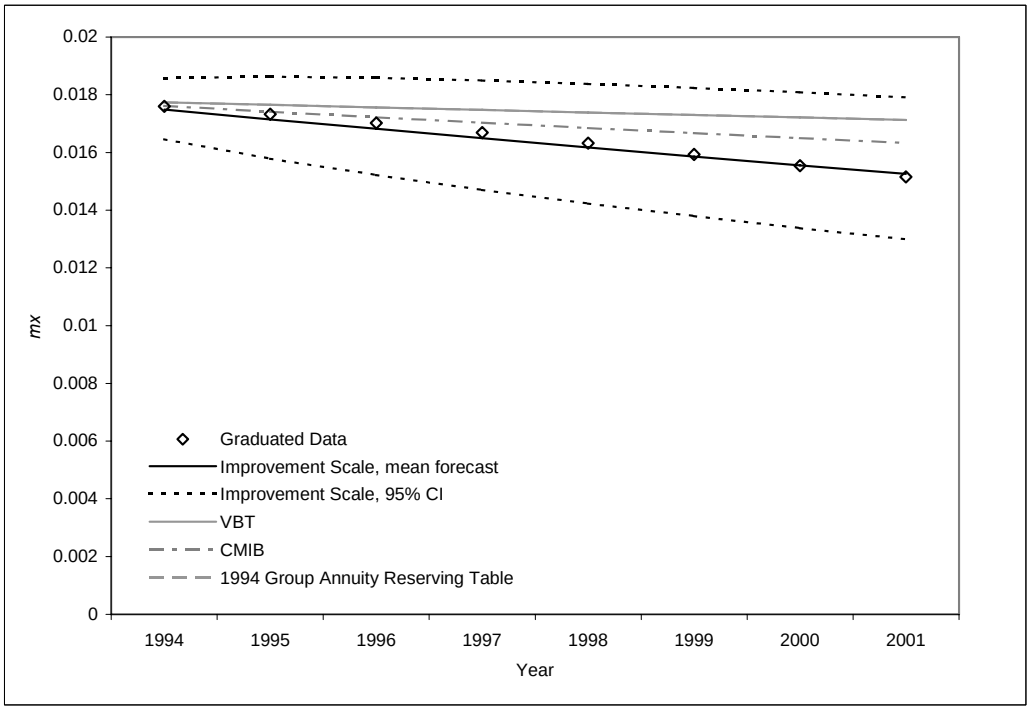


Age 30

Fig 4.8. Ex post analysis of insured lives improvement scales, female.

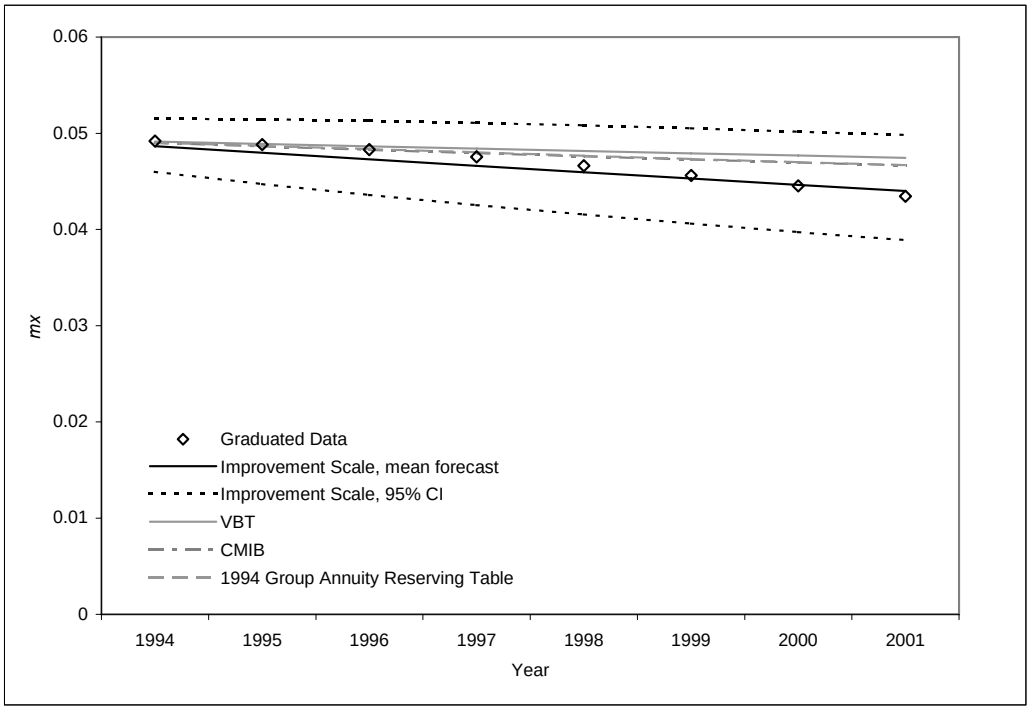


Age 50

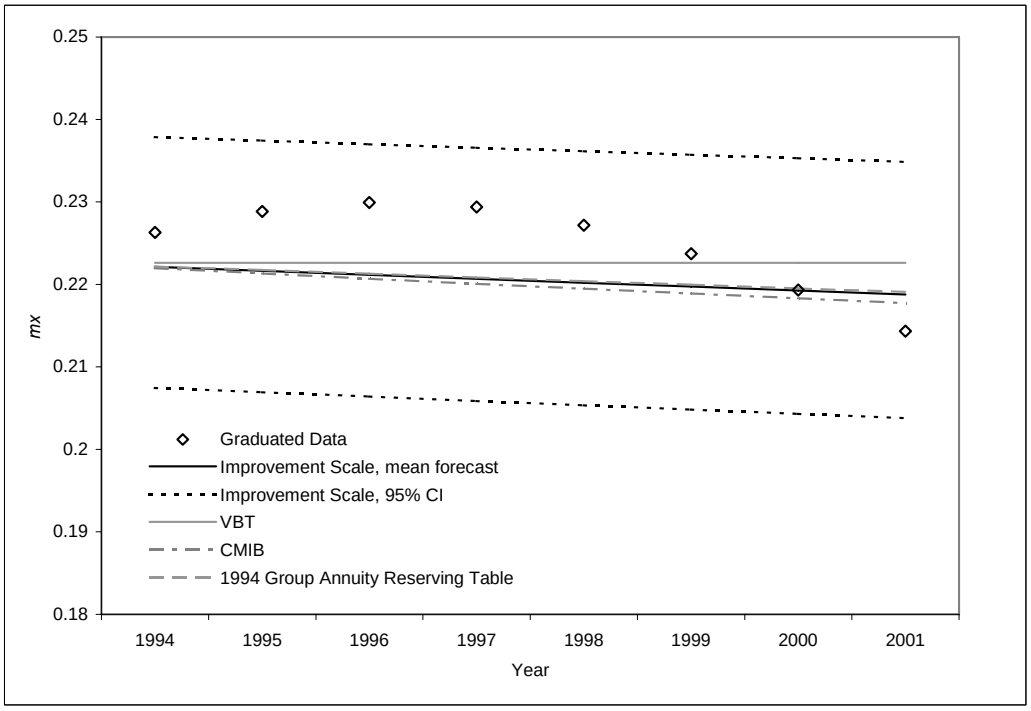


Age 70

Fig 4.8 (cont'd). Ex post analysis of insured lives improvement scales, female.



Age 80



Age 95

Fig 4.8 (cont'd). Ex post analysis of insured lives improvement scales, female.

4.3 Projecting Life Expectancy

To illustrate the effect of the projected mortality improvement, we show in Figures 4.9(a)-(c) below the mean values of the life expectancy of insured lives, at ages 15, 35 and 65. It should be noted that these are mean estimates, and that there is of course considerable uncertainty around the projections.

