Session 3

Panel: Yield Curve Projection Techniques

Distributions Based on Historical Probabilities
Thomas M. McComb

Yield Curve Projection Techniques
Gregory D. Jacobs

Arbitrage-Free Interest Rate Models
Steven P. Miller

DISTRIBUTIONS BASED ON HISTORICAL PROBABILITIES

MR. THOMAS M. MCCOMB: The following is a practical discussion of ways to project yield curves based on historical probabilities. Included in this discussion will be ways to produce yield curves from predetermined distributions such as required by the New York Regulation 126. The discussion will include distribution of long-term and short-term U.S. bond yields as well as intermediate bond yields, yields from stocks and projected increases in the consumer price index.

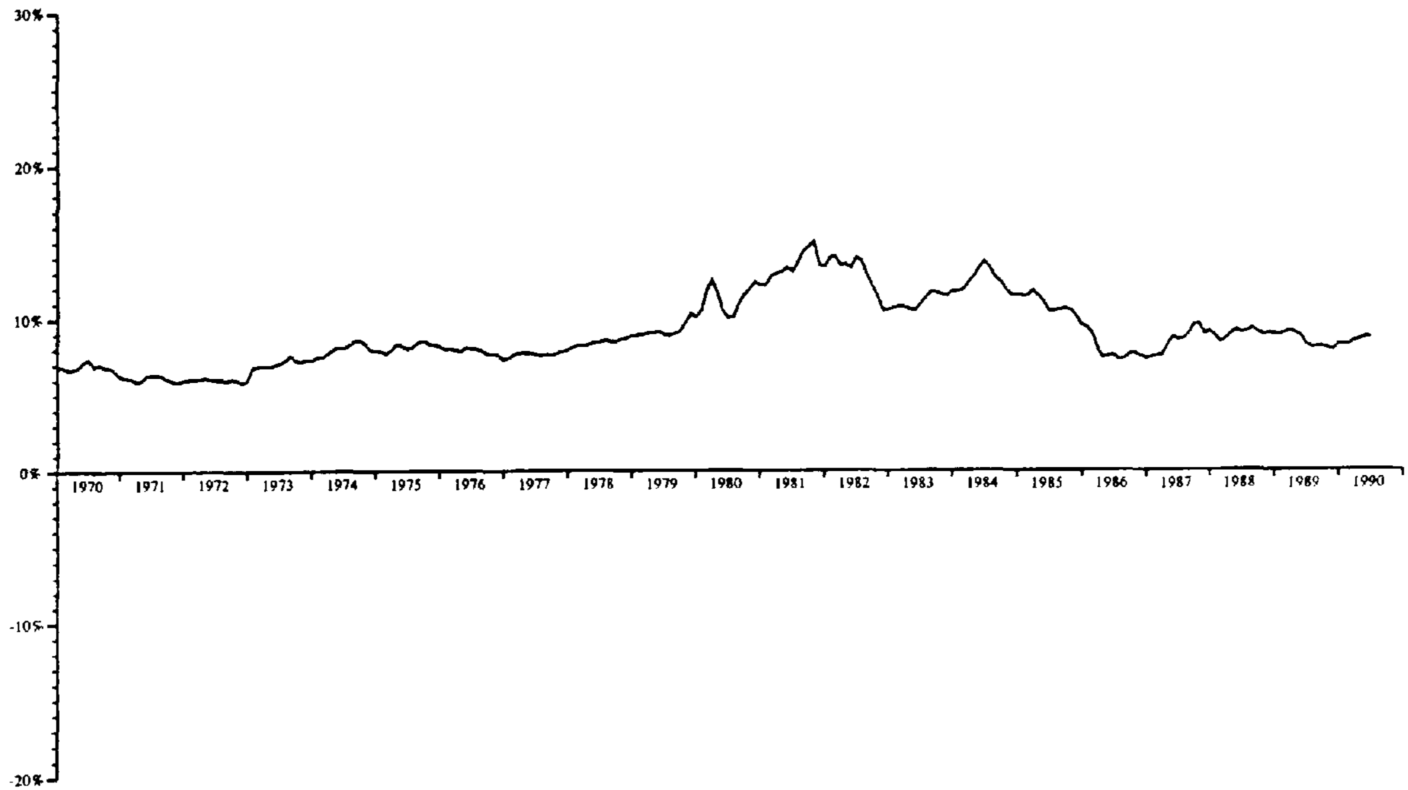
Chart 1 plots the rates on long-term U.S. bonds from 1970 through June 1990.

Interestingly enough the yield in June 1990 is about where the yield was in 1970. Until late 1979, the curve is relatively smooth, but in October 1979, when Chairman Paul Voeckler of the Federal Reserve Board disengaged the money supply, the yield rates began to behave more erratically. This continued throughout the Reagan years but would appear to have leveled out somewhat in 1990.

Any nonadjusted statistical function might be biased if based upon a pure historical analysis because of a long-term upward or downward trend. For example, if we had started an historical study in 1981, when long-term bond rates hit 14.97%, and ended in 1989, when

CHART 1

PLOT OF HISTORICAL RATES 1970 - 1990
Annual Long-Term U.S. Bond Yields



they were 8.26%, we would assume that bond rates tend to decrease. We must produce a function that has no long-term upward or downward bias.

We assume that the long-term bond rate during any month is related as a percentage of the bond rate during the previous month. By successive approximations we determine a constant factor such that the mean of the ratio of successive long-term yields, adjusted by this constant, is one.

Our purpose is to find a variable for which we can compute the mean and standard deviation. Once this variable is determined we can use the Knuth Method (Donald E. Knuth, *The Art of Computer Programming*). This procedure is described in the cookbook included at the end of my presentation.

The resulting variable is relatively random as illustrated by the following histogram in Chart 2.

Short-term bond yields, based on the 90-day bond rate, can be compared with long-term rates as shown in Chart 3. Here the difference between the short-term rates and the long-term bond rates are the "hairs" on the curve. It is logical to assume that the magnitude of the difference is related to the magnitude of the long-term bond rate. While there may be a cyclical pattern, it is difficult to reduce this cyclical pattern to any reasonable formula.

CHART 2

HISTOGRAM OF LONG-TERM YIELD DEVIATIONS

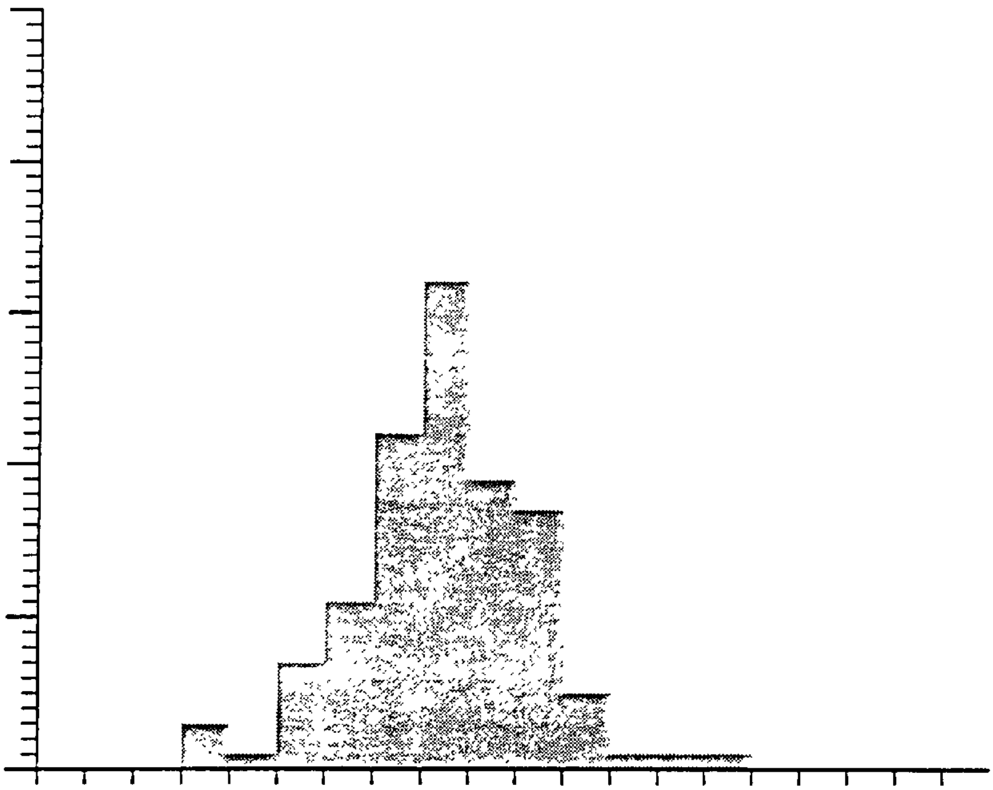
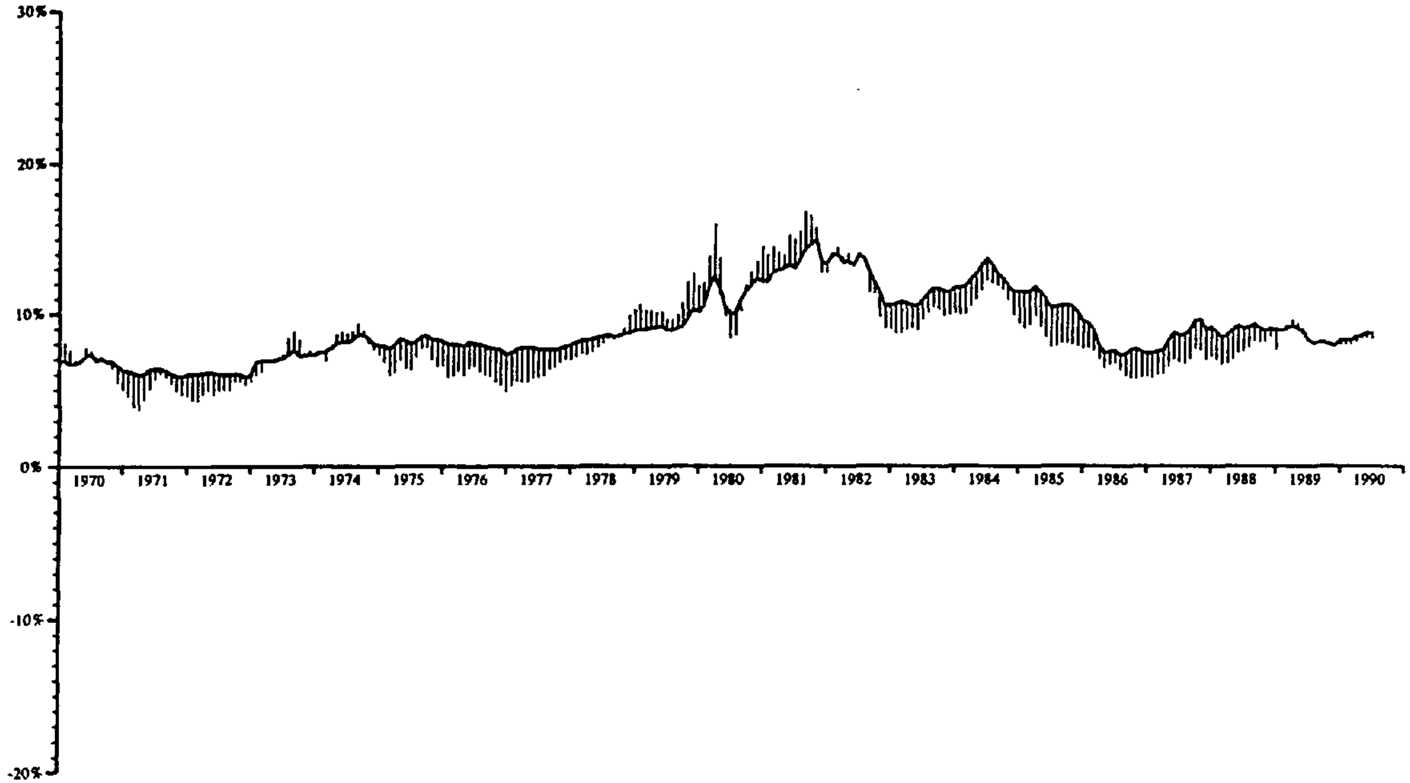


CHART 3

PLOT OF HISTORICAL RATES 1970 - 1990
Annual Long- and Short-Term U.S. Bond Yields



Many argue that rate curve inversions are more apt to occur when rates are high. This was true, of course, for the inversion that occurred from 1979 to 1981. But yield curves in 1973 and 1974 were not particularly high, and there was a yield curve inversion.

We ignore whether or not the curve is inverted and treat, as a variable, the absolute value of the difference of the long-term rate and the short-term rate. We consider the ratio of this absolute value to the long-term rate as a variable and find its mean and deviation. We can then use Knuth's Method again to produce a set of projected values.

Since a normal noninverted curve tends to stay normal and an inverted curve tends to stay inverted, although not at the same persistency, we can determine the probability that a normal curve will not invert and the probability that an inverted curve will stay inverted.

For durations more than 90 days but less than 30 years, assume that the intervening rate is the long-term rate plus or minus a portion of the absolute value of the difference between the two rates. Assume also that the portion for any duration is an exponential function given in the cookbook. We can then assume an exact fit for the values for one year and determine the value of A and B sufficient to produce a complete yield curve as described in the cookbook.

We define stock yields to be the net change in Standard & Poors (S & P) stock values plus the total dividend yield on S & P stocks. Chart 4 graphs these stock yields for the same period.

It is also helpful to compare the bond yields to the long-term stock yields (Chart 5). Note, however, that we are dealing with monthly stock yields and annual bond yields.

In order to get a reasonable variable, we convert the annual bond yield to a monthly yield and determine the variable, which is the ratio of one plus the monthly stock yield to one plus the monthly bond yield. Again we use Knuth's Method to produce a set of projected values. The histogram of stock yield deviations in Chart 6 is relatively normal as shown.

For some purposes it is necessary to make a projection of CPI increases. Chart 7 shows equivalent annual CPI increases for the period 1970 to 1990. Chart 8 is the same graph superimposed with the long- and short-term bond yields. The CPI curve is clearly volatile although not nearly as volatile as stock yields. With the exception of 1973 through 1975 there is a pretty good correlation. Actually the best correlation is to the higher of the long-term or short-term bond yields, i.e., when the yield curve is inverted, correlation should be to the short-term yield rate. Logically our variable is a constant multiple of the bond rate plus or minus a deviation. We can solve for the constant multiple using the least squares

CHART 4

PLOT OF HISTORICAL RATES 1970 - 1990
Monthly Net S & P Stock Yields

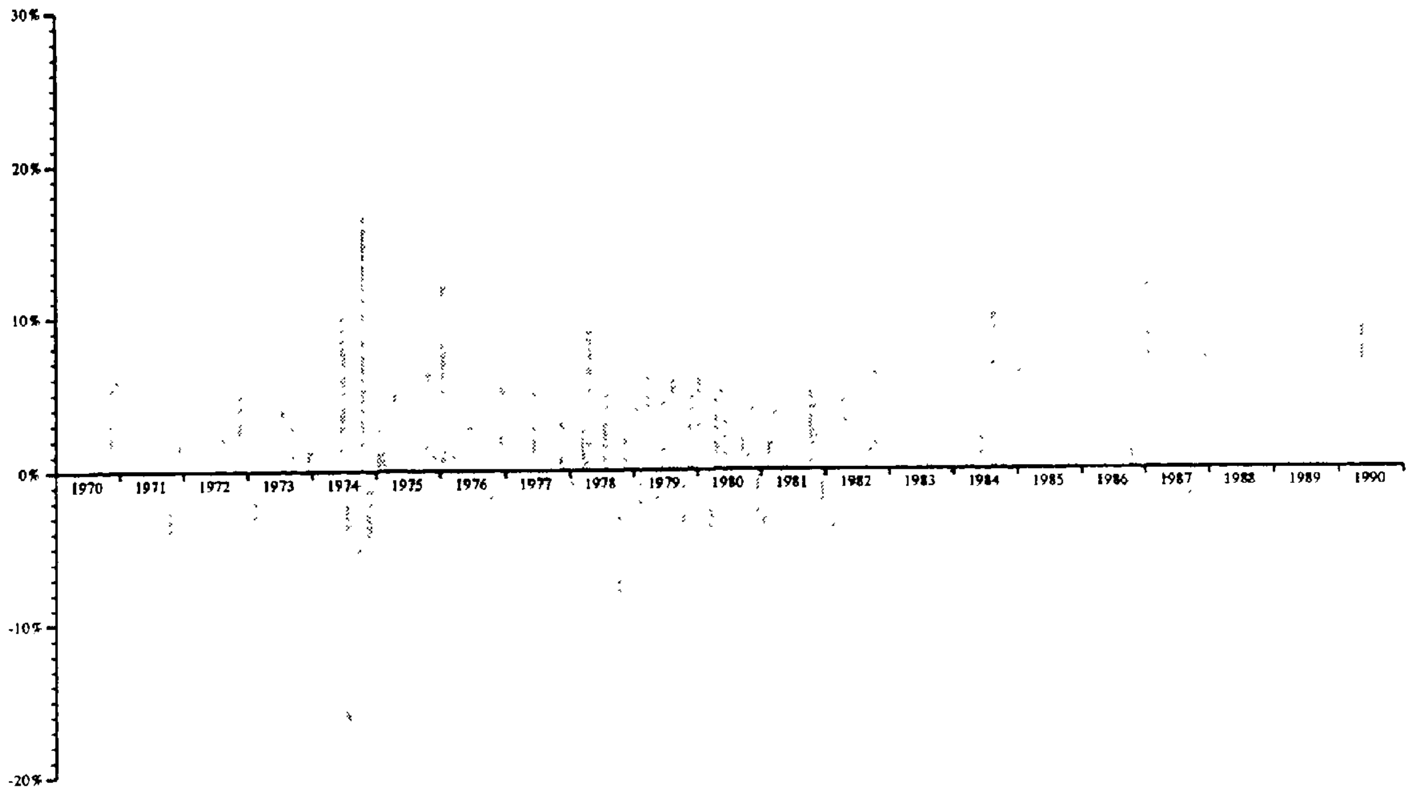


CHART 5

PLOT OF HISTORICAL RATES 1970 - 1990
Monthly Net S & P Stock Yields/Long-Term Bond Yields

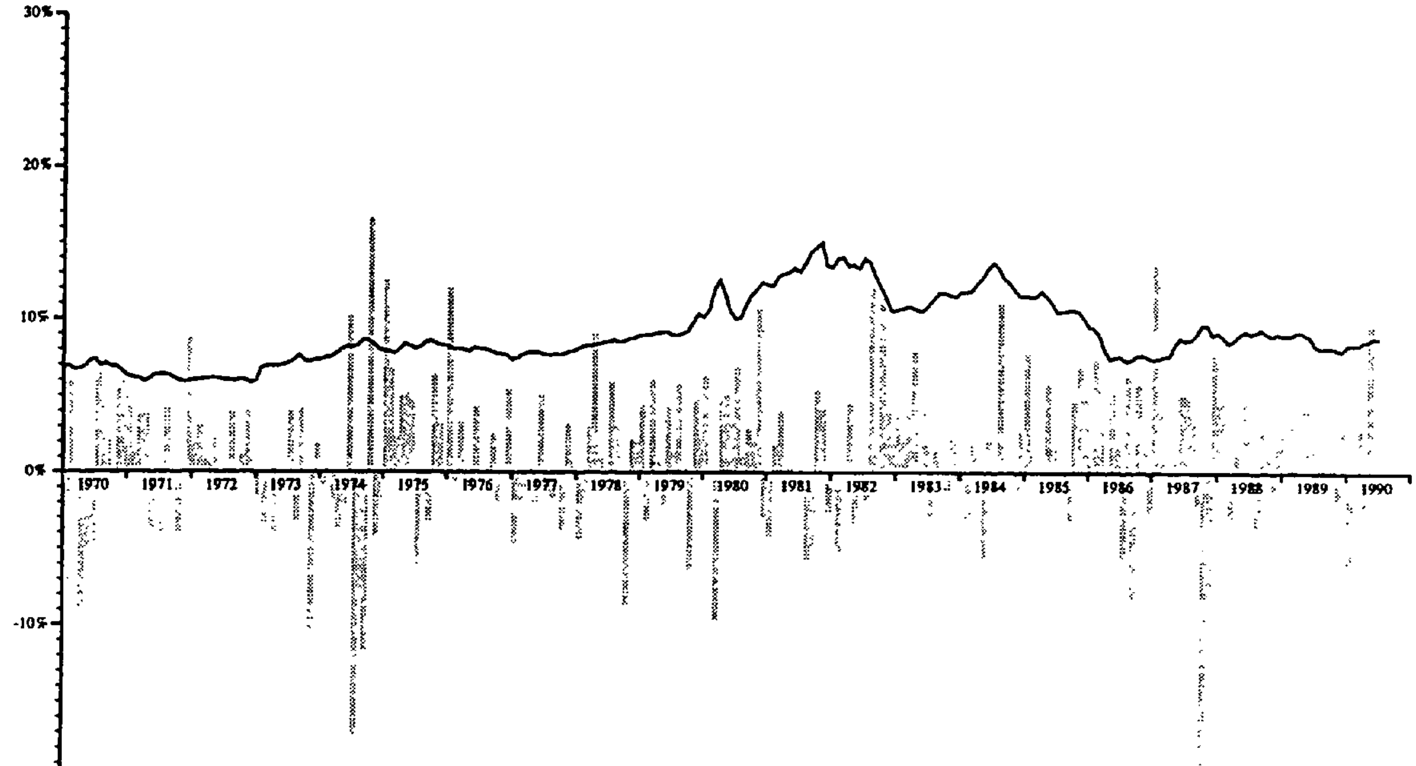


CHART 6

HISTOGRAM OF STOCK YIELD DEVIATIONS

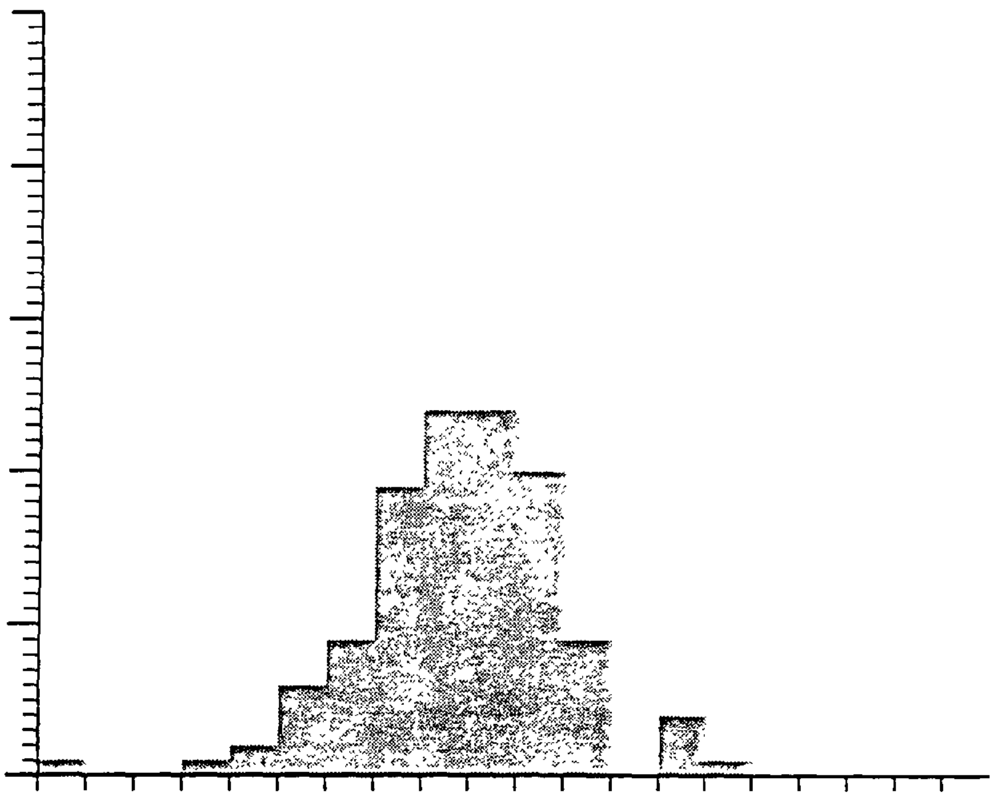


CHART 7

PLOT OF HISTORICAL RATES 1970 - 1990
Equivalent Annual CPI Increases

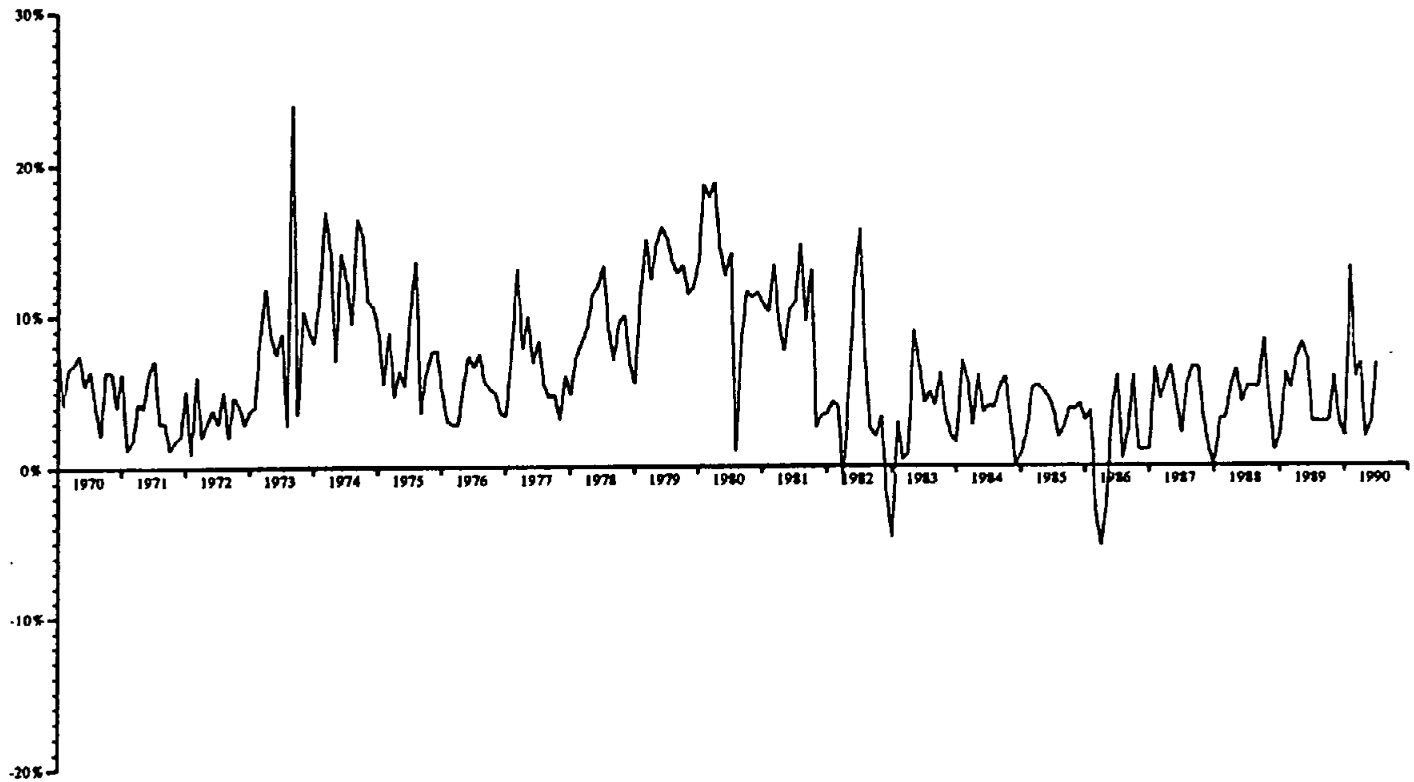
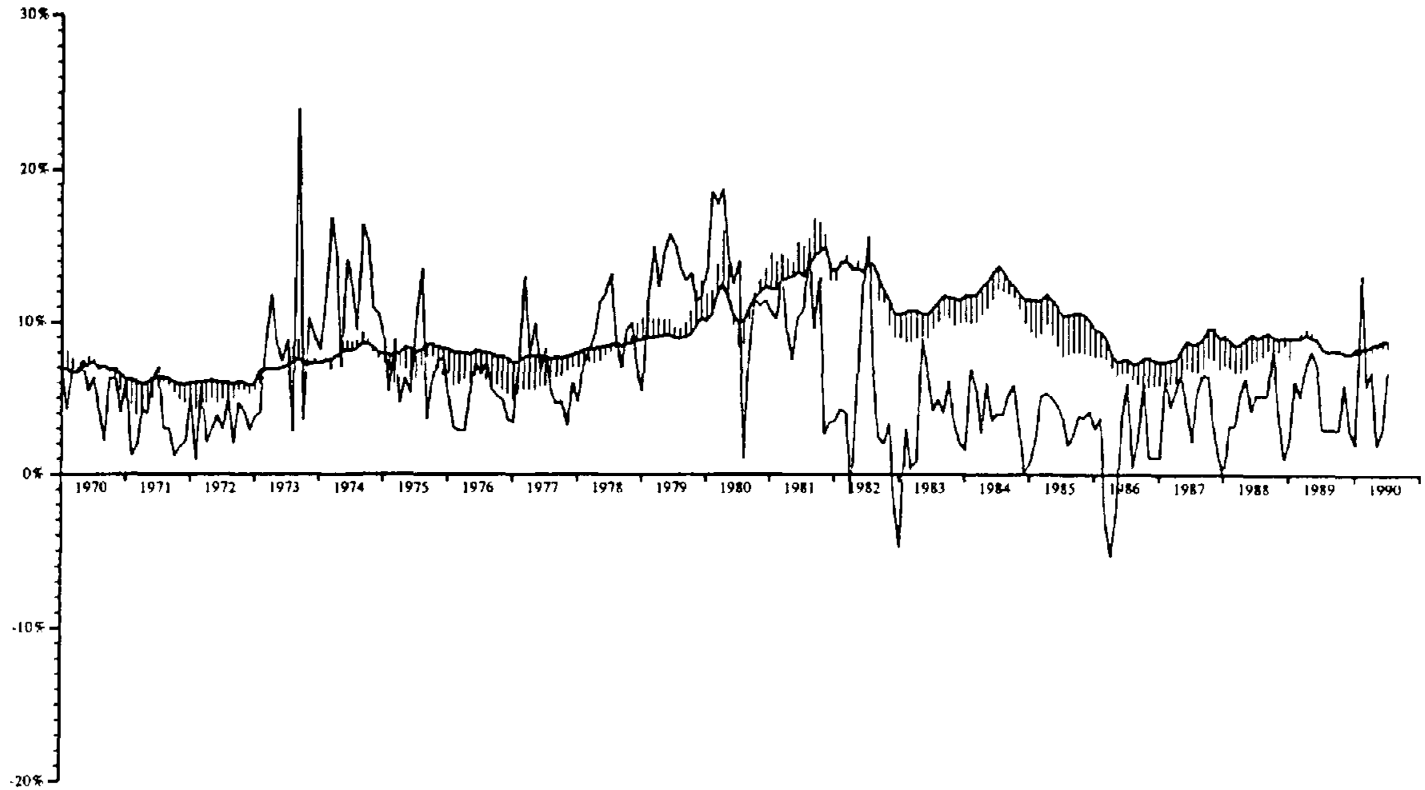


CHART 8

PLOT OF HISTORICAL RATES 1970 - 1990
Equivalent Annual CPI Increases/Bond Yields



method and treat the deviation as a variable. We can then determine the mean and standard deviation of this variable and use Knuth's Method to produce projected values.