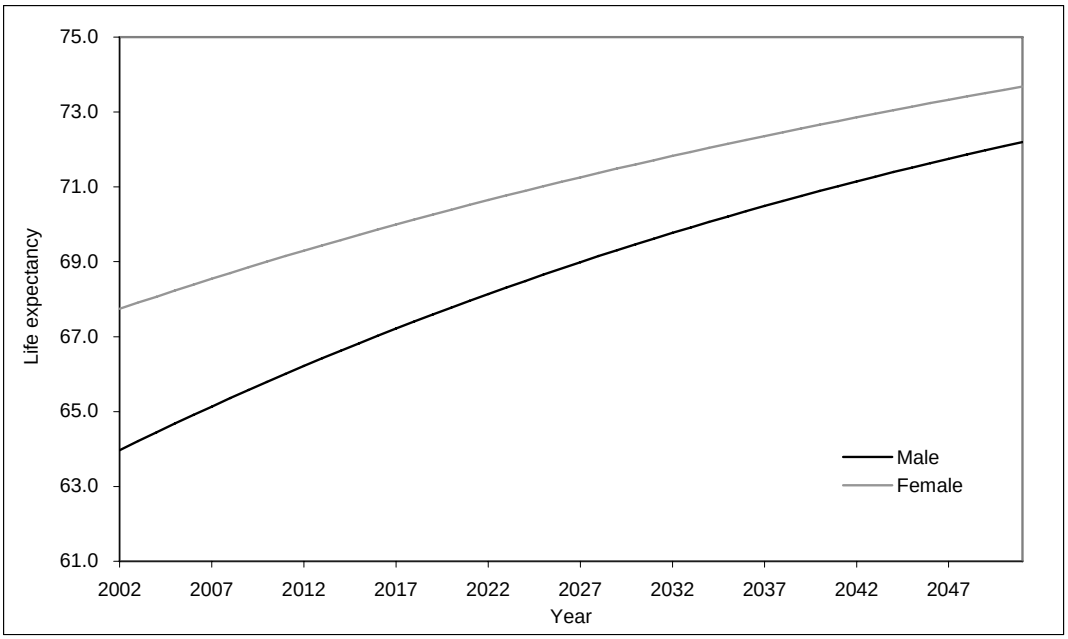


Fig 4.9(a). Mean life expectancy projection, insured lives age 15

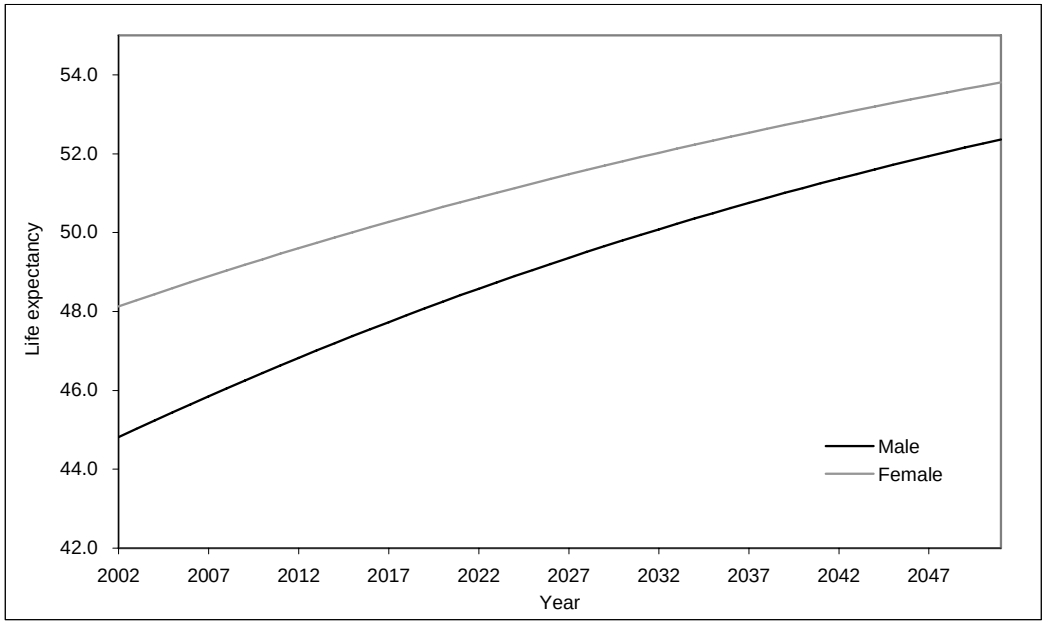


Fig 4.9(b). Mean life expectancy projection, insured lives age 35

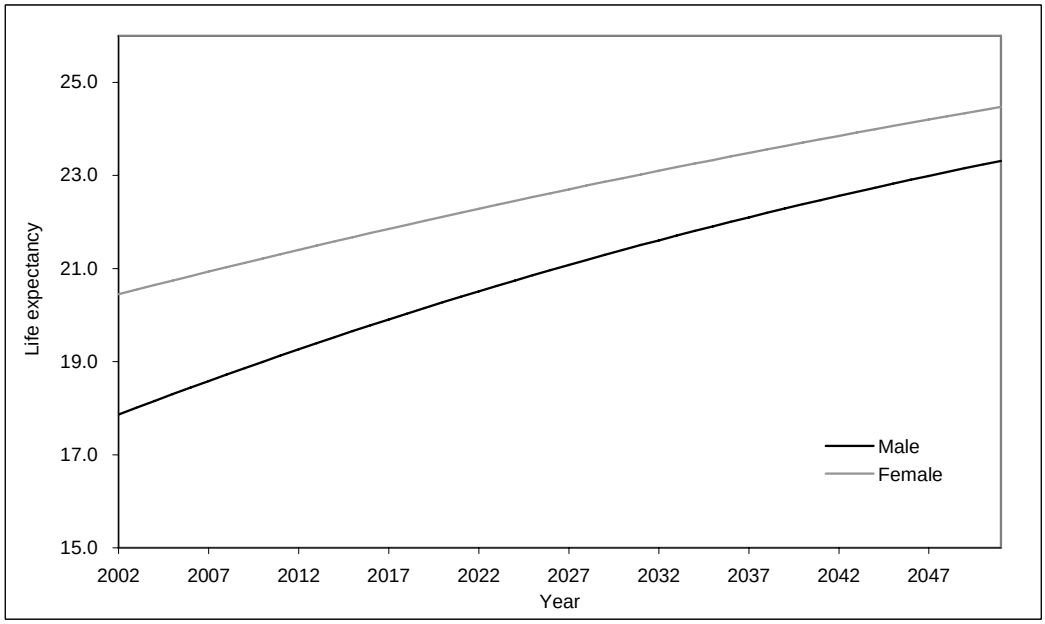


Fig 4.9(c). Mean life expectancy projection, insured lives age 65

4.4 The Effect of Duration

Recall that the Canadian insured lives data is segregated by duration 1, 2, …, 15 and ultimate (16+). Let us denote the central death rate, the number of deaths and the number of exposure-to-risk at duration d by $m_{x,t}^{d}$, $D_{x,t}^{d}$ and $E_{x,t}^{d}$ respectively. In this section, we aim to derive a relationship between $m_{x,t}^{16+}$ and $m_{x,t}^{d}$, for all $d = 1, 2, \ldots, 15$.

This problem has been considered by various researchers. Currie and Waters (1991) considered a graduation of mortality data simultaneously by attained age and duration. This simultaneous graduation requires the assumption that all the deaths and all the exposure-to-risk for attained age x and duration 16+ are concentrated at an average duration $d_{x,t}^{*}$, which may be estimated by

$$\hat{d}_{x,t}^{*} = \frac{1}{E_{x,t}^{16+}}\left[\sum_{d=16}\left(d+\frac{1}{2}\right)E_{x,t}^{d}\right]. \qquad (4.14)$$

As the mortality data available to us gives no information of $E_{x,t}^{d}$ for $d > 16$, the methodology is not applicable.

Panjer and Tan (1995) used a two-stage approach to model the ratio of select to ultimate mortality. In the first stage, they related the first policy year mortality to the ultimate mortality by a logistic-linear function, and, based on this function, they graduated the first year death rates. In the second stage, the rest of the select mortality table was determined as a weighted average between the graduated first policy year mortality and the ultimate mortality, with weights determined by an appropriate power function.

Renshaw and Haberman (1997) proposed a similar method. Instead of modeling directly the select to ultimate mortality ratio, they considered its logarithm, which should demonstrate a higher extent of linearity. In addition, rather than treating mortality in the first policy year in a preferential manner, they modeled death rates at all duration in a consistent way. We shall use Renshaw and Haberman's method in calibrating the effect of selection in the mortality improvement scales.

In contrast to the Lee-Carter, this method assumes that $E_{x,t}^{d}$ is random but $D_{x,t}^{d}$ is not. Let us define a random variable

$$Y_{x,t}^{d} = \frac{E_{x,t}^{d}}{D_{x,t}^{d}}, \tag{4.15}$$

and assume that

$$E\left(Y_{x,t}^{d}\right) = \frac{1}{m_{x,t}^{d}}, \tag{4.16}$$

and

$$Var\left(Y_{x,t}^{d}\right) = \frac{\varphi\left(1/m_{x,t}^{d}\right)^{2}}{D_{x,t}^{d}}. \tag{4.17}$$

This gives $\left(1/m_{x,t}^{d}\right)^{2}$ as the variance function, $D_{x,t}^{d}$ as the prior weight, and φ as the scale parameter. Using the variance stabilizing transformation

$$Q_{x,t}^{d} = \ln\left(Y_{x,t}^{d}\right), \tag{4.18}$$

then

$$E\left(Q_{x,t}^{d}\right) \approx \ln\left(\frac{1}{m_{x,t}^{d}}\right), \tag{4.19}$$

and

$$Var\left(Q_{x,t}^{d}\right) \approx \frac{\varphi}{D_{x,t}^{d}}. \tag{4.20}$$

Under the variance stabilizing transformation, $Var\left(Q_{x,t}^{d}\right)$ is free of $E\left(Q_{x,t}^{d}\right)$. This is particularly useful in situations when there is a paucity of data, and partly explains the motivation of using the logarithmic transformation. Denote the graduated central death rates for the ultimate duration by $\tilde{m}_{x,t}^{16+}$, and let

$$Z_{x,t}^{d} = Q_{x,t}^{16+} - Q_{x,t}^{d}, \tag{4.21}$$

which gives

$$E\left(Z_{x,t}^{d}\right) \approx \ln\left(m_{x,t}^{d}\right) - \ln\left(m_{x,t}^{16+}\right) = \upsilon_{x,t}^{d}, \tag{4.22}$$

and

$$Var\left(Z_{x,t}^{d}\right) \approx \varphi\left(\frac{D_{x,t}^{d} + D_{x,t}^{16+}}{D_{x,t}^{d} D_{x,t}^{16+}}\right). \tag{4.23}$$

For each t and d, we estimate $v_{x,t}^{d}$ by a linear predictor, with prior weights $\frac{D_{x,t}^{d} D_{x,t}^{16+}}{D_{x,t}^{d} + D_{x,t}^{16+}}$. Assuming that the mortality selection process is effective, we anticipate that

$$v_{x,t}^{0} \le v_{x,t}^{1} \le ... \le v_{x,t}^{15} \le 0, \tag{4.24}$$

for fixed x and t.

Exploratory graphical analyses provided by Figures 4.9 indicate that the logarithm of the select to ultimate mortality ratio fluctuates around a constant over all ages, for fixed t and d. This motivates us to use a linear predictor in the form of

$$\hat{v}_{x,t}^{d} = \hat{\theta}_{t}^{d} \quad . \tag{4.25}$$

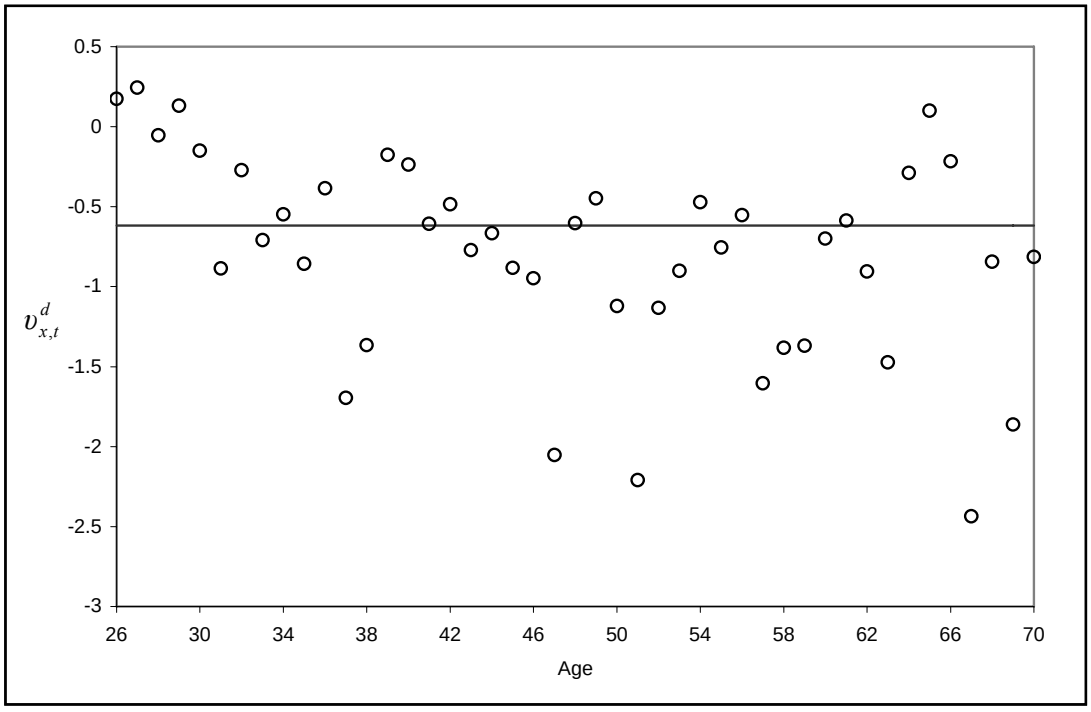
Estimate of θ_{t}^{d} and its standard error can be readily computed by the principle of weighed least squares. Analysis of residuals (not shown) supports the choice of this form of linear predictor.