It is possible to plot bond yields, stock yields and equivalent CPI increases. Chart 9 is an historical plot of these rates for the period 1970 to 1990. The primary purpose of this graph is to compare with projected graphs in the future.

Finally I show in tabular form, the analysis of yield rate variances and CPI deviations for the period from January 1, 1980 to December of 1989 (Chart 10). All factors and constants and observed and expected values of intermediate bond yields are also shown.

In making the historical study I chose to take the 10-year period beginning in 1980. This work was actually completed for a December 1989 study.

In the case of the seven prescribed trials for New York, the long-term bond yield rate is not stochastically projected but is predetermined. The NAIC Model Act makes similar provision for some studies. Nevertheless, the other rates for these predetermined scenarios, that is, the short-term rates, stock rates and CPI rates need to be stochastically projected. Chart 11 shows a summary of 107 stochastic trials, the last seven of which are in accordance with the New York regulation.

CHART 9

PLOT OF HISTORICAL RATES 1970 - 1990
Annual Long & Short Term U.S. Bond Yields, Monthly Net S & P Stock Yields, Equivalent Annual CPI Increases

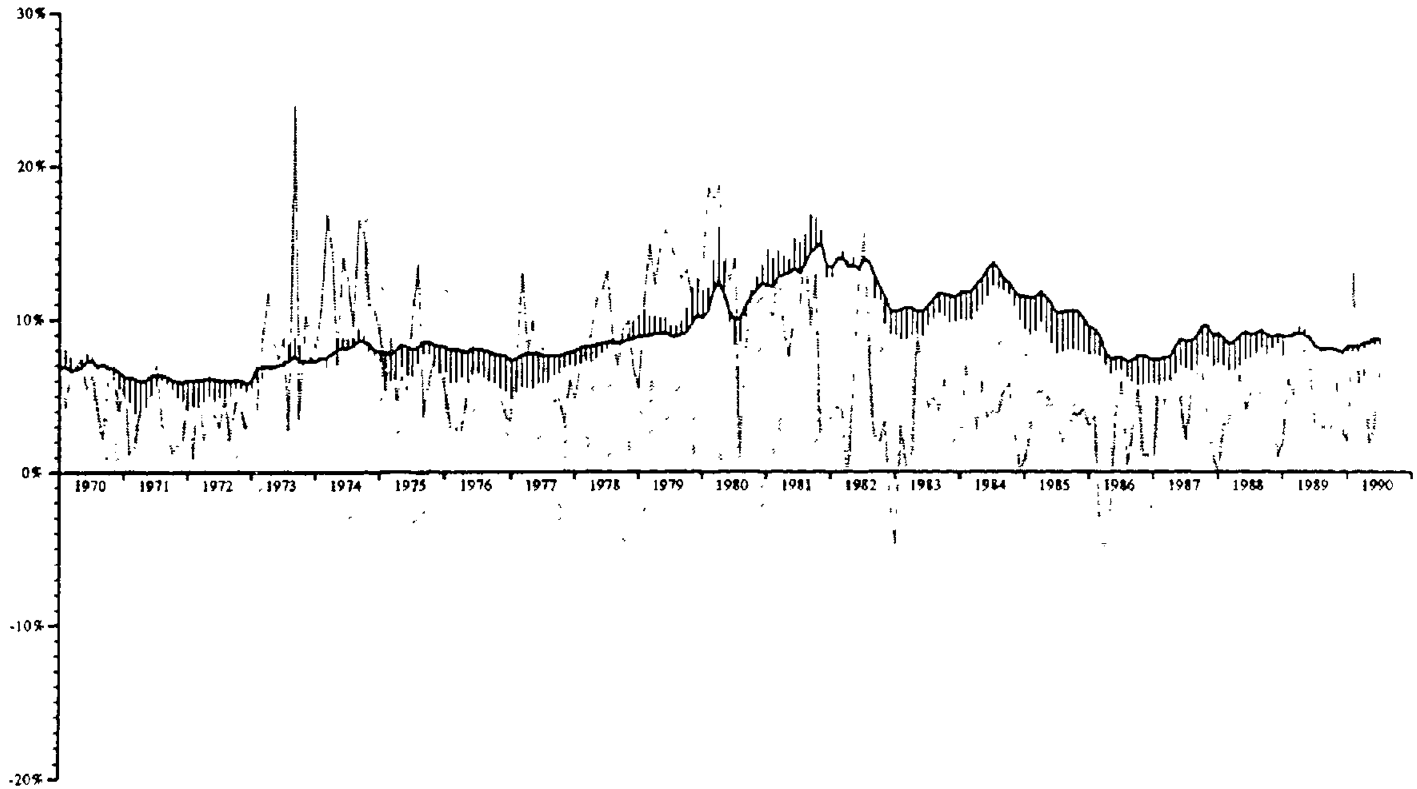


CHART 10

Analysis of Yield Rate Variances and CPI Deviations for Period From 1/80 to 12/89

	Long Term Yield		Short Term Dev'n		Stock Yield		Cost of Living Increase		
Mo Yr	L.T. Rate	Multiple/ Last Rate	S.T. Dev'n	Multiple/ L.T. Rate	Mo'ly Yield	Multiple/ L.T. (1+r)	C.P.I. Index	Annual Change	Deviation/ Expected
1/80	10.530	1.041618576	-1.660	0.157644824	6.202	1.053196356	77.910	18.606	-0.076680967
2/80	11.850	1.120102668	-1.850	0.156118143	-0.012	0.990592151	78.980	17.784	-0.054918401
3/80	12.490	1.052013411	-4.040	0.323458767	-9.704	0.894147196	80.120	18.764	-0.039326456
4/80	11.650	0.936712565	-1.080	0.092703863	4.605	1.036487852	81.020	14.345	-0.029225376
5/80	10.510	0.907194247	2.830	0.269267364	5.126	1.042541476	81.820	12.514	-0.030839326
6/80	10.010	0.955271410	2.930	0.292707293	3.135	1.023183181	82.720	14.028	-0.050467781
7/80	10.070	1.006652755	2.190	0.217477656	6.927	1.060754709	82.790	1.020	0.080151324
8/80	10.930	1.082445524	0.900	0.082342177	0.996	1.001267430	83.330	8.114	0.016929705
9/80	11.560	1.055727526	1.100	0.095155709	2.916	1.019820778	84.090	11.510	-0.011382163
10/80	11.910	1.029585283	-0.420	0.035264484	1.994	1.010420657	84.830	11.087	-0.000233836
11/80	12.370	1.037431401	-2.010	0.162489895	10.616	1.095461414	85.600	11.453	0.014495588
12/80	12.200	0.987690611	-2.790	0.228688525	-3.002	0.960719727	86.330	10.728	0.027222785
1/81	12.140	0.996081545	-3.060	0.252059308	-4.184	0.949054874	87.030	10.176	0.034622807
2/81	12.750	1.048377474	-1.350	0.105882353	1.735	1.007226964	87.940	13.295	-0.006434576
3/81	12.910	1.012608843	0.210	0.016266460	4.001	1.029539818	88.600	9.387	0.021961902
4/81	13.010	1.008048410	-0.540	0.041506533	-1.950	0.970557325	89.140	7.564	0.045938015
5/81	13.310	1.022514470	-3.440	0.258452292	0.240	0.992016064	89.870	10.282	0.047468212
6/81	13.060	0.982915746	-1.280	0.098009188	-0.632	0.983567418	90.640	10.780	0.020864919
7/81	13.610	1.040481187	-1.460	0.107274063	0.201	0.991411630	91.680	14.672	-0.011501490
8/81	14.340	1.051190609	-1.490	0.103905160	-5.791	0.931628026	92.380	9.557	0.046465007
9/81	14.660	1.021636334	0.460	0.031377899	-4.922	0.940002574	93.320	12.918	0.002362377
10/81	14.970	1.020499227	1.620	0.108216433	5.374	1.041561109	93.520	2.602	0.108295612
11/81	13.470	0.905926569	2.910	0.216035635	4.109	1.030184126	93.780	3.388	0.086983353
12/81	13.380	0.994370842	2.340	0.174887892	-2.556	0.964296074	94.050	3.510	0.084951251
1/82	13.920	1.038761157	0.560	0.040229885	-1.273	0.976605747	94.380	4.293	0.081970331
2/82	14.040	1.008794421	1.610	0.114672365	-5.564	0.934077296	94.690	4.014	0.085839407
3/82	13.450	0.960945322	0.900	0.066914498	-0.509	0.984502344	94.590	-1.260	0.133280220
4/82	13.530	1.006307566	1.060	0.078344420	4.487	1.033878983	94.990	5.194	0.069456075
5/82	13.270	0.982509951	1.790	0.134890731	-3.433	0.955694671	95.920	12.402	-0.004957913
6/82	13.950	1.049027737	1.360	0.097491039	-1.521	0.974131161	97.090	15.660	-0.031435181
7/82	13.740	0.986454426	1.180	0.085880640	-1.789	0.971629513	97.630	6.882	0.054460492
8/82	12.800	0.935847594	5.050	0.394531250	12.110	1.109903602	97.830	2.486	0.089987008
9/82	12.100	0.948754108	4.250	0.351239669	1.219	1.002601249	97.990	1.980	0.088764325
10/82	11.460	0.950367854	3.430	0.299301920	11.453	1.104498617	98.260	3.357	0.069254977
11/82	10.550	0.924863479	2.610	0.247393365	4.013	1.031472674	98.090	-2.056	0.115225103
12/82	10.550	1.000864996	2.570	0.243601896	1.918	1.010697048	97.690	-4.785	0.142512187
1/83	10.670	1.011736875	2.610	0.244611059	3.704	1.028315421	97.930	2.988	0.065854339
2/83	10.800	1.012494914	2.910	0.269444444	2.286	1.014155496	97.960	0.368	0.093221117
3/83	10.770	0.998192052	2.350	0.218198700	3.684	1.028039726	98.030	0.861	0.088025526
4/83	10.570	0.983091221	2.430	0.229895932	7.855	1.069556769	98.730	8.913	0.005704984
5/83	10.530	0.997244241	2.070	0.196581197	-0.887	0.982895336	99.260	6.635	0.028126326
6/83	10.930	1.037135940	1.950	0.178408051	3.871	1.029769983	99.600	4.189	0.056182383
7/83	11.330	1.035720946	2.200	0.194174757	-2.958	0.961779216	100.000	4.927	0.052387207
8/83	11.730	1.034406258	2.560	0.218243819	1.487	1.005532854	100.330	4.033	0.064921167
9/83	11.680	0.996725111	2.950	0.252568493	1.362	1.004331816	100.830	6.147	0.043329893
10/83	11.560	0.991001464	2.900	0.250865052	-1.171	0.979321657	101.100	3.261	0.071111746
11/83	11.430	0.990078076	2.620	0.229221347	2.095	1.011783585	101.270	2.037	0.082190408
12/83	11.760	1.028274128	2.820	0.239795918	-0.531	0.985516468	101.400	1.551	0.090003457

CHART 10 -- Continued

Analysis of Yield Rate Variances and CPI Deviations for Period From 1/80 to 12/89

	Long Term Yield		Short Term Dev'n		Stock Yield		Cost of Living Increase		
Mo Yr	L.T. Rate	Multiple/ Last Rate	S.T. Dev'n	Multiple/ L.T. Rate	Mo'ly Yield	Multiple/ L.T. (1+r)	C.P.I. Index	Annual Change	Deviation/ Expected
1/84	11.770	1.001589698	2.850	0.242141037	-0.574	0.985083089	101.970	6.958	0.036026040
2/84	11.820	1.004821225	2.690	0.227580372	-3.512	0.955938634	102.440	5.673	0.049321097
3/84	12.310	1.040134704	2.560	0.207961007	1.728	1.007485886	102.670	2.728	0.083174136
4/84	12.680	1.029241298	3.040	0.239747634	0.925	0.999259275	103.170	6.003	0.053741135
5/84	13.260	1.043988038	3.310	0.249622926	-5.552	0.934730440	103.470	3.546	0.083518488
6/84	13.690	1.031314622	3.920	0.286340394	2.143	1.010566941	103.810	4.015	0.082681820
7/84	13.310	0.974428813	3.010	0.226145755	-1.244	0.977329792	104.140	3.882	0.080603801
8/84	12.680	0.955822309	2.080	0.164037855	11.010	1.099110946	104.580	5.190	0.061875866
9/84	12.390	0.979020176	2.120	0.171105730	0.023	0.990541260	105.080	5.891	0.052264454
10/84	11.900	0.963132444	2.370	0.199159664	0.371	0.994349540	105.340	3.010	0.076674125
11/84	11.520	0.970356408	3.100	0.269097222	-1.136	0.979697759	105.340	0.000	0.103363640
12/84	11.520	1.000795924	3.770	0.327256944	2.619	1.016908119	105.410	0.800	0.095360252
1/85	11.450	0.995006621	3.690	0.322270742	7.777	1.068077463	105.610	2.301	0.079728218
2/85	11.470	1.002464577	3.250	0.283347864	1.214	1.003022685	106.050	5.116	0.051758071
3/85	11.810	1.028999507	3.240	0.274343776	0.070	0.991434085	106.510	5.331	0.052654864
4/85	11.470	0.973350610	3.470	0.302528335	-0.103	0.989971319	106.950	5.071	0.052200210
5/85	11.050	0.965854465	3.490	0.315837104	5.757	1.048373197	107.350	4.582	0.053330940
6/85	10.450	0.948880822	3.440	0.329186603	1.558	1.007202967	107.680	3.752	0.056244218
7/85	10.500	1.005449369	3.450	0.328571429	-0.147	0.990256255	107.850	1.911	0.075101257
8/85	10.560	1.006333218	3.380	0.320075758	-0.855	0.983190442	108.080	2.589	0.068856596
9/85	10.610	1.005391004	3.530	0.332704995	-3.120	0.960692936	108.420	3.841	0.056788770
10/85	10.500	0.990943841	3.330	0.317142857	4.601	1.037342839	108.750	3.714	0.057069339
11/85	10.060	0.960713939	2.860	0.284294235	6.838	1.059879821	109.120	4.160	0.048663424
12/85	9.530	0.950308963	2.460	0.258132214	4.824	1.040318458	109.390	3.010	0.055408779
1/86	9.390	0.986836105	2.350	0.250266241	0.556	0.998067363	109.720	3.681	0.047444659
2/86	8.920	0.952772833	1.890	0.211883408	7.454	1.066916138	109.420	-3.232	0.112356883
3/86	7.960	0.896967504	1.370	0.172110553	5.566	1.048943654	108.920	-5.348	0.124898632
4/86	7.390	0.931730840	1.330	0.179972936	-1.134	0.982803384	108.680	-2.612	0.092430381
5/86	7.520	1.018256419	1.400	0.186170213	5.304	1.046696480	109.020	3.819	0.029279342
6/86	7.570	1.007634877	1.360	0.179656539	1.687	1.010705209	109.550	5.992	0.007998786
7/86	7.270	0.962758219	1.430	0.196698762	-5.588	0.938614677	109.590	0.439	0.060839982
8/86	7.330	1.009236443	1.760	0.240109141	7.395	1.067637865	109.790	2.212	0.043647736
9/86	7.620	1.039534145	2.430	0.318897638	-8.263	0.911773120	110.320	5.949	0.008878899
10/86	7.700	1.011342434	2.520	0.327272727	5.759	1.051072549	110.420	1.093	0.058156703
11/86	7.520	0.978524531	2.170	0.288563830	2.427	1.018099791	110.520	1.092	0.056551595
12/86	7.370	0.981861089	1.880	0.255088195	-2.552	0.968722419	110.620	1.091	0.055215646
1/87	7.390	1.003849191	1.940	0.262516915	13.437	1.127650229	111.200	6.476	0.001542473
2/87	7.540	1.020876376	1.950	0.258620690	3.941	1.033132574	111.600	4.403	0.023623154
3/87	7.550	1.002477054	1.990	0.263576159	2.877	1.022548910	112.100	5.511	0.012634420
4/87	8.250	1.090723948	2.490	0.301818182	-0.900	0.984474933	112.700	6.615	0.007870174
5/87	8.780	1.062988973	3.030	0.345102506	0.852	1.001471883	113.100	4.343	0.035346609
6/87	8.570	0.977966498	2.880	0.336056009	5.032	1.043147726	113.300	2.143	0.055466886
7/87	8.640	1.008928629	2.860	0.331018519	5.055	1.043320115	113.800	5.426	0.023261523
8/87	8.970	1.037815070	2.970	0.331103679	3.717	1.029771893	114.400	6.514	0.015347458
9/87	9.590	1.067374799	3.270	0.340980188	-2.189	0.970674130	115.000	6.478	0.021261969
10/87	9.610	1.002949317	3.210	0.334027055	-21.497	0.779050132	115.300	3.176	0.054468666
11/87	8.950	0.934827340	3.140	0.350837989	-8.236	0.911108434	115.400	1.046	0.069846794
12/87	9.120	1.019285249	3.320	0.364035088	7.590	1.068103187	115.400	0.000	0.081829548