Next, we carry out the following procedures to diagnose the behavior of θ_{t}^{d} over different policy years.

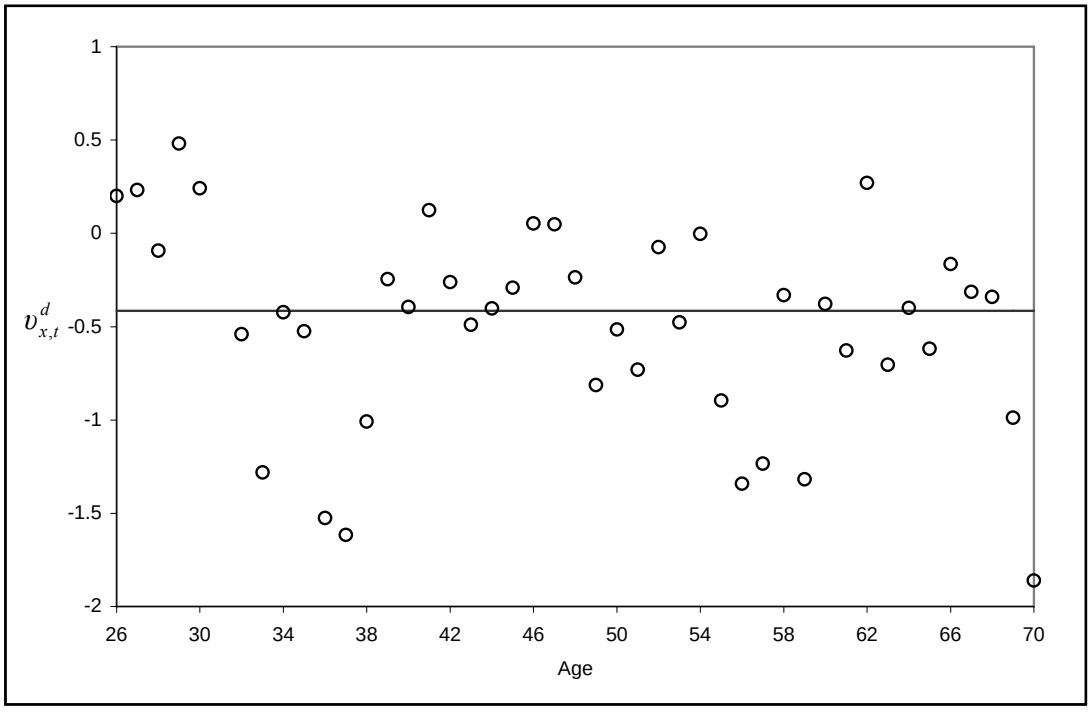
For each d

i. We sum $E_{x,t}^{d}$ and $D_{x,t}^{d}$ over t = 1982, 1983, …, 2001. Using the summation and the steps mentioned, we arrive at an authoritative value of θ_{t}^{d}. This value is denoted by θ^{d} and is assumed to be invariant over time.

ii. For each t, we estimate θ_{t}^{d} and its standard error. Then, we use the usual t-test to test the hypothesis: $\theta_{t}^{d} = \theta^{d}$. Rejection of the hypothesis implies that θ_{t}^{d} may be dependent on time.

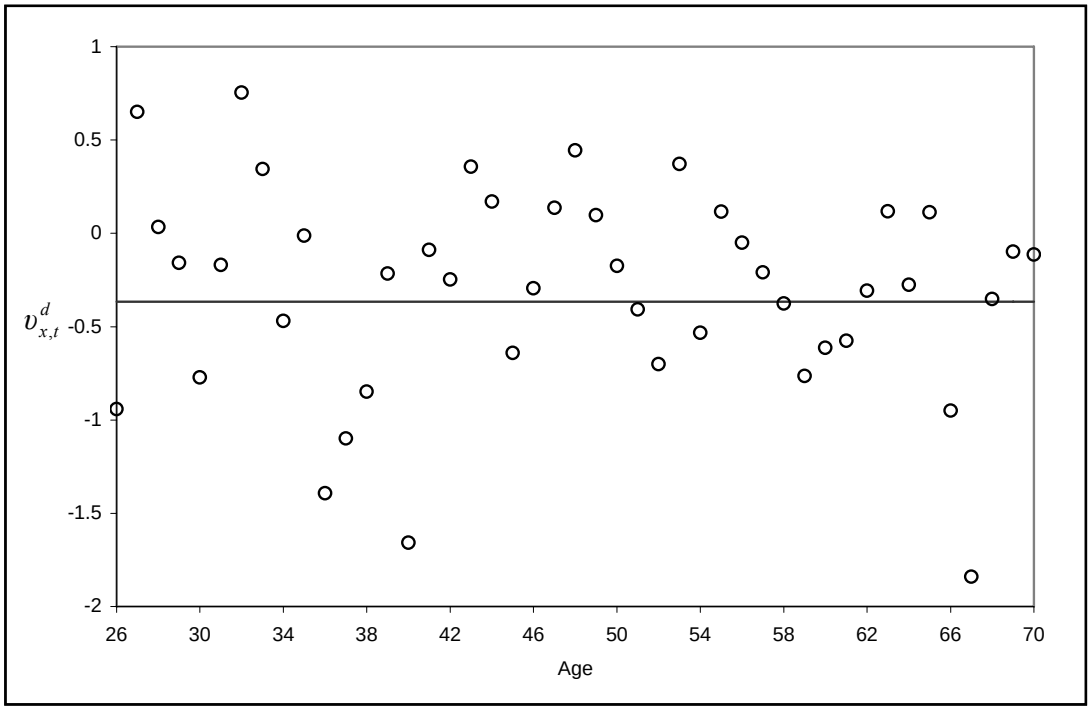


$d = 1$

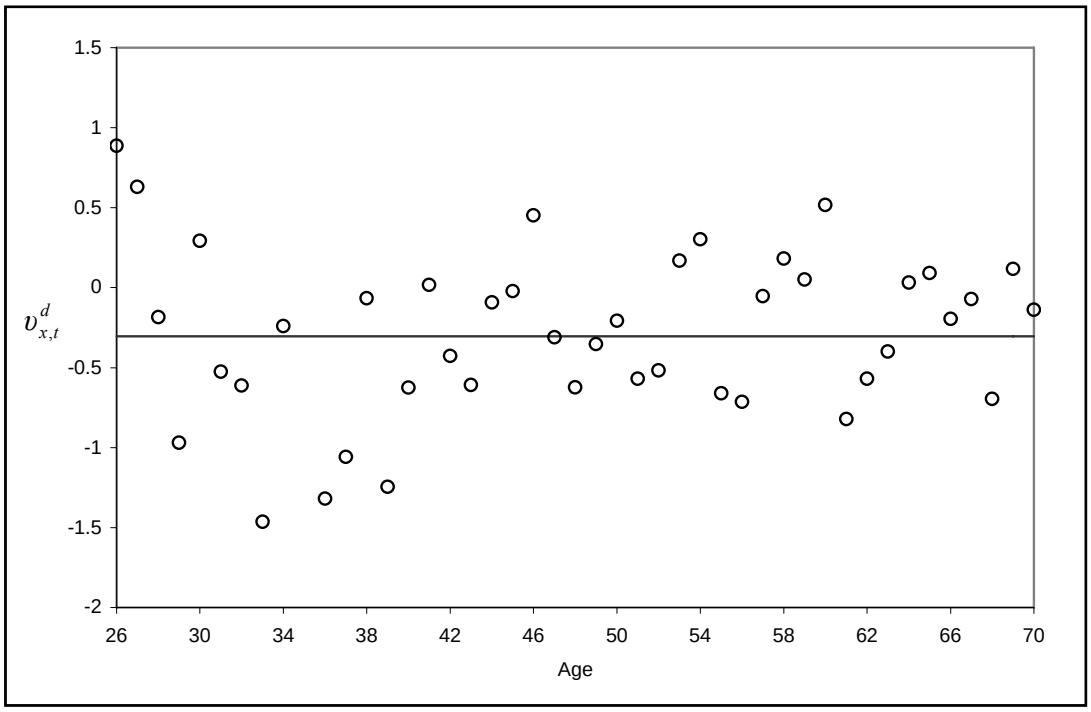


$d = 3$

Fig. 4.10. Trends of $v_{x,t}^{d}$ over age under different values of *d*, males, policy year 2001 – 2002. The solid line represents the linear predictor estimate $\hat{\theta}_t^d$.

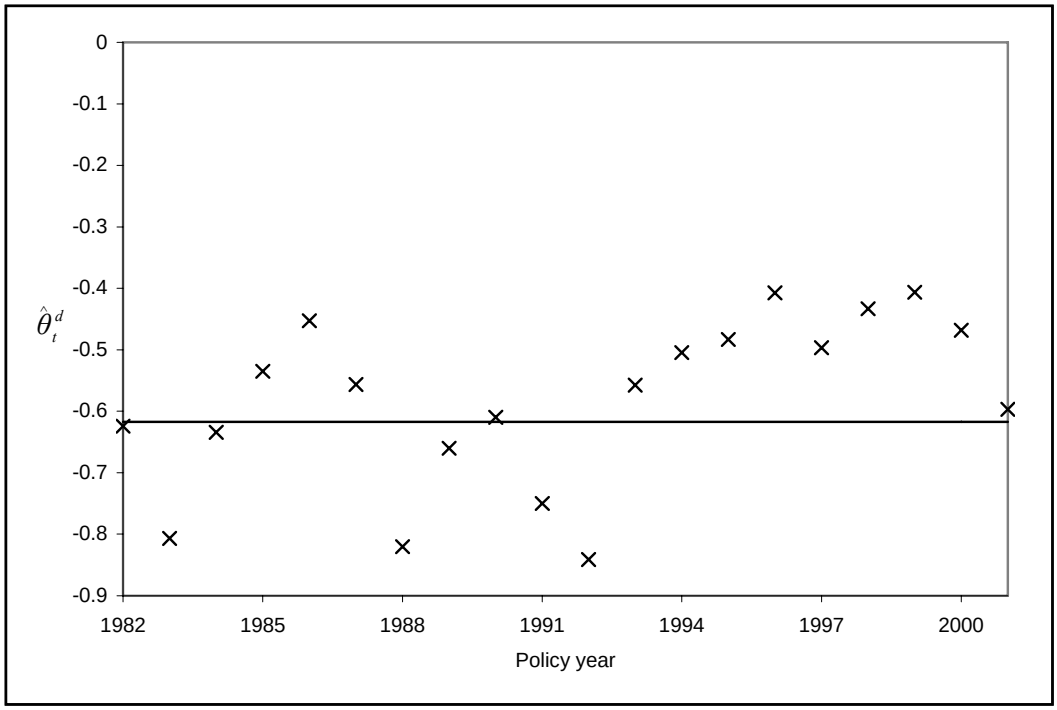


$d = 6$

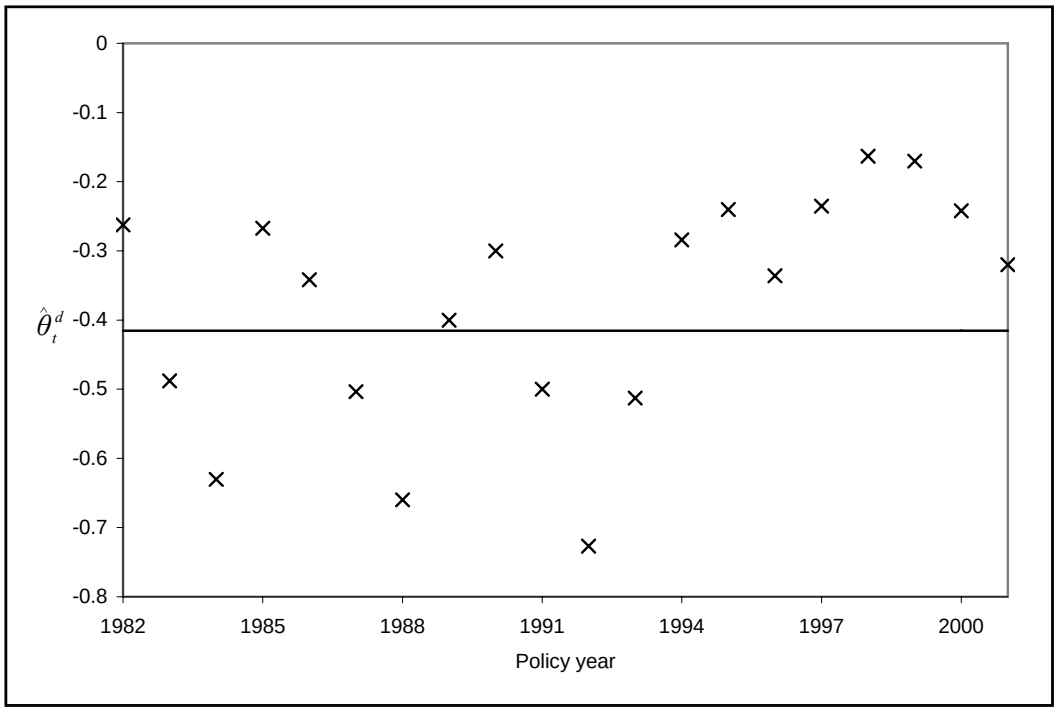


$d = 11$

Fig. 4.10 (cont'd). Trends of $\hat{v}^d_{x,t}$ over age under different values of *d*, males, policy year 2001 – 2002. The solid line represents the linear predictor estimate, $\hat{\theta}^d_t$.

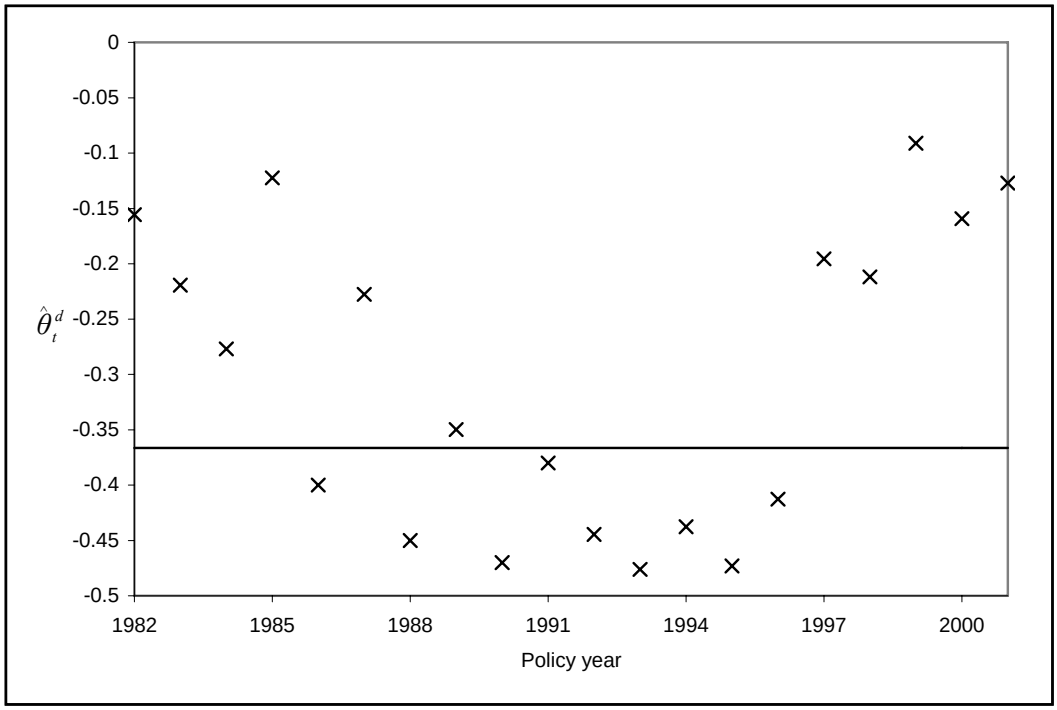


$d = 1$

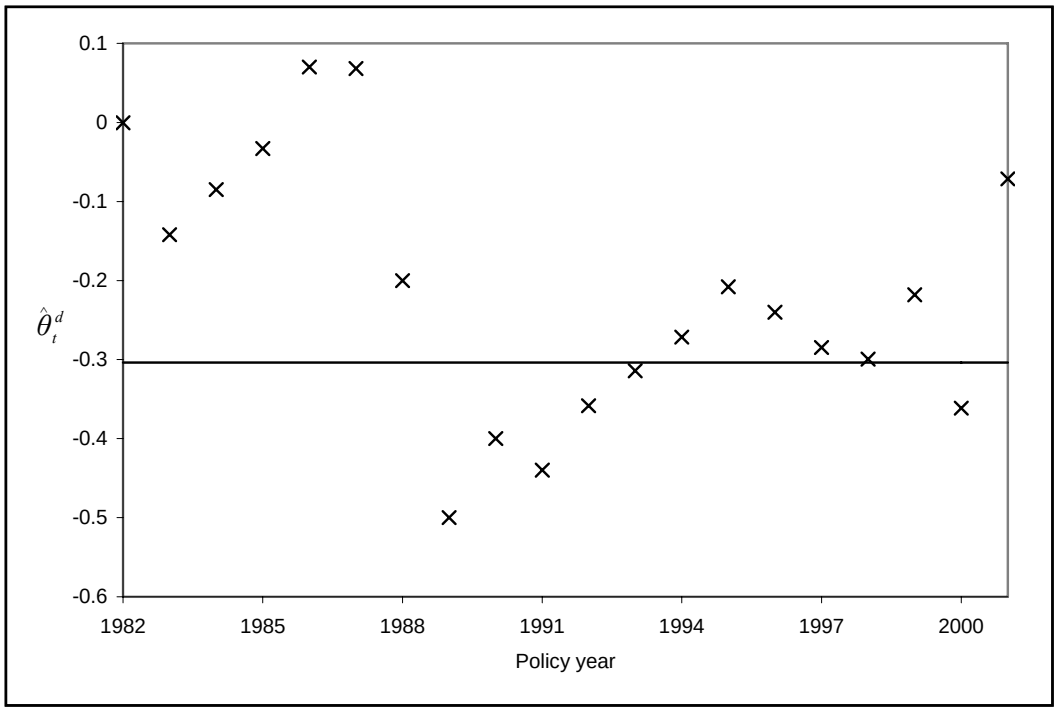


$d = 3$

Fig. 4.11. Trends of $\hat{\theta}_t^d$ over time under different values of *d*, males. The solid line represents the estimate of the authoritative value $\hat{\theta}^d$.



$d = 6$



$d = 11$

Fig. 4.11 (cont'd). Trends of $\hat{\theta}_t^d$ over time under different values of *d*, males. The solid line represents the estimate of the authoritative value $\hat{\theta}^d$.

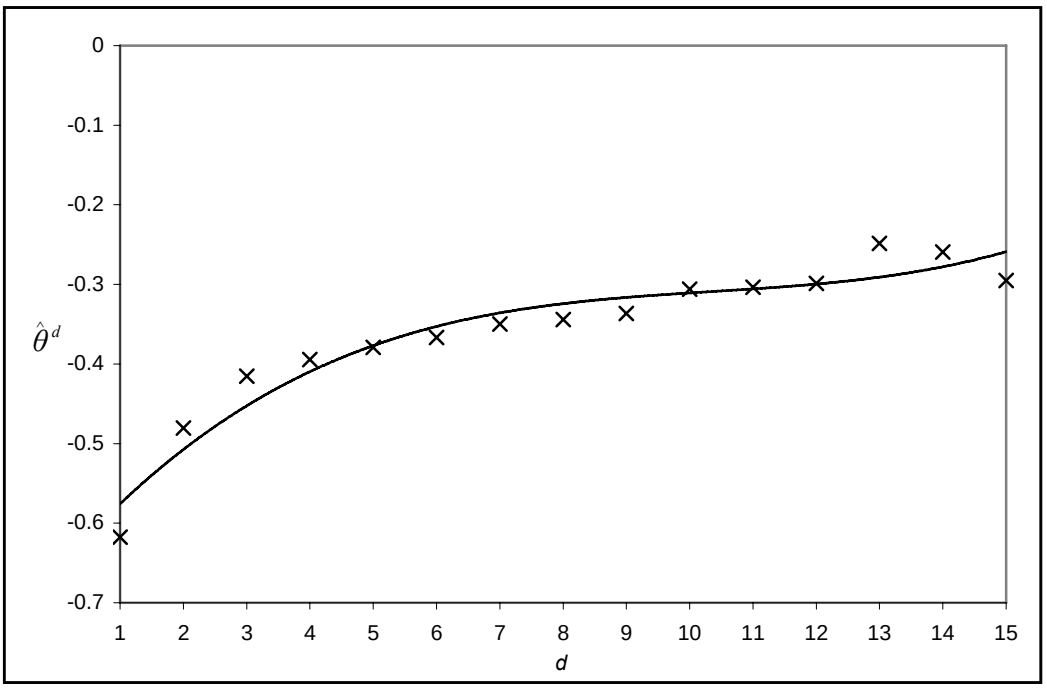


Fig. 4.12. The trend of $\hat{\theta}^d$ over *d*. The solid line represents the smoothed values of $\hat{\theta}^d$.