CHART 10 -- Continued

Analysis of Yield Rate Variances and CPI Deviations for Period From 1/80 to 12/89

	Long Term Yield		Short Term Dev'n		Stock Yield		Cost of Living Increase		
Mo Yr	L.T. Rate	Multiple/ Last Rate	S.T. Dev'n	Multiple/ L.T. Rate	Mo'ly Yield	Multiple/ L.T. (1+r)	C.P.I. Index	Annual Change	Deviation/ Expected
1/88	8.830	0.970364953	2.930	0.331823330	4.343	1.036098234	115.700	3.165	0.047581742
2/88	8.420	0.956222048	2.730	0.324228029	4.474	1.037725382	116.000	3.156	0.043986230
3/88	8.630	1.025116872	2.940	0.340672074	-3.050	0.962835287	116.500	5.297	0.024464857
4/88	8.950	1.036746364	3.030	0.338547486	1.235	1.005144308	117.100	6.358	0.016720606
5/88	9.230	1.031095443	2.960	0.320693391	0.629	0.998913761	117.500	4.177	0.041046979
6/88	9.000	0.976997367	2.500	0.277777778	4.619	1.038703726	118.000	5.228	0.028476787
7/88	9.140	1.015970230	2.410	0.263676149	-0.243	0.990325703	118.500	5.205	0.029959639
8/88	9.320	1.019921813	2.300	0.246781116	-3.554	0.957324673	119.000	5.182	0.031799434
9/88	9.060	0.974128521	1.830	0.201986755	4.275	1.035240866	119.800	8.372	-0.002431778
10/88	8.890	0.982930784	1.550	0.174353206	2.892	1.021643264	120.200	4.081	0.038955051
11/88	9.020	1.015092367	1.340	0.148558758	-1.587	0.977072874	120.300	1.003	0.070903125
12/88	8.960	0.994600399	2.270	0.253348214	1.770	1.010448500	120.500	2.013	0.060260382
1/89	8.920	0.996712083	0.640	0.071748879	3.445	1.027110576	121.100	6.141	0.018620202
2/89	9.000	1.009647526	0.520	0.057777778	3.252	1.025131546	121.600	5.069	0.030066272
3/89	9.170	1.019174503	0.340	0.037077426	0.297	0.995663684	122.300	7.131	0.010969590
4/89	9.030	0.986295210	0.330	0.036544850	3.485	1.027421311	123.100	8.138	-0.000360016
5/89	8.830	0.979669252	0.440	0.049830125	4.073	1.033417206	123.800	7.041	0.008815187
6/89	8.270	0.939791990	0.050	0.006045949	3.336	1.026540217	124.100	2.947	0.044733017
7/89	8.080	0.978890368	0.160	0.019801980	2.747	1.020838496	124.400	2.940	0.043100426
8/89	8.120	1.005898782	0.210	0.025862069	4.598	1.039196984	124.700	2.933	0.043531165
9/89	8.150	1.004679617	0.430	0.052760736	0.484	0.998300690	125.000	2.925	0.043871828
10/89	8.000	0.983315381	0.410	0.051250000	0.289	0.996478627	125.600	5.915	0.012635070
11/89	7.900	0.989038038	0.230	0.029113924	-1.692	0.976870678	125.900	2.904	0.041841080
12/89	8.260	1.045142559	0.620	0.075060533	2.692	1.020150571	126.100	1.923	0.054882979
FACTOR		0.999992807							0.897253815
MEAN		1.000000000		0.211200238		1.0060349855			0.045117970
SD		0.037797630		0.100958786		0.0460022759			0.038827318

PERCENT WHICH INVERT 1.904761905 %
PERCENT WHICH DISINVERT 20.000000000 %

	Observed	Expected
1 YR	57.546	57.546
2 YR	76.797	76.586
3 YR	82.962	84.241
5 YR	91.163	90.913
7 YR	98.392	93.931
10 YR	101.867	96.261

a=1.9054440637E-2 b=-.57163321913

CHART 11

Summary of 107 Stochastic Trials
(101-107 are New York Prescribed)

	---Long Term Bonds--			--Short Term Bonds--			----S & P Stocks----			C.P.I.
Tr.	Equiv.	High	Low	Equiv.	High	Low	Equiv.	High	Low	Equiv.
1	12.81	17.45	8.33	11.00	21.06	5.05	18.90	14.00	-12.14	7.44
2	5.26	8.80	2.97	4.35	9.38	1.81	20.72	15.24	-11.83	-0.04
3	7.33	10.97	5.40	6.07	11.17	2.96	11.92	14.08	-14.41	2.50
4	4.42	7.93	3.09	3.71	11.47	1.86	6.25	12.74	-11.19	-0.93
5	7.15	12.85	2.75	5.78	12.00	1.74	20.92	13.86	-11.63	2.07
6	8.67	13.93	4.30	6.87	14.02	2.40	13.08	12.93	-13.45	2.99
7	9.42	12.94	6.69	7.48	15.29	4.06	17.75	13.19	-15.12	4.07
8	10.48	15.86	7.00	8.79	16.47	4.47	18.70	13.99	-9.71	5.24
9	8.09	11.64	6.26	6.68	12.06	3.73	10.48	15.34	-11.78	3.00
10	10.08	13.49	7.24	8.17	17.95	4.66	10.72	12.57	-8.61	4.50
11	6.67	8.76	5.29	5.92	11.45	3.49	11.93	14.95	-11.00	1.46
12	8.16	10.85	5.53	6.52	11.18	3.57	11.35	11.40	-11.17	2.76
13	9.98	14.50	6.80	8.54	16.69	3.84	15.66	13.67	-8.93	4.70
14	7.92	12.09	4.14	6.69	14.44	2.44	11.24	13.53	-12.29	3.18
15	5.74	8.38	3.91	4.86	10.58	2.63	10.44	13.62	-15.27	0.72
16	6.64	9.61	3.89	5.41	12.58	2.43	10.61	13.16	-11.07	1.53
17	10.48	14.47	7.41	8.63	16.69	4.98	11.14	14.65	-14.09	5.04
18	8.55	12.92	6.48	7.14	14.03	3.61	7.91	18.15	-12.22	3.33
19	7.81	9.59	5.88	6.40	11.58	4.07	14.24	14.08	-12.51	2.45
20	8.88	11.40	6.66	7.76	14.62	4.00	22.47	15.14	-11.73	3.68
21	5.25	10.57	2.56	4.26	9.59	1.53	11.60	14.92	-9.79	-0.01
22	10.61	16.57	6.11	8.83	17.96	4.01	19.55	18.02	-9.28	5.45
23	12.85	22.34	8.33	10.92	22.25	4.15	21.18	16.86	-12.92	7.32
24	5.31	8.66	3.41	4.51	8.96	2.08	15.06	17.47	-9.22	0.35
25	10.27	13.93	6.52	8.40	18.16	4.57	21.08	14.14	-10.07	4.80
26	7.35	9.14	6.06	6.11	11.16	3.68	25.66	16.98	-10.90	2.18
27	10.42	18.13	7.29	8.45	18.60	5.04	22.36	22.87	-9.32	4.98
28	8.24	10.66	5.58	6.70	12.28	3.80	17.34	13.07	-9.93	2.92
29	19.14	26.39	8.60	16.10	29.92	5.83	31.76	15.26	-14.10	12.65
30	4.92	8.85	2.34	3.98	9.59	1.28	5.03	12.15	-12.50	0.18
31	12.49	16.71	8.41	10.31	19.81	5.81	19.72	15.68	-12.44	7.07
32	8.53	12.32	6.34	7.11	14.63	3.30	14.02	11.63	-17.52	3.29
33	6.17	10.88	3.57	5.44	9.92	2.02	16.81	14.39	-13.79	1.06
34	7.14	10.33	4.49	5.93	11.70	2.95	12.71	14.36	-12.46	2.11
35	9.92	13.45	6.87	7.75	12.57	4.56	20.26	16.15	-9.79	4.20
36	11.51	18.38	4.63	9.25	22.27	3.01	24.42	13.55	-11.64	6.09
37	9.18	11.53	6.58	7.69	14.43	4.13	10.38	15.04	-14.42	4.05
38	2.83	7.93	1.50	2.31	10.30	1.00	4.19	11.76	-14.48	-2.15
39	10.66	15.63	7.25	8.83	13.79	5.09	13.88	14.52	-10.33	5.46
40	4.47	8.57	2.15	3.64	10.72	1.20	6.27	11.04	-10.93	-0.31
41	7.21	9.66	5.01	6.01	12.00	3.51	12.53	13.71	-9.92	1.80
42	6.23	10.16	3.27	5.19	10.78	2.16	13.42	12.65	-13.27	1.46
43	7.56	11.46	4.15	6.67	13.82	2.97	13.08	16.83	-10.97	2.71
44	5.69	11.21	4.03	4.71	10.64	2.29	6.70	17.03	-14.53	0.72
45	7.05	9.32	4.33	5.65	9.68	2.68	7.50	16.80	-11.64	1.76
46	8.52	12.31	5.99	7.06	15.34	3.75	21.72	13.52	-11.52	3.14
47	7.48	11.39	4.22	6.24	13.45	2.72	9.50	13.44	-15.97	2.38
48	6.26	9.06	4.04	5.27	10.94	2.40	12.15	13.80	-11.62	0.89
49	3.50	8.62	1.58	3.01	11.00	1.03	9.97	11.89	-12.96	-1.28
50	8.95	12.39	6.37	7.22	15.24	3.78	13.54	12.32	-13.42	3.46
51	7.93	11.65	4.73	6.51	11.48	3.03	18.39	14.45	-14.23	2.84
52	4.61	8.54	3.32	4.00	10.13	1.90	5.05	18.48	-11.73	-0.13
53	3.75	8.24	1.43	3.15	9.92	0.94	9.53	13.85	-14.06	-1.60
54	6.21	8.88	3.71	5.25	11.99	2.18	8.94	14.18	-12.23	1.33
55	6.68	9.11	4.96	5.39	9.99	2.83	14.20	14.04	-11.55	1.58

CHART 11 -- Continued
Summary of 107 Stochastic Trials
(101-107 are New York Prescribed)

	---Long Term Bonds--			--Short Term Bonds--			----S & P Stocks----			C.P.I.
Tr.	Equiv.	High	Low	Equiv.	High	Low	Equiv.	High	Low	Equiv.
56	9.13	13.02	6.67	7.41	15.47	3.48	13.66	15.00	-11.62	3.70
57	11.73	15.04	7.54	9.83	16.69	5.12	17.98	14.10	-12.88	6.34
58	4.53	8.16	2.84	3.81	10.78	1.77	13.47	12.81	-10.24	-0.28
59	5.87	8.90	3.86	4.77	12.44	2.34	7.12	12.36	-14.67	0.54
60	15.31	22.18	7.71	12.48	27.20	5.20	26.49	14.91	-8.94	9.65
61	4.86	8.65	2.56	3.90	10.20	1.60	13.92	12.36	-10.58	-0.37
62	6.36	9.41	3.54	5.32	12.73	2.52	20.04	14.62	-12.00	1.12
63	6.29	9.71	4.07	5.26	10.48	2.30	4.04	13.22	-13.18	0.73
64	7.49	10.96	5.95	6.27	10.86	3.38	9.79	14.03	-15.44	2.17
65	7.77	10.47	5.74	6.47	12.38	3.35	25.75	12.64	-7.70	2.31
66	6.63	9.20	4.57	5.42	11.64	2.96	14.19	13.53	-9.24	1.56
67	5.53	9.72	3.94	4.43	10.94	2.39	15.52	14.05	-12.71	-0.18
68	6.02	8.19	3.52	4.89	10.92	2.00	11.82	13.88	-11.52	0.81
69	7.67	10.70	5.31	6.18	12.29	2.92	9.23	14.46	-14.24	2.55
70	8.53	12.96	5.80	6.82	12.06	3.58	16.43	13.68	-10.21	3.46
71	13.45	25.78	8.49	11.22	22.01	4.88	26.80	15.28	-10.45	7.63
72	8.56	11.12	6.16	7.06	12.16	4.13	20.22	15.19	-12.43	3.44
73	6.71	9.02	4.89	5.44	12.04	3.34	12.19	12.82	-12.03	1.44
74	7.75	12.97	3.46	6.24	11.80	2.27	17.01	15.20	-13.63	2.21
75	12.60	20.15	6.62	10.84	26.50	5.43	20.64	12.83	-9.47	7.64
76	11.20	16.07	8.03	9.38	16.92	5.17	12.97	17.18	-17.71	5.76
77	7.37	11.19	4.94	6.15	11.12	2.89	20.29	18.26	-9.89	1.99
78	8.17	11.33	5.07	6.80	13.36	3.17	16.96	12.90	-12.85	3.04
79	7.59	10.89	5.66	6.42	12.23	2.96	9.06	14.11	-13.88	2.24
80	5.97	8.35	3.25	5.10	11.57	2.39	17.24	14.90	-11.85	1.19
81	10.96	18.67	7.04	8.78	16.82	3.13	22.02	13.34	-7.89	5.58
82	10.95	21.87	7.27	8.93	22.83	4.45	11.94	11.91	-10.13	5.64
83	6.82	9.86	3.99	6.02	11.45	2.63	12.61	14.85	-14.44	2.10
84	10.37	14.26	6.18	8.29	16.56	3.80	17.78	14.44	-10.24	4.95
85	11.49	20.78	7.05	9.45	23.76	3.98	11.78	15.29	-10.12	5.92
86	7.52	11.87	3.02	6.16	15.01	1.51	12.67	16.54	-15.98	2.57
87	8.92	13.35	6.18	7.60	16.57	3.78	19.70	13.67	-10.05	3.44
88	5.73	10.59	2.15	5.13	13.68	1.32	5.30	15.12	-12.73	0.70
89	8.33	12.11	5.44	7.00	13.53	3.57	7.47	11.80	-13.05	2.85
90	13.41	22.24	6.94	10.90	20.50	4.31	19.79	15.49	-9.98	7.12
91	4.11	8.95	2.29	3.50	12.15	1.46	11.45	14.32	-11.31	-0.81
92	5.14	8.20	3.24	4.16	8.42	2.10	15.83	11.22	-9.24	0.04
93	8.29	9.86	6.88	7.30	12.97	4.45	11.35	12.09	-17.41	3.58
94	14.90	22.24	8.27	12.76	26.95	5.34	14.23	14.41	-16.98	9.11
95	10.08	19.75	6.29	8.18	16.06	4.17	19.97	14.87	-9.30	4.90
96	4.05	8.05	1.52	3.29	10.15	1.05	8.27	14.83	-11.11	-0.40
97	5.31	10.61	2.63	4.44	9.44	1.80	12.28	13.50	-12.14	0.28
98	7.12	9.11	5.60	5.79	9.74	2.88	8.28	14.14	-14.58	1.83
99	5.27	8.20	3.30	4.31	9.86	1.91	13.11	15.28	-10.87	0.51
100	10.62	18.07	6.50	8.72	18.21	3.68	17.86	11.97	-10.76	5.13
Mean	8.12			6.73			14.52			2.88
101	8.26	8.26	8.26	7.21	11.83	3.97	9.70	18.13	-12.83	3.15
102	12.01	13.26	8.30	9.57	17.33	5.24	21.29	15.37	-10.73	6.37
103	4.89	8.22	4.00	3.76	11.16	1.70	13.17	12.66	-12.47	-0.70
104	9.50	13.26	8.26	7.67	11.75	4.35	20.81	17.72	-10.89	4.29
105	7.03	8.26	4.00	5.96	12.36	2.68	10.29	15.85	-11.17	1.57
106	11.26	11.26	11.26	9.11	14.87	5.60	17.31	12.17	-10.47	5.79
107	5.26	5.26	5.26	4.28	6.75	2.24	14.30	12.98	-11.02	0.51

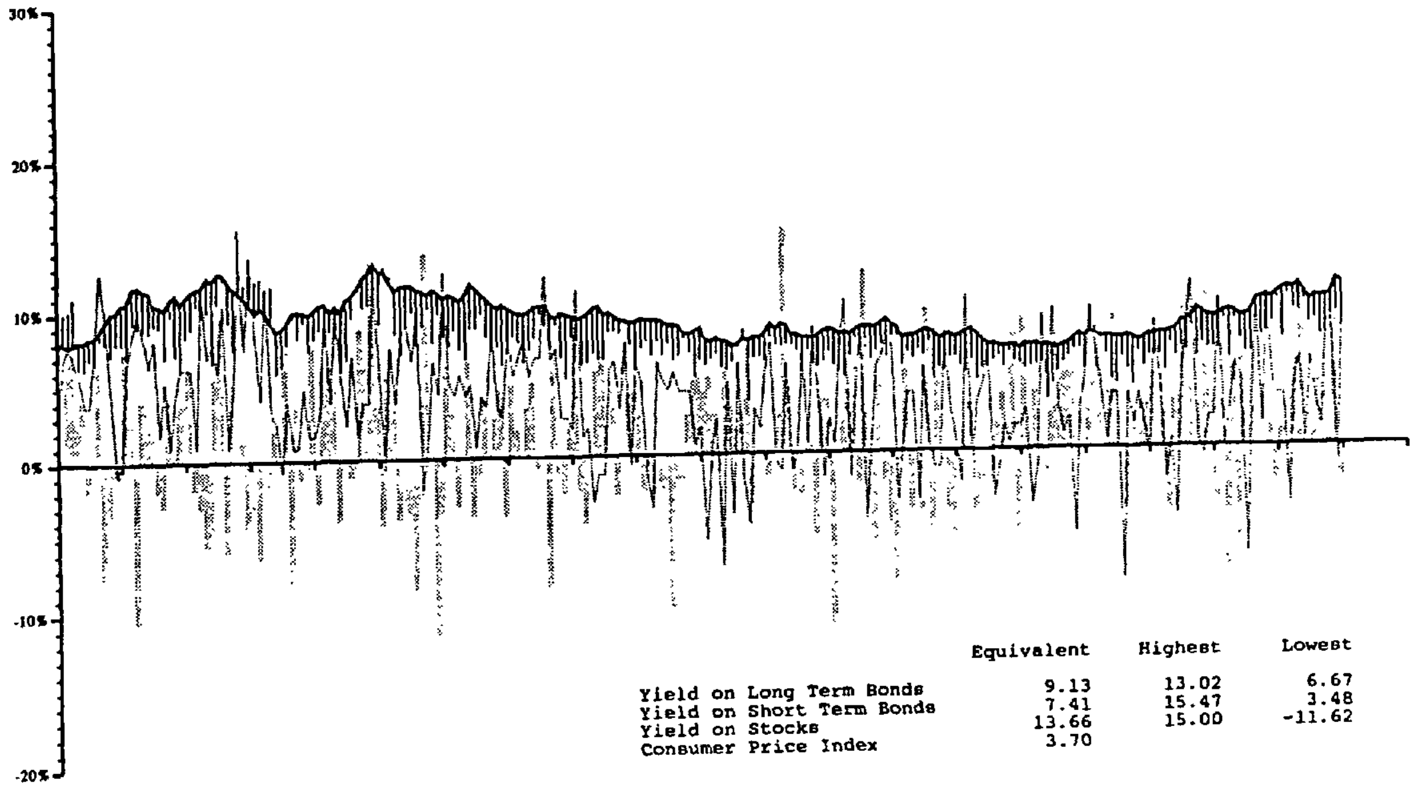
It would not be reasonable to show graphs of all of these projections. I do include some graphs of the projections, one of which is a New York prescribed projection. These graphs may be compared with the composite graph shown previously. Trial number 56 (Chart 12) is a fairly inconsequential projection which, while uninspiring, is consistent with the historical study. Trial number 5 (Chart 13) illustrates a utopian projection where after 13 years the yields rates reduce to what they were in the wonderful years of the 1950s. Trial number 94 (Chart 14) is a nightmare where inflation averages over 9% for 20 years and stock yields are less than the yields on long-term bonds.

It should be noted that I have only projected yields on U.S. government obligations. Since companies will own other than U.S. obligations, it's necessary to equate the yields on non-U.S. government obligations to the rate of government obligations. Further, we have projected stock yields in total, including dividends and market increases. These must be broken apart in order to match the dividend yields on a particular portfolio such that the aggregate is consistent with the assumptions.

I conclude with a cookbook which summarizes the methodology used.

CHART 12

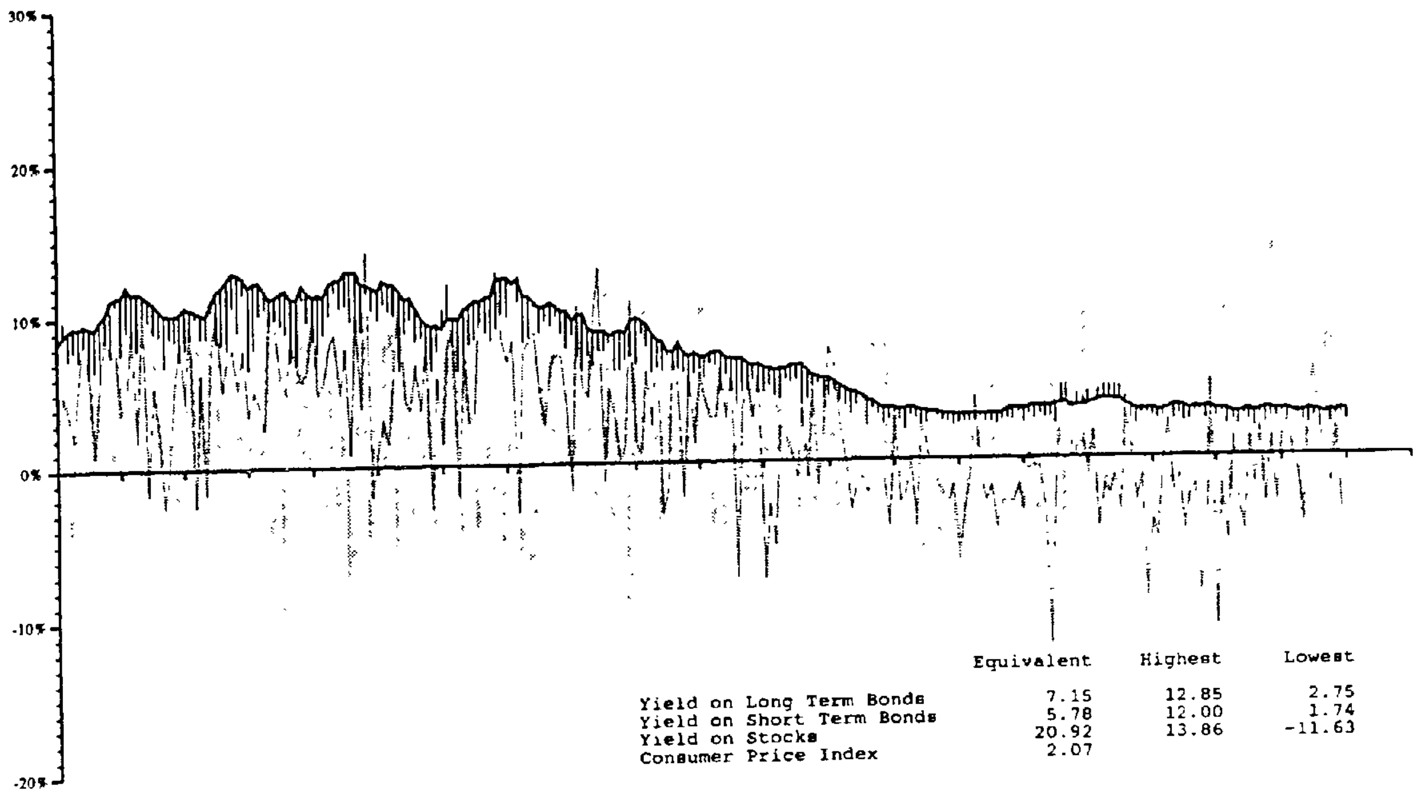
Plot of Projected Rates Commencing 1/90
Trial Number 56



	Equivalent	Highest	Lowest
Yield on Long Term Bonds	9.13	13.02	6.67
Yield on Short Term Bonds	7.41	15.47	3.48
Yield on Stocks	13.66	15.00	-11.62
Consumer Price Index	3.70		

CHART 13

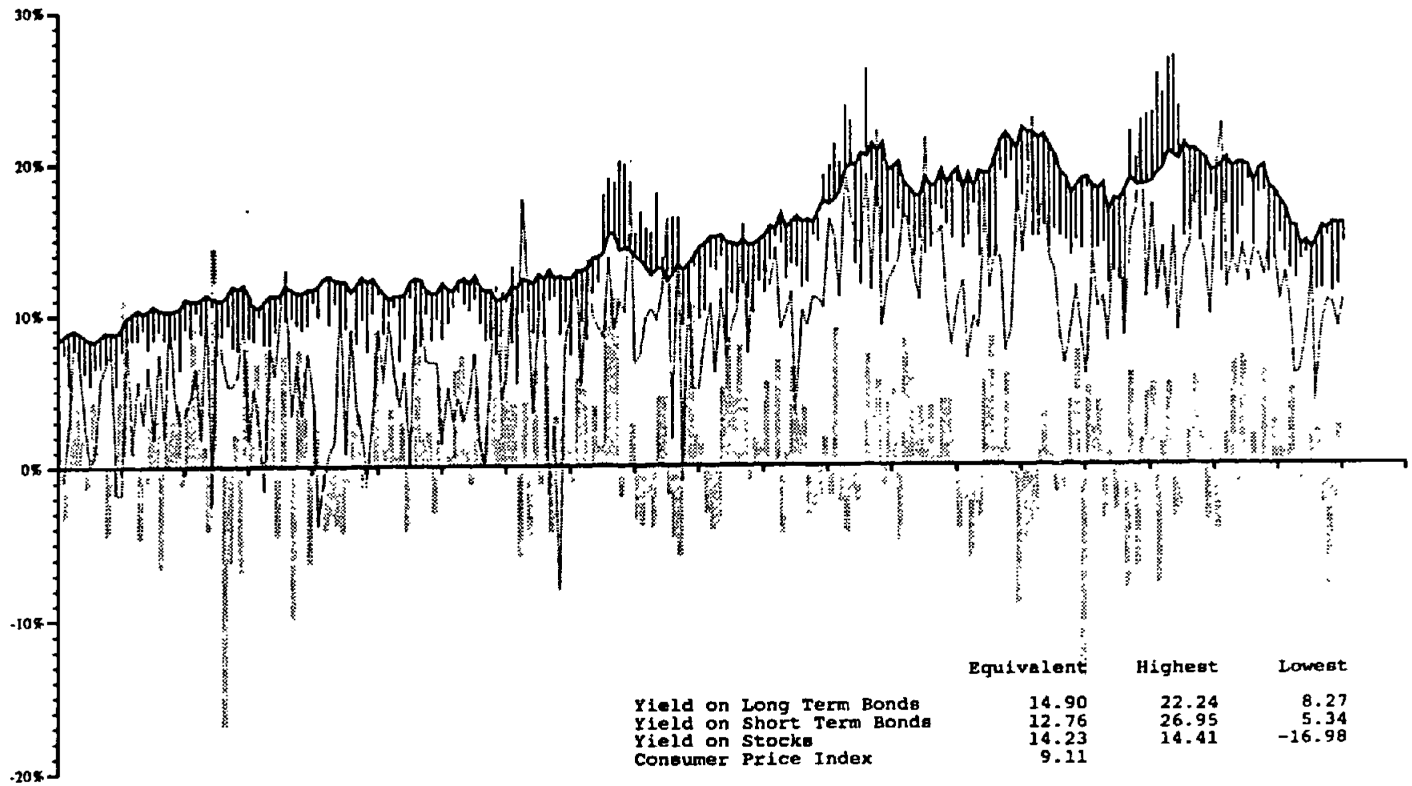
Plot of Projected Rates Commencing 1/90
Trial Number 5



	Equivalent	Highest	Lowest
Yield on Long Term Bonds	7.15	12.85	2.75
Yield on Short Term Bonds	5.78	12.00	1.74
Yield on Stocks	20.92	13.86	-11.63
Consumer Price Index	2.07		

CHART 14

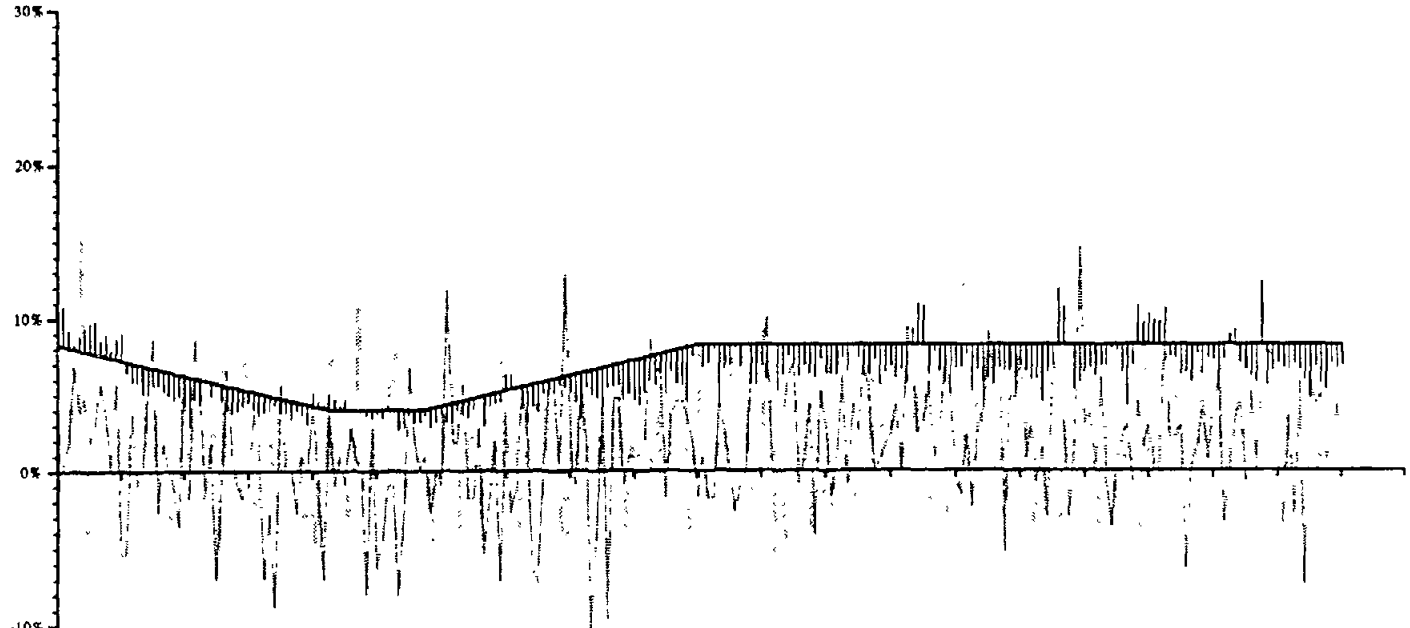
Plot of Projected Rates Commencing 1/90
Trial Number 94



	Equivalent	Highest	Lowest
Yield on Long Term Bonds	14.90	22.24	8.27
Yield on Short Term Bonds	12.76	26.95	5.34
Yield on Stocks	14.23	14.41	-16.98
Consumer Price Index	9.11		

CHART 15

Plot of Projected Rates Commencing 1/90
Trial Number 105



	Equivalent	Highest	Lowest
Yield on Long Term Bonds	7.03	8.26	4.00
Yield on Short Term Bonds	5.96	12.36	2.68
Yield on Stocks	10.29	15.85	-11.17
Consumer Price Index	1.57		

Cookbook for History-Based Yield Curves, Etc.