The above is illustrated graphically in Figures 4.10 to 4.12. For all *d*, the null hypothesis is not rejected, suggesting that we may assume θ_t^d be invariant over time. Then, by equation (4.22), we have

$$\hat{m}_{x,t}^d = \tilde{m}_{x,t}^{16+} \exp\left(\theta^d\right), \tag{4.26}$$

for all *x*, *t*, and *d*. This leads to

$$IS_{2001}^{\text{insured},\,d}\left(x,s\right) = IS_{2001}^{\text{insured}}\left(x,s\right), \tag{4.27}$$

where $IS_{2001}^{\text{insured},\,d}\left(x,s\right)$ and $IS_{2001}^{\text{insured}}\left(x,s\right)$ are the insured lives improvement scales for duration $d = 0, 1, .., 15$, and the ultimate duration respectively. Summing up, we have no statistical evidence to support the provision of separate mortality improvement scales for different durations.

4.5 The Effect of Smoker Status

Parametric models were used in previous graduations of insured lives mortality in both the United Kingdom and Canada. CMIB (2001) attempted to quote linear relationships linking the observed mortality rates for smokers and non-smokers over 1991 – 1994 and 1995 – 1998 to those in the CMIB "92" Series base tables. The linkage is based on the formula

$$q_x = \lambda_1 + (1 - \lambda_2) q_x^*, \tag{4.28}$$

where q_x is the graduated smoker / non-smoker experience, q_x^* is the value of probability of death from the "92" Series base tables, and λ_1 and λ_2 are constants, independent of x. Separate relationship was derived for experiences at various durations. However, in terms of statistical tests of goodness of fit, the performance of this formula on the UK insured lives data was unsatisfactory, and therefore the CMIB did not recommend reliance on this formula to calculate separate mortality rates for smokers and non-smokers.

In the graduation of Canadian individual insurance experience, Panjer and Tan (1995) modeled the ratios of smokers / non-smoker to aggregate mortality rates. They characterized the ratios by:

$$q_x = r(x) q_x^*, \tag{4.29}$$

where

$$r(x) = 1 + l_1 (105 - x)^{l_2} x^{l_3}, \tag{4.30}$$

where a, b and k are constants that are free of x. In this specification, $r(x)$ tends to 1 at the youngest and highest ages and reaches a maximum at some intermediate ages. The ratios are assumed to be the same for all durations. Graphical analysis suggests that the fitting of equations (4.29) and (4.30) to the Canadian insured lives data is adequate.

Panjer and Tan's methodology may be introduced to the calibration of improvement scales. If the linkage specified by equation (4.29) is persistent, the improvement scales for the insured smoker / non-smoker are the same as that for the aggregate, due to the cancellation of $r(x)$[3]. However, it may be possible that the linkage is dependent on time. In this case, we may rewrite equation (4.30) as

[3] We assume that equation (4.16) is applicable to both q_x and m_x.

$$r(x,t) = 1 + l_{1,t}(105 - x)^{l_{2,t}} x^{l_{3,t}}. \tag{4.31}$$

Then,
$$IS_{2001}^{\text{insured, smoker}}(x,s) = \frac{r(x,2001+s)}{r(x,2001)} IS_{2001}^{\text{insured}}(x,s), \tag{4.32}$$

where $IS_{2001}^{\text{insured, smoker}}(x,s)$ and $IS_{2001}^{\text{insured}}(x,s)$ are the improvement scales for the insured lives (smoker) and the aggregate insured lives. Hence, the adjustment for the effect of smoker status are determined by the dynamics of parameters $l_{1,t}$, $l_{2,t}$,and $l_{3,t}$.

	Smoker	Non-smoker	Unclassified
		Male	
1992-1993	12,802	20,300	1,616,410
1996-1997	25,531	23,194	1,538,654
2001-2002	146,829	317,862	1,447,933
		Female	
1992-1993	1,617	10,057	663,043
1996-1997	21,869	8,482	775,397
2001-2002	111,637	220,350	830,030

Table 4.1. Number of exposure to risk, ultimate, segregated by smoker status.

	Smoker	Non-smoker	Unclassified
		Male	
1992-1993	170	156	23,679
1996-1997	173	157	25,499
2001-2002	1,481	1,660	25,154
		Female	
1992-1993	21	33	5,007
1996-1997	107	27	8,654
2001-2002	738	622	9,952

Table 4.2. Number of deaths, ultimate, segregated by smoker status.

Tables 4.1 and 4.2 illustrate the challenges in trending parameters $l_{1,t}$, $l_{2,t}$,and $l_{3,t}$. Most of the data collected in the beginning of 1990s are of unknown smoker status, for example, in policy year 1992 – 1993, the number of exposure-to-risk with identified smoker status accounts for no more than two percent of the aggregate number of exposure-to-risk. The situation is not much improved in 2001-2, when 75% of the

exposure and 89% of deaths are unclassified for the male lives. The sparse data available which is reliably categorized by smoker status renders inappropriate any attempt to separate the smokers and non-smokers.

Consequently, we are not able to draw any conclusion on the behavior of parameters $l_{1,t}$, $l_{2,t}$,and $l_{3,t}$ over time, and we leave this to the future, when more information on the relative mortality of smokers and non-smokers is available. We urge the industry to much higher standards of data collection in this area.

5. CONCLUSION

In this first part of this report we examined Canadian population mortality, using 81 years of data up to 2001. We compared the semi-parametric approach of Currie *et al* (2004) with a parametric Lee-Carter methodology. We concluded that the Currie approach was highly suitable for graduation, but less useful for projection due to a strong influence of very recent mortality trends.

However, the Lee –Carter method in its original form is very restrictive. We have adapted it to allow for over-dispersion by using the negative binomial distribution in place of the Poisson distribution. This was supported by the model diagnostics, and resulted in somewhat wider confidence interval.

We used outlier analysis to explore shocks and structural shifts in the Canadian mortality experience. The evidence for such shocks is not very strong. However, there is a clear highly significant shock in the US data from the 1918 flu pandemic. To allow for the possibility of future epidemics, a flu pandemic residual was included in the bootstrap procedure for estimating the uncertainty in the mortality projection. It should be noted though that the residuals are assumed to be independent, so although the flu pandemic residual is included, it has no more weight than any of the other residuals. If the pandemic becomes more likely over time (as some experts suggest) and we are 'due' another outbreak imminently, then the short term uncertainty is greater than that emerging from the bootstrap procedure in Section 3.5.

Using the Lee-Carter model, we proposed a mortality improvement formula for

population mortality that is multiplicative in the force of mortality, which is equivalent to saying it is exponential in the survival probabilities.

We then propose a relatively simple but flexible model for the relationship between population and insured lives mortality. The combined effect of this model with the results already obtained for the population mortality projection gave the final formula proposed for the projection of insured lives mortality:

$$p(x,s) = p_x^{Is(x,s)} \tag{5.1}$$

$$Is(x,s) = \exp\left(\hat{z}_x s + k\sqrt{\hat{u}_{1,x} + \hat{u}_{2,x} s}\right) \tag{5. 2}$$

where x is the age, s is the projection term, z_x , $u_{1,x}$ and $u_{2,x}$ are parameters explained in Sections 3 and 4, with estimates tabulated in Appendix D, and k is a standard normal deviate to allow for a specified margin for adverse deviation.

In Section 4 we discuss selection, and conclude that there is no evidence to apply different improvement factors for different durations. We also consider smoker/non-smoker issues, and explain why the data on smoker/non-smoker differential mortality is far too sparse for any conclusions.

6. REFERENCES

American Academy of Actuaries (2002). *Final Report of the American Academy of Actuaries' Commissioners Standard Ordinary Task Force.* Available at www.actuaries.org.

Akaike, H. (1974). A New Look at the Statistical Model Identification. *IEEE Transactions on Automatic Control* , **AC-19**, 716-723.

Anscombe. F.J. (1950). Sampling Theory of the Negative Binomial and Logarithmic Series Distributions, *Biometrika*, **36**, 358-382.

Bourbeau, R. and Desjardins, B. (2002). Dealing with Problems in Data Quality for the Measurement of Mortality at Advanced Ages in Canada. *North American Actuarial Journa*l, **1**, 1-25.

Bowers, Jr., N.L., Gerber, H.U., Hickman, J.C., Jones, D.A., Nesbitt, C.J. (1997). *Actuarial Mathematics*. Schaumburg, IL: Society of Actuaries.

Box, G.E.P. and Jenkins, G.M. (1976). *Time Series Analysis Forecasting and Control.* 2nd ed., San Francisco: Holden-Day.

Brass, W. (1975). *Methods for Estimating Fertility and Mortality From Limited and Defective Data.* Chapel Hill, North Carolina: Carolina Population Center, Laboratories for Population Statistics.

Brouhns, N, Denuit, M. and Keilegom, I.V. (2005). Bootstrapping the Poisson Log-bilinear Model for Mortality Forecasting. *Scandinavian Actuarial Journal*, **3**, 212-224.

Brouhns, N., Denuit, M. and Vermunt, J.K. (2002). A Poisson Log-bilinear Regression Approach to the Construction of Projected Lifetables. *Insurance: Mathematics and Economics*, **31**, 373-393.

Buettner, T. (2002). Approaches and Experiences in Projecting Mortality Patterns for the Oldest-old. *North American Actuarial Journal*, **6**, 14-29.

Chen, C. and Liu, L.M. (1993). Joint Estimation of Model Parameters and Outlier Effects in time series. *Journal of American Statistical Association*, **88**, 284-297.

Chia, N.C. and Tsui, A.K.C. (2003). Life Annuities and Compulsory Savings and Income

Adequacy of the Elderly in Singapore. *Journal of Pension Economics and Finance*, **2**, 41-65.

Coale, A. and Guo, G. (1989). Revised Regional Model Life Tables at Very Low Mortality, *Population Index*, **55**, 614-643.

Coale, A. and Kisker, E. (1990). Defects in Data on Old-age Mortality in the United States: New Procedures for Calculating Mortality Schedules and Life Tables at the Highest Ages. *Asian and Pacific Population Forum*, **4**, 1-31.

Continuous Mortality Investigation Bureau (1999). Standard Tables of Mortality Based on the 1991-94 Experiences. CMI Report no. 17. London: Institute of Actuaries and Faculty of Actuaries.

Continuous Mortality Investigation Bureau (2001). Mini-Graduations of the Mortality Experience of Smokers and Non-Smokers for Assured Lives. CMI Report no. 20. London: Institute of Actuaries and Faculty of Actuaries.

Continuous Mortality Investigation Bureau (2002). An Interim Basis for Adjusting the "92" Series Mortality Projections for Cohort Effects. CMI Working Paper no. 1. London: Institute of Actuaries and Faculty of Actuaries.

Continuous Mortality Investigation Bureau (2005). Projecting Future Mortality: Towards a proposal for a stochastic methodology. CMI Working Paper no. 15. London: Institute of Actuaries and Faculty of Actuaries.

Cox, D.R. (1983). Some Remarks on Overdispersion. *Biometrika*, **70**, 269-274.

Currie, I.D. and Durban, M. (2002). Flexible Smoothing with P-splines: a Unified Approach. *Statistical Modelling*, **2**, 333-349.

Currie I. D., Durban, M. and Eilers, P. H. C. (2004). Smoothing and Forecasting Mortality Rates, *Statistical Modelling*, **4**, 279-298.

Currie, I.D. and Waters, H.R. (1991). On Modelling Select Mortality. *Journal of the Institute of Actuaries*, **118**, 453-481.

Edwards, R.D. and Tuljapurkar, S. (2005). Inequality in Life Span and Mortality Convergence across Industrialized Countries. In Program on the Global Demography of Aging, Harvard School of Public Health. Available at http://www.globalhealth.harvard.edu/PGDA_Seminars.aspx

Eilers, P.H.C. and Marx, B.D. (1996). Flexible Smoothing with B-splines ad penalties.

Statistical Science, **18**, 251-262.

Hardy, M.R. (2003). *Investment Guarantees: Modeling and Risk Management for Equity-linked Life Insurance.* Hoboken, New Jersey: John Wiley & Sons.

Heligman, L and Pollard, J.H. (1980). The Age Pattern of Mortality. *Journal of the Institute of Actuaries*, **107**, 437-455.

Himes, C.L., Preston, S.H., and Condran, G.A. (1994). A Relational Model of Mortality at Older Ages in Low Mortality Countries. *Population Studies*, **48**, 269-291.

Hörmann, W. (1993). The Transformed Rejection Method for Generating Poisson Random Variables, *Insurance: Mathematics and Economics*, **12**, 39-45

Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org or www.humanmortality.de (data downloaded on 25 May 2005).

Keyfitz N. (1982). Choice of Function for Mortality Analysis: Effective Forecasting Depends on a Minimum Parameter Representation. *Theoretical population Biology*, **21**, 329-352.

Lee, R.D. and F. Nault. (1993). "Modeling and Forecasting Provincial Mortality in Canada." Presented at the World Congress of the International Union for the Scientific Study of Population, August 24–September 1, Montreal.

Lee R. D and Carter L. (1992). Modeling and Forecasting U.S. Mortality. *Journal of the American Statistical Association*, **87**, 659 – 671.

Li, S.H. and Chan, W.S. (2005a). Outlier Analysis and Mortality Forecasting: the United Kingdom and Scandinavian countries. *Scandinavian Actuarial Journal*, **3**, 187-211.

Li, S.H. and Chan, W.S. (2005b). The Lee-Carter Model for Forecasting Mortality revisited. Living to 100 and Beyond Symposium Monograph. Available at: http://www.soa.org/ccm/content/research-publications/library-publications/monographs/life-monographs/living-to-100-and-beyond-monograph/

Liu, L.M. and Hudak, G.B. (1994). *Forecasting and Time Series Analysis Using the SCA Statistical System.* Chicago: Scientific Computing Associates.

McCullagh, P. and Nelder, J.A. (1989). *Generalized Linear Models*, 2nd edition, Chapman and Hall.

National Center for Health Statistics (2004a). *HIST290 Death Rates for Selected Causes by 10-Year Age Groups, Race, and Sex: Death Registration States, 1900-32, and United States, 1933-98.* Available at http://www.cdc.gov/nchs/datawh/statab/unpubd/mortabs/hist290.htm. Accessed January 25, 2004.

National Center for Health Statistics (2004b). *GMWK23R Death Rates by 10-Year Age Groups: United States and Each State, 1999-2001.* Available at http://www.cdc.gov /nchs/datawh/ statab/unpubd/mortabs.htm Accessed January 25, 2004.

Panjer, H.H. and Russo, G. (1992). Parametric Graduation of Canadian Individual Insurance Mortality Experience: 1982-1988. *Proceedings of The Canadian Institute of Actuaries*, **23**, 378-449.

Panjer, H.H. and Tan, K.S. (1995). *Graduation of Canadian Individual Insurance Mortality Experience: 1986 – 1992.* Canadian Institute of Actuaries.

Pollard, J.H. (1988). On the Derivation of a Full Life Table from Mortality Data Recorded in Five-year age Groups. *Mathematical Population Studies*, **2**, 1-44.

Renshaw, A.E. and Haberman, S., (2003a), On the forecasting of mortality reduction factors. *Insurance: Mathematics and Economics*, 32, 379- 401.

Renshaw, A.E. and Haberman, S., (2003b), Lee-Carter mortality forecasting: a parallel generalized linear modelling approach for England and Wales mortality projections. *Journal of the Royal Statistical Society*, 52, 119 -137.

Richards, S.J., Kirkby, J.G. and Currie, I.D. (2005). The Importance of Year of Birth in Two-Dimensional Mortality Data. Paper Presented to the Institute of Actuaries. Available at http://www.actuaries.org.uk/files/pdf/sessional/sm20051024.pdf.

Ruppert, D. (2002). Selecting the Number of Knots for Penalized Splines. *Journal of Computational and Graphical Statistics*, **11**, 735-757.

Society of Actuaries Group Annuity Valuation Table Task Force. (1995). 1994 Group Annuity Mortality Table and 1994 Group Annuity Reserving Table, *TSA*, **XLVII**, 865-915.

Statistics Canada (1994). *Selected Mortality Statistics, Canada, 1921-1990.* Ottawa, Ontario: Statistics Canada.

Statistics Canada (2004) *CANSIM Table 051-001—Estimates of Population by Age and Sex for Canada, the Provinces and the Territories, Annual.* Available at

http://cansim2.statcan.ca/cgi-win/cnsmcgi.exe?Lang=E&RootDir=CII/&ResultTemplate=CII/CII_pick&Array_Pick=1&ArrayId=510001. Accessed January 25, 2004.

Schwarz, G. (1978). Estimating the Dimension of a Model. *Annals of Statistics*, **6**, 461-464.

Tabeau, E. (2001). A Review of Demographic Forecasting Models for Mortality. In Tabeau, E., Jeths, A.V.D.B. and Heathcote, C. (eds.), *Forecasting Mortality in Developed Countries – Insights from a Statistical, Demographic and Epidemiological Perspective.* Dordrecht / Boston / London: Kluwer Academic Publishers.

Tsay, R.S. (1986). Time Series Model Specification in the Presence of Outliers. *Journal of American Statistical Association*, **81**, 132-141.

United Nations. (1997). Report of the Working Group on Projecting Old-Age Mortality and Its Consequences. New York: United Nations.

U.S. Census Bureau (2004). *Monthly Population Estimates, 1990–2000.* Available at www.census.gov/popest/archives/1990s/nat_detail.html. Accessed January 25, 2004.

Vincent, P. (1951). La Mortalité des Vieillards. *Population*, **6**, 181–204.

Wikipedia (2004). Spanish Flu. Retrieved May 30, 2004, from Wilipedia Encyclopedia Online. Available at http://en.wikipedia.org/wiki/Aftermath_of_World_War_I#Influenza_pandemic

Willekens, F.J. (1990). Demographic Forecasting; State-of-art and Research Needs. In: C.A. Hazeu and G.A.B. Frinkin (eds.): *Emerging Issues in Demographic Research*. Elsevier Science Publishers B.V., pp. 9-75.

Wilmoth, J.R. (1993). Computational Methods for Fitting and Extrapolating the Lee-Carter Model of Mortality Change. Technical report. Department of Demography. University of California, Berkeley.

7. APPENDIX A: ILLUSTRATIVE BASE TABLES, CANADIAN POPULATION, 2001

	Male	Female		Male	Female
0	0.005945	0.004579	50	0.003566	0.002282
1	0.000647	0.000531	51	0.003895	0.002516
2	0.000211	0.000174	52	0.004257	0.002776
3	0.000142	0.000116	53	0.004665	0.003060
4	0.000143	0.000117	54	0.005131	0.003365
5	0.000161	0.000130	55	0.005675	0.003690
6	0.000161	0.000130	56	0.006307	0.004034
7	0.000146	0.000119	57	0.007028	0.004403
8	0.000128	0.000105	58	0.007832	0.004807
9	0.000116	0.000095	59	0.008709	0.005255
10	0.000115	0.000094	60	0.009640	0.005762
11	0.000128	0.000104	61	0.010623	0.006337
12	0.000159	0.000124	62	0.011676	0.006983
13	0.000209	0.000156	63	0.012826	0.007701
14	0.000284	0.000197	64	0.014110	0.008490
15	0.000384	0.000241	65	0.015575	0.009349
16	0.000505	0.000279	66	0.017246	0.010281
17	0.000637	0.000307	67	0.019127	0.011298
18	0.000766	0.000324	68	0.021213	0.012417
19	0.000872	0.000332	69	0.023488	0.013655
20	0.000931	0.000332	70	0.025926	0.015037
21	0.000935	0.000329	71	0.028534	0.016586
22	0.000899	0.000323	72	0.031353	0.018322
23	0.000847	0.000318	73	0.034438	0.020271
24	0.000797	0.000315	74	0.037862	0.022461
25	0.000766	0.000317	75	0.041714	0.024924
26	0.000761	0.000325	76	0.046054	0.027707
27	0.000775	0.000337	77	0.050914	0.030870
28	0.000802	0.000354	78	0.056321	0.034489
29	0.000838	0.000375	79	0.062295	0.038654
30	0.000875	0.000398	80	0.068858	0.043476
31	0.000911	0.000424	81	0.076171	0.049040
32	0.000946	0.000453	82	0.084537	0.055409
33	0.000983	0.000488	83	0.094363	0.062632
34	0.001024	0.000529	84	0.106201	0.070741
35	0.001072	0.000579	85	0.120742	0.079748
36	0.001130	0.000638	86	0.138300	0.089706
37	0.001199	0.000708	87	0.158823	0.100698
38	0.001279	0.000786	88	0.181978	0.112817
39	0.001372	0.000870	89	0.207025	0.126160
40	0.001479	0.000956	90	0.232874	0.140838
41	0.001601	0.001044	91	0.259193	0.156983
42	0.001741	0.001134	92	0.286246	0.174750
43	0.001897	0.001228	93	0.314547	0.194313
44	0.002073	0.001330	94	0.344886	0.215878
45	0.002269	0.001444	95	0.378133	0.239662
46	0.002485	0.001573	96	0.413836	0.265793
47	0.002722	0.001720	97	0.450628	0.294337
48	0.002981	0.001885	98	0.486638	0.325312
49	0.003263	0.002072	99	0.519494	0.358682