Long-Term U.S. Treasury Bond Yields

1. Determine a bias factor in long-term bond yields such that any upward or downward trend over the entire period studied is eliminated. (For this purpose use 30-year U.S. Treasury yield rates after February 1970, or 20-year U.S. Treasury yield rates for prior months). The bias factor is determined by successive approximation. Compute a variable for each month equal to the ratio of the current long-term bond yield to the product of the previous long-term bond yield times the bias factor and compute the mean value of this variable. Adjust as needed until the mean value is one.

2. Determine the standard deviation of this variable in the usual manner.

3. Construct a set of projected long-term yield rates using two randomly generated values, Z1 and Z2, such that $0 < Z1 < 1$ and $0 < Z2 < 1$. Each long-term yield rate is the previous long-term yield rate times a factor M computed as follows:

$$R1 = 2 \bullet Z1-1$$

$$R2 = 2 \bullet Z2-1$$

$$M = \mu+\sigma \bullet R1 \bullet \sqrt{\{-2 \bullet \log(s) /s\}}$$

Where $s = R1^2 + R2^2, s < 1$

If the value of s is equal to or greater than one, select new values of Z1 and Z2.

Short-Term U.S. Treasury Bond Yields

1. Compute a variable equal to the absolute value of the difference between the long-term bond yield over the short-term bond yield each month divided by the long-term bond yield for such month. (For this purpose use 90-day Treasury bill rates as short-term rates).

2. Determine the percentage of months in the study when the yield curve "inverts" and the percentage when an inverted yield curve "disinverts."

3. Determine the mean and standard deviation of this variable in the usual manner.

4. Construct a set of projected absolute values of the differences between long-term and short-term rates in the same manner as used to determine long-term rates.

5. Determine whether the yield curve is inverted using a randomly generated value compared to the percentages described in (2).

6. For durations more than 90 days but less than 30 years, assume that the intervening rate is the long-term rate plus or minus a portion of the absolute value of the difference between the long-term rate and the short-term rate according to whether the yield curve is or is not inverted. Assume that the portion for any duration is an exponential function:

$$p = e^{(a+b/y)}$$

where y is the bond duration in years.

Assume an exact fit between observed and expected values where $y=1$. The values of a and b can be determined as follows:

$$\mu = \Sigma\{(r_1-r_{.25})/(r_{30}-r_{.25})\} \ / \ n$$

$$a = -\log(\mu)/29$$

$$b = -30 \bullet a$$

where r is the observed yield rate for the duration indicated and n is the number of months included in the historical study.

Total Stock Yields

1. Compute a variable equal to the ratio of one plus the monthly net stock yield (dividends and stock value increase or decrease) divided by the twelfth root of one

plus the long-term bond yield. (Note that the ratio is of one plus the two rates, not the ratio of the two rates themselves).

2. Determine the mean and standard deviation of this variable in the usual manner.

3. Construct a set of values of the projected monthly stock yield in the same manner as used to determine long-term rates. (Note that the stock yield is the product of the random variable times one plus the long-term yield rate, less one).

CPI Index

1. Compute a variable equal to the equivalent annual change in the CPI index each month.

2. Assume that each CPI variable is equal to a constant multiple of the corresponding higher of long-term or short-term bond yields for each month plus a deviation.

$$f(x) - r^{max}_{(x)} \bullet m + b$$

Solve for the constant multiple, m, using the least squares method, and compute a second variable, b(x), equal to the monthly deviation.

3. Determine the mean and standard deviation of the second variable, b(x), in the usual manner.

4. Construct a set of values of projected equivalent annual changes in the CPI index and the resulting CPI index for each month in the same manner used to determine long-term rates.

For the seven prescribed scenarios (New York 126 and the proposed NAIC Model Act) fix the long-term yields programmatically and compute the remaining rate according to the "cookbook."

YIELD CURVE PROJECTION TECHNIQUES

MR. GREGORY D. JACOBS: As Tom mentioned, I'm going to be talking about something I guess I talked about awhile ago, and Tom affectionately called it the M&R approach. I assume others have adopted it; it's too formal to call the M&R approach. It is another approach to projecti[illegible] project future asset and liability cash flows. Let's call it the yield-curve [illegible] approach. It is, in fact, a stochastic process.

Let's start with some [illegible] .eld curves in Chart 1. Yield curve one just [illegible] five happens to be a high yield curve, yield c[illegible] sent. In the real world, it's much more robust than [illegible] mple. This is the universe of yield curves of which we will project [illegible] se yield curves.

The next thing that we have to come up with using this particular approach is what we call transition probability. Transition probability is a Markov chain, statistical sort of

CHART 1

SIMPLE EXAMPLE

YIELD CURVE UNIVERSE

CURVE #	MATURITY 1 YEAR	5 YEAR	10 YEAR
1	3%	5%	7%
2	5	6.5	8
3*	7	8	9
4	10	10	10
5	14	12	11

* STARTING YIELD CURVE

TRANSITION PROBABILITY

TO CURVE	MOVE FROM CURVE 1	2	3	4	5
1	75%	25%	-	-	-
2	25	50	25%	-	-
3	-	25	50	25%	-
4	-	-	25	50	25%
5	-	-	-	25	75

terminology. It says if we are starting in some yield curve, we need to know the probabilities to move from one to the other yield curve. Again in this very simplistic example, Chart 1 shows that we started with yield curve three. We have a 50% chance of staying in yield curve three for the next year or the next period or the next quarter, whatever it is we're projecting. We have a 25% chance of jumping to the top yield curve or up one yield curve and a 25% chance of jumping down. When we get up into the corners, it's a little interesting. This particular sort of example is probably not a good example. It says once we get into the lowest yield curve or into the highest yield curve, there's a high chance that we're going to stay. That's not very good when you see the real one that I've developed. You'll see that it's a little bit different than that. But again, this is just an illustration.

The mechanics are as follows. Let's pretend there's a four-sided die, because this is purely a random number generator. We roll a four-sided die. If a "one" appears, we move from curve three to two, a 25% chance. If a "two" or a "three" appears, we stay at curve three. If a "four" appears we move up to curve four. That's what happens inside the computer model. If you have a transition probability that is not in 25% increments but is in 5% increments, then you have a 20-sided die. Simple mechanics provide a very simple computer routine to generate random numbers based on your transition probabilities to move you through the various yield curves in each period. The magic is in what happens when the yield curve is set. This is universal to all systems that I've seen out there. It's not unique to M&R by any stretch of the imagination.

When a new yield curve is set, the interest-sensitive assumptions are triggered. We have a new asset earnings rate, obviously, because now we have new investment rates of which we're buying new assets at those new investment rates. We possibly have a new investment strategy. Some companies set a different strategy when it's a high interest rate or inverted yield curve or a variety of other issues, but we set a new investment strategy. Certainly the market value has changed. Market values of all of our assets are driven by the Treasuries. Calls and prepayments possibly occur depending on the specific formulas you have set up. You probably have a new inflation rate, if your inflation rate is tied to some outside index. Your credit rates probably are new because they're either going to be tied off of your portfolio or some outside rate. Your market or competition rate is going to change, presumably. Lapse rates will change because of the two above changes. When all is said and done, new cash flows are computed. That is the entire yield curve projection process and the results of those yield curve projections processes.

We do this for one period. At the end of the period, we throw the die again. We move from the ending curve last period to the new curve this period; we go through all the mechanics. We do that for x years, and we have a trial, 20 years, 30 years, whatever the case may be. Then we repeat the process. We go back to the starting yield curve, which in our simple example yield curve is three. We set up some new random numbers, and we go through the trials again. There's no "normal" number of trials. We often run 50 trials because they fit on a page nicely. To be statistically significant, I believe it's been stated in the past that you need at least 200 trials. I know that in some work we've done for banks, we've literally done 2,000 trials to make them feel warm and fuzzy. They like details. That's how the transition probability approach works.

The result of all of this stuff is that each move is independent of the previous move. I personally believe that interest-rate movements this month have absolutely no dependence on interest-rate movements last month or the month before. In this particular approach every move is independent of the other move. Each move is also independent of the cash-flow results of the company. All of these interest-rate projections are done before any cash flows are computed. So, if you have positive cash flows or negative cash flows, they don't influence at all what the yield curve is going to be next period. Each trial is totally independent of the other trial except that the starting point is the same. The reason the starting point is the same is because we have to start with today's yield curve or the valuation date yield curve. This is classic Markov chain.

Let's look at another example. "Where do you get the yield curve universes?" is the first question that one should ask. The second question one should ask is, "Where do you get the transition probabilities?" Here's how I do it. We have historical Treasury interest rates since January 1965 for 90-day, three-year, 10-year, and 30-year Treasuries. (See Chart 2.) From that, we looked at what has happened to interest rates during the 1980s. (See Chart 3.) What is shown are the high interest rate, the low interest rate, the mean of those interest rates, and the standard deviation of those interest rates. With this factual information from the Federal Reserve Statistical Release, I set the yield curve universe. (See Chart 4.) My low yield curve just so happens to be the lowest interest rates that were exhibited during the 1980s. The highest yield curve happens to be the highest interest rates that I saw during the 1980s. I arbitrarily chose to segment this out into 19 yield curves. Oftentimes I use 19 or 25 or some odd number, so I have just as many above and just as many below the mean, and my midpoint is the mean. In between those, I do some interpolations, some sort of simple arithmetic curve-fitting, to fit yield curves in the other nineteen slots. This is how I set my yield curves. (See Chart 5.)

This is the way it actually looks in our computer system. Across the top we see the number of the beginning curve, and down the side we have the number of the ending curve. The numbers in the middle represent a 36% chance of staying where you are, a 22.5% chance of moving up one curve or down one curve, an 8% chance of moving up two curves or

CHART 2

EFFECTIVE YIELDS ON U.S. GOVERNMENT SECURITIES -- TREASURIES

MONTH	YEAR	90 DAY	3 YEAR	10 YEAR	30 YEAR
JAN	1965	3.92	4.08	4.23	4.24
FEB	1965	3.92	4.06	4.23	4.22
MAR	1965	4.06	4.12	4.24	4.23
APR	1965	4.07	4.13	4.21	4.21
MAY	1965	4.01	4.13	4.23	4.21
JUN	1965	3.94	4.13	4.23	4.21
JUL	1965	3.91	4.04	4.23	4.21
AUG	1965	3.90	4.10	4.24	4.22
SEP	1965	4.01	4.22	4.30	4.30
OCT	1965	4.14	4.36	4.39	4.36
NOV	1965	4.17	4.42	4.43	4.37
DEC	1965	4.24	4.52	4.51	4.42
JAN	1966	4.60	5.00	4.64	4.52
FEB	1966	4.81	4.98	4.73	4.59
MAR	1966	4.82	5.09	4.97	4.81
APR	1966	4.67	4.85	4.66	4.63
MAY	1966	4.82	4.95	4.78	4.67
JUN	1966	4.80	5.15	4.81	4.74
JUL	1966	4.71	5.26	4.99	4.81
AUG	1966	4.91	5.33	5.06	4.83
SEP	1966	5.26	6.25	5.47	5.05
OCT	1966	5.51	5.47	4.91	4.80
NOV	1966	5.41	5.40	4.88	4.71
DEC	1966	5.35	5.47	5.07	4.84
JAN	1967	4.97	4.97	4.62	4.59
FEB	1967	4.65	4.66	4.55	4.47
MAR	1967	4.65	4.80	4.73	4.70
APR	1967	4.23	4.28	4.52	4.59
MAY	1967	3.79	4.48	4.74	4.77
JUN	1967	3.57	4.52	4.89	4.88
JUL	1967	3.92	5.12	5.25	5.09
AUG	1967	4.24	5.06	5.16	5.03
SEP	1967	4.51	5.39	5.26	5.13
OCT	1967	4.50	5.41	5.33	5.18
NOV	1967	4.70	5.65	5.65	5.45
DEC	1967	5.11	5.74	5.78	5.64
JAN	1968	5.20	5.82	5.72	5.58
FEB	1968	5.03	5.48	5.55	5.36
MAR	1968	5.20	5.59	5.55	5.38
APR	1968	5.34	5.73	5.75	5.57
MAY	1968	5.71	5.96	5.70	5.46
JUN	1968	5.88	5.98	5.71	5.48
JUL	1968	5.52	5.77	5.57	5.35
AUG	1968	5.33	5.37	5.28	5.17
SEP	1968	5.39	5.41	5.33	5.23
OCT	1968	5.33	5.39	5.39	5.34
NOV	1968	5.69	5.62	5.49	5.45
DEC	1968	5.72	5.66	5.63	5.65
JAN	1969	6.48	6.36	6.04	5.99
FEB	1969	6.43	6.27	6.09	6.09
MAR	1969	6.44	6.43	6.25	6.14
APR	1969	6.22	6.28	6.42	6.14
MAY	1969	6.08	6.25	6.16	5.94
JUN	1969	6.32	6.68	6.51	6.32
JUL	1969	6.32	7.22	6.55	6.27
AUG	1969	7.37	7.35	6.55	6.23
SEP	1969	7.20	7.36	6.68	6.25
OCT	1969	7.45	8.15	7.33	6.67
NOV	1969	7.29	7.41	6.92	6.50
DEC	1969	7.85	7.84	7.28	6.69

DATA SOURCE : FEDERAL RESERVE STATISTICAL RELEASE

CHART 2 -- Continued

EFFECTIVE YIELDS ON U.S. GOVERNMENT SECURITIES -- TREASURIES

MONTH	YEAR	90 DAY	3 YEAR	10 YEAR	30 YEAR
JAN	1970	8.37	8.39	7.54	6.91
FEB	1970	8.25	8.33	7.64	6.94
MAR	1970	7.14	7.30	6.94	6.50
APR	1970	6.62	7.15	7.01	6.55
MAY	1970	7.20	7.95	7.56	6.97
JUN	1970	7.20	7.94	7.87	7.37
JUL	1970	6.54	7.92	7.70	7.15
AUG	1970	6.62	7.61	7.33	6.86
SEP	1970	6.50	7.40	7.49	6.94
OCT	1970	5.99	7.04	7.24	6.75
NOV	1970	6.03	6.91	7.32	6.92
DEC	1970	5.28	5.62	6.51	6.32
JAN	1971	4.97	5.78	6.47	6.35
FEB	1971	4.22	5.30	6.08	6.04
MAR	1971	3.44	4.86	6.10	6.20
APR	1971	3.66	4.48	5.56	5.86
MAY	1971	4.08	5.73	6.04	6.07
JUN	1971	4.41	5.86	6.30	6.22
JUL	1971	5.22	6.52	6.50	6.40
AUG	1971	5.44	6.84	7.02	6.34
SEP	1971	4.40	5.85	6.33	6.01
OCT	1971	4.67	5.78	6.06	5.88
NOV	1971	4.39	5.26	5.91	5.81
DEC	1971	4.42	5.36	6.04	5.87
JAN	1972	3.68	5.06	6.05	5.89
FEB	1972	3.35	5.31	6.29	5.98
MAR	1972	3.49	5.33	6.27	5.94
APR	1972	3.87	5.83	6.30	6.02
MAY	1972	3.65	5.58	6.20	6.02
JUN	1972	3.86	5.41	6.18	5.86
JUL	1972	3.99	5.70	6.21	5.91
AUG	1972	3.78	5.73	6.30	5.81
SEP	1972	4.60	6.01	6.46	5.77
OCT	1972	4.64	6.03	6.68	5.91
NOV	1972	4.86	6.06	6.53	5.81
DEC	1972	4.98	5.95	6.42	5.66
JAN	1973	5.27	6.04	6.43	5.91
FEB	1973	5.73	6.42	6.60	7.01
MAR	1973	6.01	6.82	6.72	7.02
APR	1973	6.62	6.83	6.76	6.97
MAY	1973	6.42	6.79	6.76	7.00
JUN	1973	7.15	6.83	6.92	7.21
JUL	1973	7.83	7.02	6.92	7.28
AUG	1973	8.70	8.04	7.35	7.72
SEP	1973	8.96	7.48	7.15	7.45
OCT	1973	7.24	6.86	6.92	7.13
NOV	1973	7.64	6.83	6.78	7.40
DEC	1973	7.57	6.83	6.78	7.23
JAN	1974	7.73	6.83	6.92	7.40
FEB	1974	7.79	6.91	7.05	7.53
MAR	1974	7.73	6.95	7.00	7.62
APR	1974	8.69	7.90	7.23	7.95
MAY	1974	9.11	8.33	7.58	8.19
JUN	1974	8.44	8.17	7.56	8.23
JUL	1974	7.79	8.39	7.63	8.22
AUG	1974	8.27	8.76	7.85	8.58
SEP	1974	9.55	8.92	8.21	8.89
OCT	1974	6.38	8.17	7.64	8.63
NOV	1974	8.12	8.01	7.46	8.30
DEC	1974	7.74	7.50	7.23	8.18

DATA SOURCE : FEDERAL RESERVE STATISTICAL RELEASE

CHART 2 -- Continued

EFFECTIVE YIELDS ON U.S. GOVERNMENT SECURITIES -- TREASURIES

MONTH	YEAR	90 DAY	3 YEAR	10 YEAR	30 YEAR
JAN	1975	7.35	7.36	7.15	8.11
FEB	1975	5.93	7.02	7.12	8.11
MAR	1975	5.54	6.61	7.01	7.98
APR	1975	5.70	7.08	7.11	8.37
MAY	1975	5.63	7.85	7.62	8.58
JUN	1975	5.33	7.30	7.52	8.51
JUL	1975	6.00	7.51	7.51	8.30
AUG	1975	6.43	7.97	7.62	8.43
SEP	1975	6.65	8.06	7.77	8.61
OCT	1975	6.81	8.42	8.26	8.73
NOV	1975	5.73	7.24	8.00	8.37
DEC	1975	5.68	7.56	8.04	8.50
JAN	1976	5.33	7.12	7.68	8.19
FEB	1976	4.80	7.00	7.52	8.17
MAR	1976	5.11	7.20	7.61	8.17
APR	1976	5.14	7.11	7.52	8.08
MAY	1976	5.08	7.07	7.56	8.16
JUN	1976	5.69	7.59	8.10	8.33
JUL	1976	5.57	7.33	8.01	8.21
AUG	1976	5.35	7.08	7.99	8.22
SEP	1976	5.27	6.81	7.79	8.05
OCT	1976	5.21	6.72	7.66	7.97
NOV	1976	5.04	6.42	7.54	7.95
DEC	1976	4.56	5.74	7.12	7.72
JAN	1977	4.80	6.32	7.34	7.43
FEB	1977	4.85	6.54	7.53	7.89
MAR	1977	4.77	6.58	7.60	7.95
APR	1977	4.71	6.42	7.51	7.89
MAY	1977	5.16	6.66	7.60	7.95
JUN	1977	5.22	6.49	7.41	7.79
JUL	1977	5.40	6.62	7.41	7.79
AUG	1977	5.72	6.91	7.54	7.83
SEP	1977	6.07	6.96	7.48	7.79
OCT	1977	6.45	7.32	7.66	7.92
NOV	1977	6.38	7.35	7.72	8.00
DEC	1977	6.35	7.43	7.84	8.10
JAN	1978	6.75	7.76	8.12	8.35
FEB	1978	6.76	7.82	8.19	8.42
MAR	1978	6.59	7.85	8.20	8.40
APR	1978	6.59	8.00	8.32	8.51
MAY	1978	6.72	8.23	8.52	8.61
JUN	1978	7.06	8.47	8.64	8.68
JUL	1978	7.37	8.72	8.83	8.84
AUG	1978	7.44	8.50	8.59	8.65
SEP	1978	8.29	8.59	8.60	8.65
OCT	1978	8.44	8.81	8.83	8.86
NOV	1978	9.16	9.24	9.00	8.94
DEC	1978	9.64	9.55	9.21	9.08
JAN	1979	9.95	9.73	9.31	9.14
FEB	1979	9.91	9.51	9.31	9.20
MAR	1979	10.09	9.60	9.33	9.23
APR	1979	10.07	9.65	9.39	9.29
MAY	1979	10.24	9.64	9.46	9.40
JUN	1979	9.62	9.15	9.11	9.12
JUL	1979	9.82	9.14	9.15	9.13
AUG	1979	10.14	9.35	9.23	9.18
SEP	1979	10.97	9.93	9.55	9.38
OCT	1979	12.60	11.25	10.57	10.09
NOV	1979	12.70	11.49	10.93	10.57
DEC	1979	12.99	11.00	10.66	10.38

DATA SOURCE : FEDERAL RESERVE STATISTICAL RELEASE

CHART 2 -- Continued

EFFECTIVE YIELDS ON U.S. GOVERNMENT SECURITIES -- TREASURIES

MONTH	YEAR	90 DAY	3 YEAR	10 YEAR	30 YEAR
JAN	1980	12.94	11.18	11.09	10.88
FEB	1980	13.93	13.25	12.80	12.50
MAR	1980	16.67	14.54	13.16	12.72
APR	1980	14.33	12.38	11.80	11.73
MAY	1980	9.09	9.66	10.44	10.63
JUN	1980	7.43	9.12	10.02	10.05
JUL	1980	8.52	9.49	10.51	10.50
AUG	1980	9.70	10.91	11.41	11.30
SEP	1980	10.98	11.91	11.84	11.66
OCT	1980	12.51	12.37	12.10	11.93
NOV	1980	14.94	13.75	13.08	12.75
DEC	1980	17.10	14.12	13.25	12.78
JAN	1981	16.46	13.43	12.97	12.51
FEB	1981	16.18	14.12	13.63	13.21
MAR	1981	14.51	13.97	13.55	13.09
APR	1981	14.89	14.59	14.15	13.64
MAY	1981	17.98	15.65	14.60	14.06
JUN	1981	16.11	14.80	13.92	13.38
JUL	1981	16.37	15.72	14.79	14.05
AUG	1981	17.04	16.64	15.50	14.67
SEP	1981	16.08	16.88	15.91	15.21
OCT	1981	14.72	16.10	15.72	15.22
NOV	1981	11.64	13.54	13.84	13.80
DEC	1981	11.63	14.13	14.19	13.90
JAN	1982	13.26	15.18	15.12	14.73
FEB	1982	14.65	15.27	14.95	14.73
MAR	1982	13.72	14.63	14.34	13.99
APR	1982	13.75	14.68	14.35	13.82
MAY	1982	13.04	14.24	14.08	13.68
JUN	1982	13.48	15.00	14.81	14.40
JUL	1982	12.20	14.49	14.44	14.01
AUG	1982	9.20	13.02	13.49	13.18
SEP	1982	8.36	12.39	12.72	12.43
OCT	1982	8.13	10.90	11.21	11.48
NOV	1982	8.53	10.23	10.83	10.82
DEC	1982	8.38	10.12	10.82	10.82
JAN	1983	8.30	9.87	10.73	10.91
FEB	1983	8.57	10.16	11.01	11.18
MAR	1983	8.84	10.08	10.79	10.91
APR	1983	8.68	10.00	10.67	10.76
MAY	1983	8.66	9.89	10.65	10.81
JUN	1983	9.32	10.59	11.14	11.23
JUL	1983	9.64	11.20	11.70	11.73
AUG	1983	9.93	11.62	12.20	12.17
SEP	1983	9.56	11.38	11.99	11.97
OCT	1983	9.16	11.17	11.87	11.92
NOV	1983	9.29	11.26	12.03	12.10
DEC	1983	9.56	11.44	12.18	12.23
JAN	1984	9.44	11.23	12.01	12.10
FEB	1984	9.66	11.36	12.19	12.31
MAR	1984	10.14	11.93	12.70	12.76
APR	1984	10.33	12.34	13.03	13.05
MAY	1984	10.48	13.16	13.86	13.88
JUN	1984	10.53	13.61	14.02	13.89
JUL	1984	10.81	13.51	13.81	13.65
AUG	1984	11.20	12.89	13.12	12.93
SEP	1984	11.09	12.72	12.91	12.67
OCT	1984	10.38	12.20	12.53	12.34
NOV	1984	9.12	11.20	11.91	11.89
DEC	1984	8.52	10.84	11.83	11.85

DATA SOURCE : FEDERAL RESERVE STATISTICAL RELEASE

CHART 2 -- Continued

EFFECTIVE YIELDS ON U.S. GOVERNMENT SECURITIES -- TREASURIES

MONTH	YEAR	90 DAY	3 YEAR	10 YEAR	30 YEAR
JAN	1985	8.19	10.70	11.70	11.78
FEB	1985	8.75	10.83	11.84	11.80
MAR	1985	9.02	11.36	12.21	12.16
APR	1985	8.40	10.77	11.76	11.80
MAY	1985	7.88	9.99	11.14	11.36
JUN	1985	7.30	9.26	10.42	10.72
JUL	1985	7.44	9.39	10.58	10.78
AUG	1985	7.51	9.53	10.60	10.84
SEP	1985	7.46	9.59	10.64	10.89
OCT	1985	7.53	9.46	10.50	10.78
NOV	1985	7.62	9.08	10.02	10.31
DEC	1985	7.43	8.58	9.47	9.77
JAN	1986	7.40	8.59	9.40	9.62
FEB	1986	7.39	8.26	8.89	9.13
MAR	1986	6.91	7.43	7.93	8.12
APR	1986	6.34	6.98	7.43	7.53
MAY	1986	6.40	7.40	7.86	7.66
JUN	1986	6.50	7.55	7.95	7.71
JUL	1986	6.10	6.98	7.43	7.40
AUG	1986	5.81	6.60	7.30	7.46
SEP	1986	5.40	6.73	7.59	7.77
OCT	1986	5.39	6.67	7.57	7.85
NOV	1986	5.57	6.56	7.38	7.66
DEC	1986	5.72	6.53	7.24	7.51
JAN	1987	5.68	6.51	7.21	7.53
FEB	1987	5.83	6.67	7.38	7.68
MAR	1987	5.80	6.69	7.38	7.69
APR	1987	6.03	7.45	8.18	8.42
MAY	1987	6.03	8.18	8.80	8.97
JUN	1987	5.95	7.97	8.58	8.75
JUL	1987	6.04	7.89	8.63	8.83
AUG	1987	6.29	8.19	8.95	9.17
SEP	1987	6.64	8.86	9.64	9.82
OCT	1987	6.72	8.94	9.75	9.84
NOV	1987	6.09	8.15	9.06	9.15
DEC	1987	6.07	8.30	9.19	9.33
JAN	1988	6.17	8.02	8.86	9.02
FEB	1988	5.95	7.52	8.38	8.61
MAR	1988	5.95	7.64	8.55	8.82
APR	1988	6.18	7.98	8.91	9.15
MAY	1988	6.55	8.41	9.30	9.44
JUN	1988	6.81	8.39	9.12	9.20
JUL	1988	7.06	8.62	9.27	9.35
AUG	1988	7.37	8.96	9.47	9.54
SEP	1988	7.51	8.59	8.96	9.08
OCT	1988	7.62	8.35	8.75	8.86
NOV	1988	8.11	8.90	9.04	9.04
DEC	1988	8.34	9.15	9.19	9.02
JAN	1989	8.62	9.13	9.02	8.82
FEB	1989	8.96	9.41	9.34	9.13
MAR	1989	9.25	9.72	9.36	9.16
APR	1989	8.67	9.24	9.07	8.93
MAY	1989	8.92	8.81	8.66	8.62
JUN	1989	8.03	8.20	8.14	8.10
JUL	1989	7.98	7.77	7.97	8.08
AUG	1989	7.90	8.37	8.25	8.20
SEP	1989	7.84	8.42	8.31	8.26
OCT	1989	7.60	7.86	7.89	7.91
NOV	1989	7.63	7.76	7.85	7.91
DEC	1989	7.68	7.90	7.93	7.98

DATA SOURCE : FEDERAL RESERVE STATISTICAL RELEASE

CHART 2 -- Continued

EFFECTIVE YIELDS ON U.S. GOVERNMENT SECURITIES -- TREASURIES

MONTH	YEAR	90 DAY	3 YEAR	10 YEAR	30 YEAR
JAN	1990	7.76	8.38	8.47	8.50
FEB	1990	7.76	8.44	8.50	8.53
MAR	1990	7.85	8.57	8.56	8.53
APR	1990	7.78	9.01	9.02	9.00
MAY	1990	7.71	8.64	8.66	8.63
JUN	1990	7.77	8.45	8.51	8.49
JUL	1990	7.43	8.12	8.37	8.44
AUG	1990	7.46	8.56	8.88	9.00
SEP	1990	7.29	8.58	8.96	9.10

DATA SOURCE : FEDERAL RESERVE STATISTICAL RELEASE

CHART 2 -- Continued

EFFECTIVE YIELDS ON U.S. GOVERNMENT SECURITIES -- TREASURIES STANDARD DEVIATION SUMMARY

	90 DAY	3 YEAR	10 YEAR	30 YEAR
1965 - SEP 1990	2.92	2.74	2.69	2.65
1970 - SEP 1990	2.96	2.60	2.42	2.31
1975 - SEP 1990	3.02	2.55	2.31	2.08
1980 - SEP 1990	3.09	2.63	2.34	2.16
1985 - SEP 1990	1.00	1.06	1.15	1.16
1990 - SEP 1990	0.19	0.22	0.22	0.25
1965 - 1969	1.03	1.00	0.81	0.71
1970 - 1974	1.76	1.09	0.59	0.87
1975 - 1979	2.20	1.29	0.91	0.64
1980 - 1984	2.92	1.98	1.50	1.30
1985 - 1989	1.05	1.13	1.23	1.23
1990 - 1990	0.19	0.22	0.22	0.25
1990 - SEP 1990	0.19	0.22	0.22	0.25
1989 - SEP 1990	0.54	0.52	0.45	0.40
1988 - SEP 1990	0.81	0.52	0.45	0.45
1987 - SEP 1990	0.98	0.67	0.58	0.55
1986 - SEP 1990	1.01	0.81	0.69	0.66
1985 - SEP 1990	1.00	1.06	1.15	1.16
1984 - SEP 1990	1.44	1.72	1.77	1.73
1983 - SEP 1990	1.44	1.73	1.79	1.75
1982 - SEP 1990	1.97	2.19	2.12	2.02
1981 - SEP 1990	2.91	2.65	2.40	2.23
1980 - SEP 1990	3.09	2.63	2.34	2.16
1979 - SEP 1990	3.00	2.53	2.26	2.10
1978 - SEP 1990	2.94	2.48	2.24	2.08
1977 - SEP 1990	3.00	2.55	2.29	2.11
1976 - SEP 1990	3.05	2.58	2.30	2.11
1975 - SEP 1990	3.02	2.55	2.31	2.08
1974 - SEP 1990	2.94	2.51	2.32	2.07
1973 - SEP 1990	2.88	2.52	2.35	2.11
1972 - SEP 1990	2.98	2.59	2.40	2.23
1971 - SEP 1990	3.02	2.64	2.45	2.30
1970 - SEP 1990	2.96	2.60	2.42	2.31
1969 - SEP 1990	2.91	2.57	2.43	2.34
1968 - SEP 1990	2.89	2.60	2.48	2.41
1967 - SEP 1990	2.91	2.65	2.55	2.49
1966 - SEP 1990	2.90	2.67	2.61	2.56
1965 - SEP 1990	2.92	2.74	2.69	2.65