8. APPENDIX B: IMPROVEMENT SCALES, CANADIAN POPULATION

	Male		Female			Male		Female	
Age (x)	w_x	v_x	w_x	v_x	Age (x)	w_x	v_x	w_x	v_x
0	-0.046300	0.007800	-0.040200	0.006900	50	-0.011673	0.000511	-0.016615	0.001203
1	-0.055976	0.011533	-0.049690	0.010649	51	-0.011219	0.000474	-0.016178	0.001138
2	-0.050810	0.009379	-0.045732	0.008950	52	-0.010806	0.000439	-0.015806	0.001082
3	-0.046913	0.007950	-0.042573	0.007733	53	-0.010419	0.000405	-0.015483	0.001032
4	-0.044138	0.007081	-0.040202	0.006920	54	-0.010042	0.000373	-0.015193	0.000988
5	-0.042334	0.006607	-0.038609	0.006433	55	-0.009662	0.000344	-0.014921	0.000948
6	-0.041354	0.006363	-0.037780	0.006193	56	-0.009270	0.000318	-0.014657	0.000911
7	-0.040983	0.006213	-0.037615	0.006122	57	-0.008884	0.000295	-0.014421	0.000880
8	-0.040852	0.006083	-0.037803	0.006136	58	-0.008526	0.000276	-0.014234	0.000857
9	-0.040574	0.005909	-0.038006	0.006150	59	-0.008218	0.000261	-0.014117	0.000842
10	-0.039761	0.005626	-0.037887	0.006083	60	-0.007982	0.000248	-0.014092	0.000839
11	-0.038025	0.005172	-0.037109	0.005850	61	-0.007825	0.000239	-0.014163	0.000847
12	-0.035246	0.004532	-0.035579	0.005430	62	-0.007718	0.000232	-0.014288	0.000861
13	-0.031770	0.003780	-0.033640	0.004910	63	-0.007628	0.000226	-0.014417	0.000876
14	-0.027994	0.003000	-0.031678	0.004387	64	-0.007523	0.000219	-0.014503	0.000886
15	-0.024313	0.002275	-0.030080	0.003960	65	-0.007368	0.000211	-0.014496	0.000886
16	-0.021120	0.001689	-0.029228	0.003725	66	-0.007157	0.000201	-0.014384	0.000874
17	-0.018645	0.001279	-0.029254	0.003717	67	-0.006930	0.000191	-0.014212	0.000856
18	-0.016886	0.001025	-0.029934	0.003881	68	-0.006732	0.000182	-0.014035	0.000837
19	-0.015823	0.000900	-0.031024	0.004157	69	-0.006609	0.000175	-0.013907	0.000825
20	-0.015440	0.000880	-0.032273	0.004483	70	-0.006605	0.000174	-0.013881	0.000825
21	-0.015712	0.000937	-0.033440	0.004800	71	-0.006724	0.000177	-0.013965	0.000838
22	-0.016509	0.001045	-0.034382	0.005069	72	-0.006913	0.000183	-0.014102	0.000856
23	-0.017582	0.001177	-0.035071	0.005277	73	-0.007111	0.000190	-0.014229	0.000873
24	-0.018674	0.001307	-0.035486	0.005411	74	-0.007260	0.000197	-0.014285	0.000879
25	-0.019527	0.001406	-0.035605	0.005459	75	-0.007302	0.000201	-0.014209	0.000867
26	-0.019899	0.001449	-0.035409	0.005408	76	-0.007230	0.000202	-0.013992	0.000837
27	-0.019764	0.001434	-0.034937	0.005268	77	-0.007092	0.000200	-0.013683	0.000796
28	-0.019296	0.001380	-0.034279	0.005070	78	-0.006940	0.000196	-0.013335	0.000751
29	-0.018672	0.001307	-0.033526	0.004841	79	-0.006825	0.000190	-0.012999	0.000709
30	-0.018069	0.001233	-0.032769	0.004613	80	-0.006798	0.000183	-0.012728	0.000678
31	-0.017654	0.001178	-0.032092	0.004412	81	-0.006869	0.000176	-0.012540	0.000660
32	-0.017455	0.001146	-0.031496	0.004240	82	-0.007018	0.000172	-0.012427	0.000653
33	-0.017400	0.001130	-0.030920	0.004082	83	-0.007220	0.000174	-0.012378	0.000654
34	-0.017417	0.001124	-0.030304	0.003919	84	-0.007452	0.000184	-0.012383	0.000662
35	-0.017433	0.001122	-0.029586	0.003738	85	-0.007692	0.000205	-0.012433	0.000673
36	-0.017382	0.001117	-0.028714	0.003522	86	-0.007926	0.000236	-0.012512	0.000686
37	-0.017246	0.001107	-0.027703	0.003276	87	-0.008147	0.000272	-0.012599	0.000698
38	-0.017040	0.001090	-0.026606	0.003015	88	-0.008347	0.000309	-0.012676	0.000708
39	-0.016773	0.001063	-0.025475	0.002755	89	-0.008518	0.000342	-0.012723	0.000713
40	-0.016460	0.001026	-0.024365	0.002510	90	-0.008653	0.000368	-0.012720	0.000712
41	-0.016110	0.000977	-0.023323	0.002293	91	-0.008739	0.000385	-0.012653	0.000703
42	-0.015727	0.000918	-0.022357	0.002105	92	-0.008757	0.000396	-0.012510	0.000688
43	-0.015308	0.000855	-0.021457	0.001942	93	-0.008692	0.000399	-0.012277	0.000667
44	-0.014852	0.000790	-0.020615	0.001800	94	-0.008526	0.000398	-0.011943	0.000641
45	-0.014358	0.000729	-0.019821	0.001673	95	-0.008240	0.000391	-0.011494	0.000609
46	-0.013826	0.000675	-0.019070	0.001558	96	-0.007795	0.000375	-0.010908	0.000566
47	-0.013273	0.000628	-0.018365	0.001454	97	-0.007147	0.000344	-0.010162	0.000504
48	-0.012716	0.000586	-0.017715	0.001361	98	-0.006249	0.000292	-0.009231	0.000416
49	-0.012176	0.000547	-0.017129	0.001277	99	-0.005055	0.000211	-0.008092	0.000293

9. APPENDIX C: ILLUSTRATIVE BASE TABLES

Canadian Insured Lives, Ultimate, Composite Smoker/Non-smoker, 2001

	Male	Female		Male	Female
15	0.000593	0.000242	58	0.006472	0.004580
16	0.000613	0.000249	59	0.007164	0.005068
17	0.000633	0.000257	60	0.007926	0.005607
18	0.000654	0.000265	61	0.008767	0.006200
19	0.000675	0.000273	62	0.009696	0.006855
20	0.000697	0.000282	63	0.010727	0.007576
21	0.000718	0.000292	64	0.011874	0.008370
22	0.000739	0.000302	65	0.013155	0.009243
23	0.000761	0.000312	66	0.014587	0.010203
24	0.000783	0.000324	67	0.016184	0.011261
25	0.000806	0.000336	68	0.017962	0.012429
26	0.000830	0.000350	69	0.019935	0.013720
27	0.000855	0.000365	70	0.022119	0.015152
28	0.000881	0.000381	71	0.024536	0.016742
29	0.000908	0.000399	72	0.027215	0.018515
30	0.000937	0.000420	73	0.030190	0.020497
31	0.000968	0.000443	74	0.033499	0.022719
32	0.001001	0.000468	75	0.037189	0.025220
33	0.001035	0.000497	76	0.041305	0.028040
34	0.001073	0.000529	77	0.045900	0.031219
35	0.001113	0.000565	78	0.051030	0.034808
36	0.001156	0.000606	79	0.056759	0.038860
37	0.001203	0.000651	80	0.063156	0.043435
38	0.001255	0.000702	81	0.070280	0.048592
39	0.001314	0.000759	82	0.078176	0.054382
40	0.001381	0.000822	83	0.086884	0.060856
41	0.001457	0.000892	84	0.096432	0.068060
42	0.001543	0.000970	85	0.106916	0.076021
43	0.001643	0.001057	86	0.118414	0.084806
44	0.001757	0.001154	87	0.131010	0.094486
45	0.001890	0.001263	88	0.144792	0.105137
46	0.002042	0.001384	89	0.159854	0.116841
47	0.002218	0.001520	90	0.176297	0.129683
48	0.002418	0.001672	91	0.194224	0.143755
49	0.002646	0.001842	92	0.213749	0.159152
50	0.002905	0.002033	93	0.234986	0.175974
51	0.003197	0.002247	94	0.258060	0.194329
52	0.003526	0.002486	95	0.283100	0.214326
53	0.003896	0.002753	96	0.310241	0.236081
54	0.004308	0.003049	97	0.339623	0.259715
55	0.004768	0.003376	98	0.371395	0.285354
56	0.005279	0.003739	99	0.405708	0.313126
57	0.005845	0.004139			

10. APPENDIX D: IMPROVEMENT SCALES, CANADIAN INSURED LIVES

	Male			Female		
Age (x)	z_x	$u1_x$	$u2_x$	z_x	$u1_x$	$u2_x$
0	-0.048097	0.000050	0.008209	-0.017908	0.000020	0.001410
1	-0.051806	0.000050	0.009488	-0.019879	0.000020	0.001684
2	-0.051508	0.000050	0.009378	-0.020008	0.000020	0.001685
3	-0.048780	0.000050	0.008431	-0.019010	0.000020	0.001522
4	-0.045199	0.000050	0.007200	-0.017602	0.000020	0.001304
5	-0.042325	0.000050	0.006230	-0.016491	0.000020	0.001136
6	-0.040883	0.000050	0.005773	-0.016012	0.000020	0.001068
7	-0.040415	0.000050	0.005666	-0.015971	0.000020	0.001069
8	-0.040374	0.000050	0.005712	-0.016132	0.000020	0.001099
9	-0.040211	0.000050	0.005716	-0.016263	0.000020	0.001121
10	-0.039406	0.000050	0.005494	-0.016144	0.000020	0.001098
11	-0.037802	0.000050	0.005022	-0.015774	0.000020	0.001035
12	-0.035497	0.000050	0.004389	-0.015307	0.000020	0.000960
13	-0.032596	0.000050	0.003687	-0.014897	0.000020	0.000906
14	-0.029202	0.000050	0.003007	-0.014701	0.000020	0.000903
15	-0.029211	0.000005	0.003057	-0.014534	0.000006	0.000938
16	-0.030018	0.000004	0.003364	-0.015623	0.000004	0.001063
17	-0.030819	0.000005	0.003598	-0.016680	0.000003	0.001198
18	-0.031619	0.000007	0.003785	-0.017719	0.000002	0.001345
19	-0.032421	0.000008	0.003949	-0.018757	0.000001	0.001506
20	-0.033229	0.000009	0.004115	-0.019809	0.000000	0.001682
21	-0.034042	0.000010	0.004300	-0.020885	0.000000	0.001874
22	-0.034853	0.000010	0.004502	-0.021982	0.000001	0.002081
23	-0.035653	0.000009	0.004714	-0.023095	0.000002	0.002300
24	-0.036433	0.000009	0.004933	-0.024220	0.000004	0.002532
25	-0.037184	0.000010	0.005151	-0.025351	0.000006	0.002775
26	-0.037894	0.000012	0.005362	-0.026482	0.000009	0.003026
27	-0.038542	0.000015	0.005557	-0.027603	0.000012	0.003285
28	-0.039107	0.000020	0.005727	-0.028706	0.000014	0.003548
29	-0.039569	0.000026	0.005861	-0.029780	0.000015	0.003813
30	-0.039905	0.000034	0.005950	-0.030817	0.000015	0.004078
31	-0.040103	0.000044	0.005990	-0.031806	0.000014	0.004340
32	-0.040161	0.000054	0.005986	-0.032738	0.000011	0.004594
33	-0.040081	0.000064	0.005944	-0.033606	0.000008	0.004836
34	-0.039866	0.000071	0.005872	-0.034399	0.000005	0.005061
35	-0.039516	0.000076	0.005775	-0.035108	0.000003	0.005264
36	-0.039039	0.000077	0.005657	-0.035732	0.000001	0.005443
37	-0.038454	0.000075	0.005515	-0.036279	0.000001	0.005600
38	-0.037780	0.000072	0.005347	-0.036758	0.000002	0.005738
39	-0.037036	0.000068	0.005149	-0.037178	0.000004	0.005861
40	-0.036241	0.000064	0.004918	-0.037550	0.000006	0.005970
41	-0.035409	0.000061	0.004658	-0.037875	0.000009	0.006067
42	-0.034544	0.000060	0.004382	-0.038145	0.000012	0.006152
43	-0.033649	0.000061	0.004101	-0.038353	0.000016	0.006222
44	-0.032729	0.000064	0.003831	-0.038489	0.000020	0.006278
45	-0.031785	0.000070	0.003584	-0.038545	0.000024	0.006318
46	-0.030830	0.000081	0.003367	-0.038502	0.000027	0.006334
47	-0.029884	0.000097	0.003176	-0.038328	0.000031	0.006312
48	-0.028970	0.000123	0.003009	-0.037993	0.000035	0.006236
49	-0.028108	0.000159	0.002861	-0.037463	0.000041	0.006090

APPENDIX D (CONT'D)

Improvement Scales, Canadian Insured Lives

	Male			Female		
Age (x)	z_x	$u_{1,x}$	$u_{2,x}$	z_x	$u_{1,x}$	$u_{2,x}$
50	-0.027321	0.000208	0.002730	-0.036708	0.000047	0.005859
51	-0.026625	0.000270	0.002614	-0.035731	0.000055	0.005546
52	-0.026033	0.000340	0.002516	-0.034575	0.000063	0.005174
53	-0.025557	0.000414	0.002438	-0.033287	0.000072	0.004766
54	-0.025209	0.000489	0.002381	-0.031912	0.000079	0.004349
55	-0.025000	0.000559	0.002350	-0.030497	0.000085	0.003946
56	-0.024941	0.000622	0.002345	-0.029084	0.000089	0.003571
57	-0.025042	0.000677	0.002369	-0.027707	0.000092	0.003230
58	-0.025312	0.000723	0.002425	-0.026404	0.000094	0.002925
59	-0.025761	0.000758	0.002515	-0.025211	0.000097	0.002660
60	-0.026397	0.000782	0.002641	-0.024163	0.000101	0.002439
61	-0.027179	0.000796	0.002797	-0.023270	0.000105	0.002260
62	-0.028025	0.000806	0.002971	-0.022519	0.000109	0.002118
63	-0.028851	0.000815	0.003148	-0.021896	0.000114	0.002007
64	-0.029571	0.000831	0.003317	-0.021386	0.000118	0.001920
65	-0.030105	0.000856	0.003463	-0.020977	0.000121	0.001851
66	-0.030420	0.000891	0.003575	-0.020644	0.000122	0.001795
67	-0.030524	0.000931	0.003647	-0.020362	0.000124	0.001749
68	-0.030427	0.000971	0.003668	-0.020102	0.000124	0.001708
69	-0.030140	0.001007	0.003630	-0.019835	0.000125	0.001667
70	-0.029671	0.001035	0.003527	-0.019535	0.000126	0.001624
71	-0.029047	0.001052	0.003366	-0.019194	0.000128	0.001575
72	-0.028303	0.001059	0.003167	-0.018820	0.000131	0.001522
73	-0.027473	0.001054	0.002952	-0.018420	0.000138	0.001463
74	-0.026594	0.001038	0.002739	-0.018000	0.000147	0.001398
75	-0.025696	0.001011	0.002548	-0.017567	0.000161	0.001327
76	-0.024768	0.000977	0.002378	-0.017111	0.000181	0.001250
77	-0.023768	0.000944	0.002212	-0.016612	0.000210	0.001169
78	-0.022655	0.000920	0.002037	-0.016052	0.000250	0.001083
79	-0.021388	0.000911	0.001836	-0.015410	0.000304	0.000993
80	-0.019931	0.000924	0.001597	-0.014670	0.000373	0.000901
81	-0.018304	0.000958	0.001330	-0.013836	0.000456	0.000807
82	-0.016558	0.001005	0.001056	-0.012928	0.000550	0.000713
83	-0.014746	0.001058	0.000797	-0.011965	0.000649	0.000621
84	-0.012921	0.001111	0.000577	-0.010966	0.000750	0.000534
85	-0.011131	0.001156	0.000415	-0.009949	0.000850	0.000451
86	-0.009406	0.001191	0.000307	-0.008934	0.000944	0.000375
87	-0.007762	0.001216	0.000239	-0.007941	0.001031	0.000307
88	-0.006216	0.001230	0.000195	-0.006988	0.001107	0.000247
89	-0.004786	0.001234	0.000162	-0.006094	0.001170	0.000196
90	-0.003486	0.001226	0.000126	-0.005276	0.001216	0.000155
91	-0.002313	0.001207	0.000087	-0.004532	0.001245	0.000123
92	-0.001258	0.001177	0.000053	-0.003852	0.001258	0.000098
93	-0.000308	0.001138	0.000028	-0.003223	0.001255	0.000080
94	0.000547	0.001089	0.000020	-0.002636	0.001237	0.000065
95	0.001315	0.001031	0.000033	-0.002082	0.001203	0.000054
96	0.001988	0.000965	0.000060	-0.001567	0.001155	0.000045
97	0.002554	0.000891	0.000090	-0.001106	0.001093	0.000038
98	0.002998	0.000812	0.000113	-0.000710	0.001018	0.000032
99	0.003309	0.000728	0.000118	-0.000393	0.000929	0.000026

11. APPENDIX E : PREVIOUS MORTALITY IMPROVEMENT SCALES

i. The AA Scale in the Society of Actuaries 1994 Group Annuity Mortality Table

In 1994, the Society of Actuaries Group Annuity Valuation Table Task Force developed a table that it recommended as suitable for a Group Annuity Reserve Valuation Standard. For projection of mortality reduction beyond 1994, the Task Force decided to use the following formula.

$$IS^{\mathrm{AA}}(x,s) = (1 - AA_x)^s, \qquad \text{(A1)}$$

where values of AA_x can be found in Society of Actuaries Group Annuity Valuation Table Task Force (1995, p.892). Separate scales for different durations and smoker statuses are not available.

ii. Improvement Factors in the Society of Actuaries 2001 Valuation Basic Experience Table

In 2002, the Society of Actuaries and the Academy of Actuaries developed the 2001 CSO tables which was intended to replace the 1980 CSO Table in the current statutory valuation structure (American Academy of Actuaries, 2002). The 2001 CSO tables consists of six tables – for each sex, there are separate tables for nonsmoker, smoker, and composite nonsmoker/smoker. Each table has values for a 25-year select period and for ultimate ages.

The entire work was divided into two pieces: the construction of the 2001 valuation basic experience table (VBT, done by the SoA Task Force), and the development of the loads (done by the Academy's Task Force). A key step in the construction of the VBT was to project the mortality experience underlying the 1990-95 Basic Mortality Tables to year 2001, the projected date at which the valuation table would be released. Having examined improvement in insured lives mortality from the 1985-90 to the 1990-95 Basic Mortality Tables and considered improvement from various non-life insurance sources, the SoA Task Force decided the following factors to reflect the improvement in the insured lives

mortality.

For males:

$$IS^{\text{VBT}}(x,s)=\begin{Bmatrix} 0 & x<45 \\ \left(1-\dfrac{0.01(x-45)}{10}\right)^{s} & 45\le x<55 \\ 0.99^{s} & 55\le x<80 \\ \left(1-\dfrac{0.01(90-x)}{5}\right)^{s} & 80\le x<90 \\ 0 & x>90 \end{Bmatrix}, \qquad \text{(A2)}$$

and for females:

$$IS^{\text{VBT}}(x,s)=\begin{Bmatrix} 0 & x<45 \\ \left(1-\dfrac{0.005(x-45)}{10}\right)^{s} & 45\le x<55 \\ 0.995^{s} & 55\le x<85 \\ \left(1-\dfrac{0.005(90-x)}{5}\right)^{s} & 85\le x<90 \\ 0 & x>90 \end{Bmatrix}, \qquad \text{(A3)}$$

Separate scales for different durations and smoker statuses are not provided.

iii. The Reduction Factors in the Institute of Actuaries "92" Series Base Table

The CMIB of the Institute of Actuaries has periodically been considering future improvements in mortality for annuitants and pensioners. In "92" Series CMIB tables (CMIB 1999), the projected mortality values are estimated by multiplying the probabilities of death by the following reduction factors (improvement scales).

$$IS^{\text{CMIB}}(x,s)=\alpha(x)+[1-\alpha(x)][1-f_n(x)]^{s/n}. \qquad \text{(A4)}$$

The value for n is fixed at 20. $\alpha(x)$ and $f_{20}(x)$ are given respectively by

$$\alpha(x) = \begin{Bmatrix} 0.13 & x < 60 \\ 1 + 0.87\dfrac{(x-110)}{100} & 60 \le x \le 110 \\ 1 & x > 110 \end{Bmatrix}, \qquad \text{(A5)}$$

$$f_{20}(x) = \begin{Bmatrix} 0.55 & x < 60 \\ \dfrac{0.55(110-x) + 0.29(x-60)}{50} & 60 \le x \le 110 \\ 0.29 & x > 110 \end{Bmatrix}. \qquad \text{(A6)}$$

The same scale is applied to both sexes. Separate scales for different durations and smoker statuses are not available.