CHART 3

EXAMPLE

Based on interest rates during the 1980s, the following facts are known:

	90-Day	3-Year	10-Year	30-Year
High	17.98%	16.88%	15.91%	15.22%
Low	5.39	6.51	7.21	7.40
Mean	9.56	10.79	11.21	11.18
Std. Dev.	3.36	2.75	2.33	2.12

CHART 4

EXAMPLE (CONTINUED)

SET YIELD CURVE UNIVERSE

CURVE #		90-Day	3-Year	10-Year	30-Year
Low	1	5.39%	6.15%	7.21%	7.40%
	2	-	-	-	-
	3	-	-	-	-
	-	-	-	-	-
	-	-	-	-	-
	-	-	-	-	-
Mean	10	9.56	10.79	11.21	11.18
	-	-	-	-	-
	-	-	-	-	-
	-	-	-	-	-
	17	-	-	-	-
	18	-	-	-	-
High	19	17.98	16.88	15.91	15.22
Today	20	7.37	8.96	9.47	9.54

Interpolate Interim Values

CHART 5

SET TRANSITION PROBABILITIES

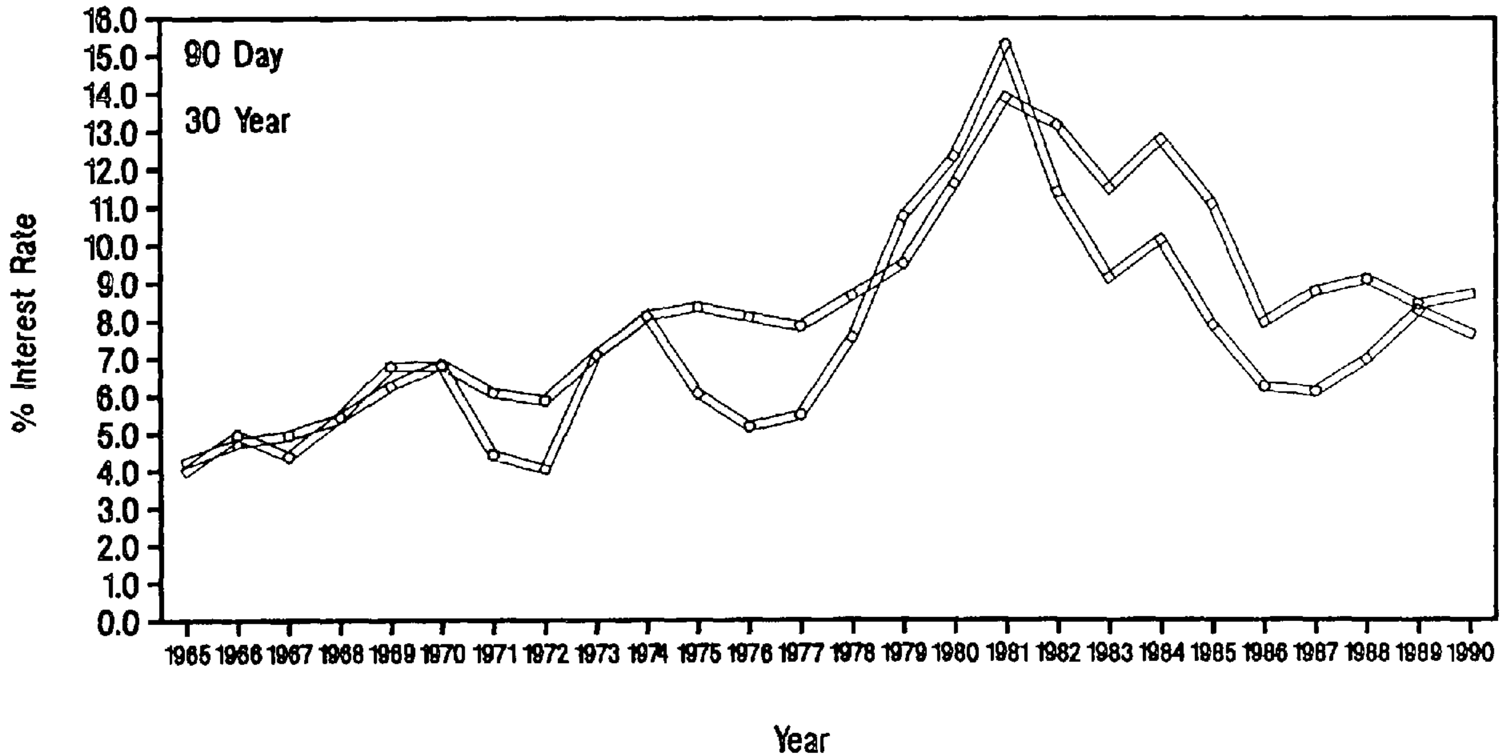
ENDING	BEGINNING CURVE																			
CURVE	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	5.0	5.0	5.0	1.5																
2	10.0	10.0	10.0	8.0	1.5															
3	25.0	25.0	25.0	22.5	8.0	1.5														1.5
4	25.0	25.0	25.0	36.0	22.5	8.0	1.5													8.0
5	20.0	20.0	20.0	22.5	36.0	22.5	8.0	1.5												22.5
6	10.0	10.0	10.0	8.0	22.5	36.0	22.5	8.0	1.5											36.0
7	5.0	5.0	5.0	1.5	8.0	22.5	36.0	22.5	8.0	1.5										22.5
8					1.5	8.0	22.5	36.0	22.5	8.0	1.5									8.0
9						1.5	8.0	22.5	36.0	22.5	8.0	1.5								1.5
10							1.5	8.0	22.5	36.0	22.5	8.0	1.5							
11								1.5	8.0	22.5	36.0	22.5	8.0	1.5						
12									1.5	8.0	22.5	36.0	22.5	8.0	1.5					
13										1.5	8.0	22.5	36.0	22.5	8.0	1.5	5.0	5.0	5.0	
14											1.5	8.0	22.5	36.0	22.5	8.0	10.0	10.0	10.0	
15												1.5	8.0	22.5	36.0	22.5	20.0	20.0	20.0	
16													1.5	8.0	22.5	36.0	25.0	25.0	25.0	
17														1.5	8.0	22.5	25.0	25.0	25.0	
18															1.5	8.0	10.0	10.0	10.0	
19																1.5	5.0	5.0	5.0	
20																				
	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100

down two curves, and a 1.5% chance of moving up three curves or down three curves. Now how did I get that? Simple math and an investigation of the interest rates during the 1980s. This just happens to normally distributed. If we had looked at a different time period or a different set of data, we probably wouldn't end up with a normal distribution. This reproduces the standard deviation. What I have done in this simple example is to come up with transition probabilities that replicate the standard deviations for the 1980s. I would then take this to my client and expose its assets and liabilities to several random trials through this process. I could then feel comfortable in determining whether or not I could have withstood what happened during the 1980s. A different time period will definitely yield different answers. There's no right answer in projecting yield curves. That's the bad news, but the good news is, there's no wrong answer either. What I've tried to do with this example through our particular approach, the yield-curve-universe transition-probability approach, is to make use of historical data and use a Markov chain process to allow the computer to conveniently project yield curves.

Now, we will look at some general observations about what's happening. These are things that ought to be important to you when you project your yield curves or attempt to project them (See Chart 6). With the same database, I looked at the mean interest rates for 30-day and 90-day Treasuries since 1965. These are the monthly average Treasuries. You

CHART 6

MEAN INTEREST RATES
(30-YEAR AND 90-DAY TREASURIES)



will note a general increase from 1965 to the peak in 1981. There has been a general decline since 1981. It now appears to be flattening out.

Chart 7 shows the number of inversions. I define an inversion as occurring when the 30-year rate is less than the 90-day rate. Now, some approaches I have seen in yield curve projections project an inordinate number of inversions, and I would submit to you that that's because the data are couched in terms of early 1980, or prior, data. If you don't believe that that's the real view of the world, then you ought to extract those and investigate your yield curve projections pretty thoroughly because inversions have not been the norm since 1981. They've certainly been the exception.

Chart 8 is similar to the last one. It shows the yield curve slope again, just looking at the 30-year rate versus the 90-day rate. Several projection techniques that I have seen some clients use is to project the slope that they put in a 90-day Treasury and one of the variables is the slope. What I've gleamed from this is that's a terribly erratic assumption, it's all over the place, and it's very difficult to be projecting what slopes are going forward because it is so erratic. Again, I believe you need to investigate the results of any projection technique you use to make sure that you have some reasonable numbers of slope. You'll notice that since 1965 the slope has never gotten greater than about 1.58 and it's

CHART 7

NUMBER OF INVERSIONS
(30-YR VS 90-DAY TREASURIES)

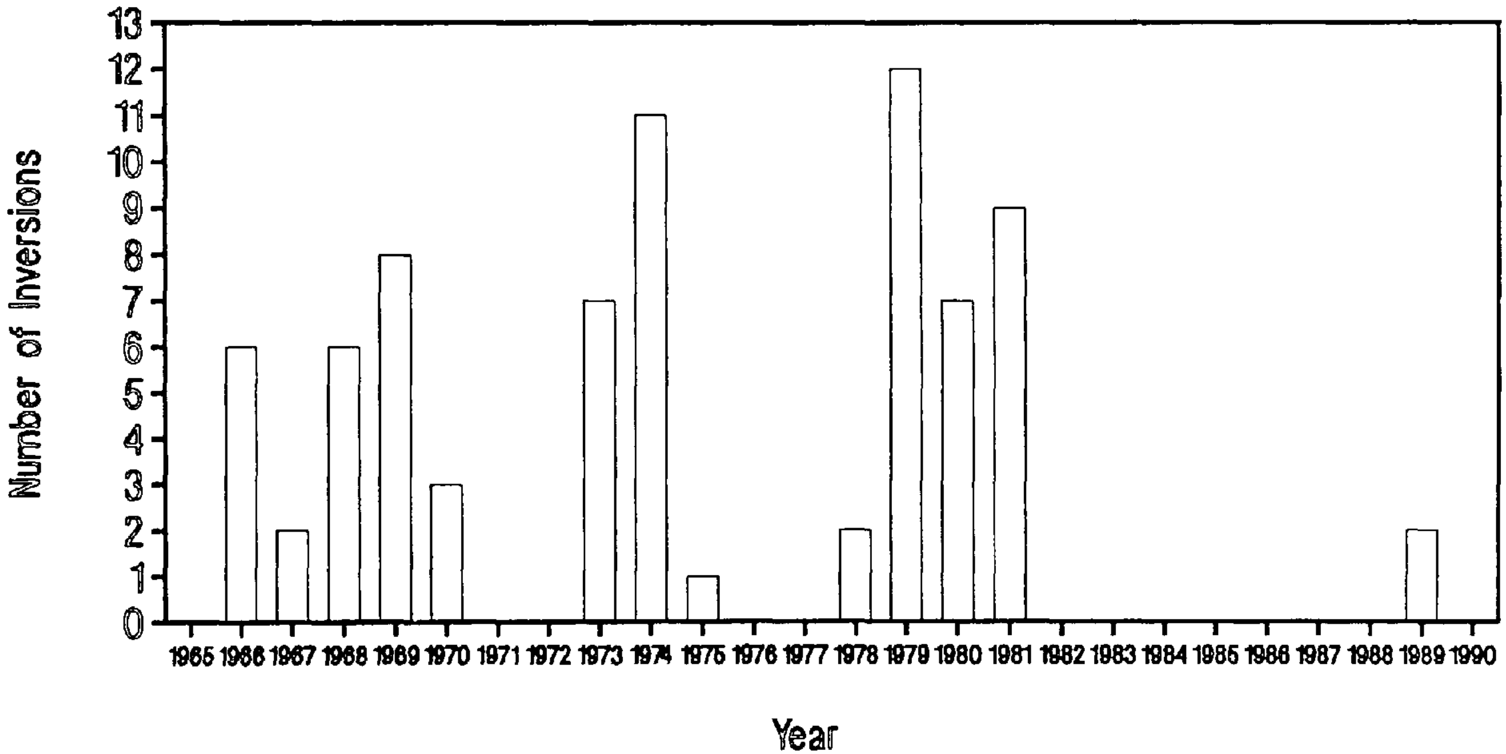
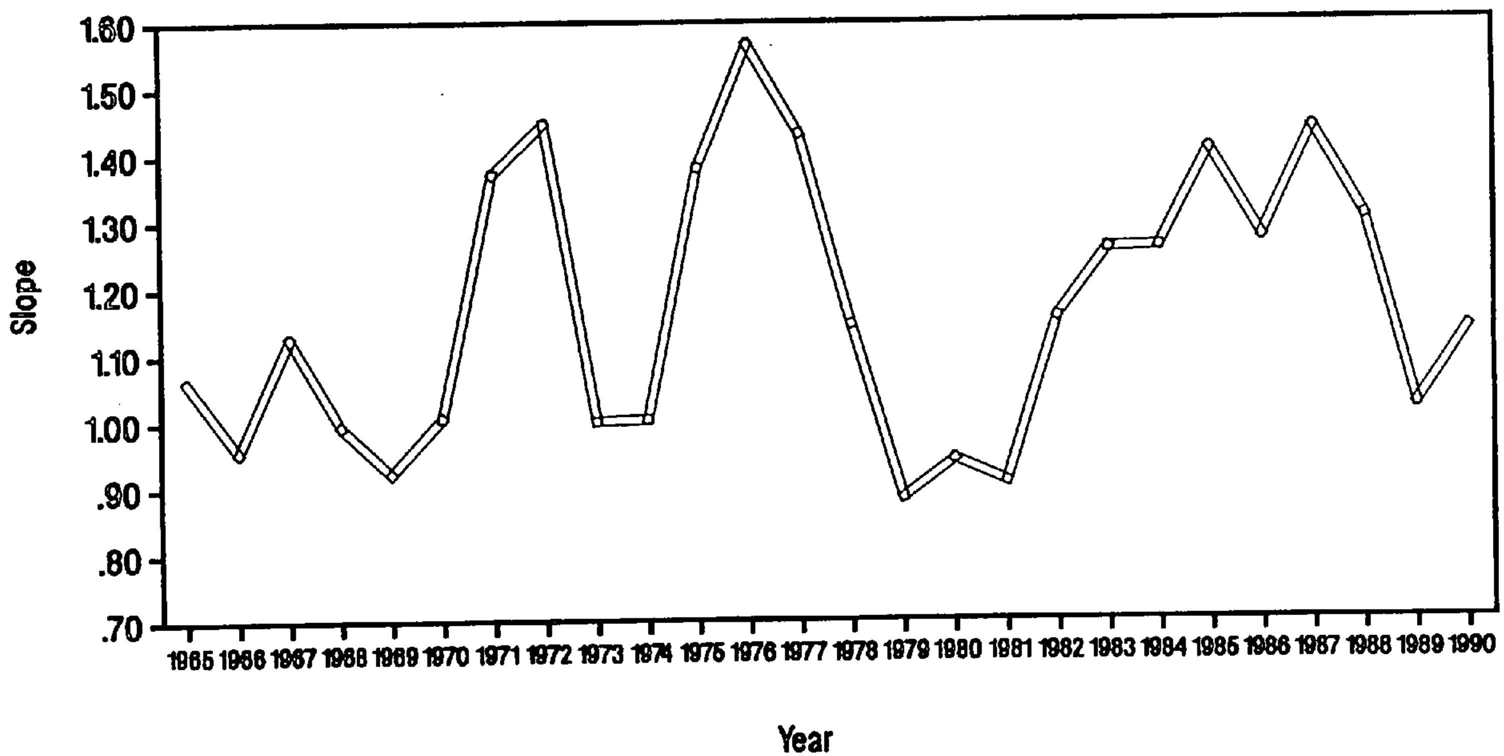


CHART 8

YIELD CURVE SLOPE
(30-YR. VS. 90-DAY TREASURIES)

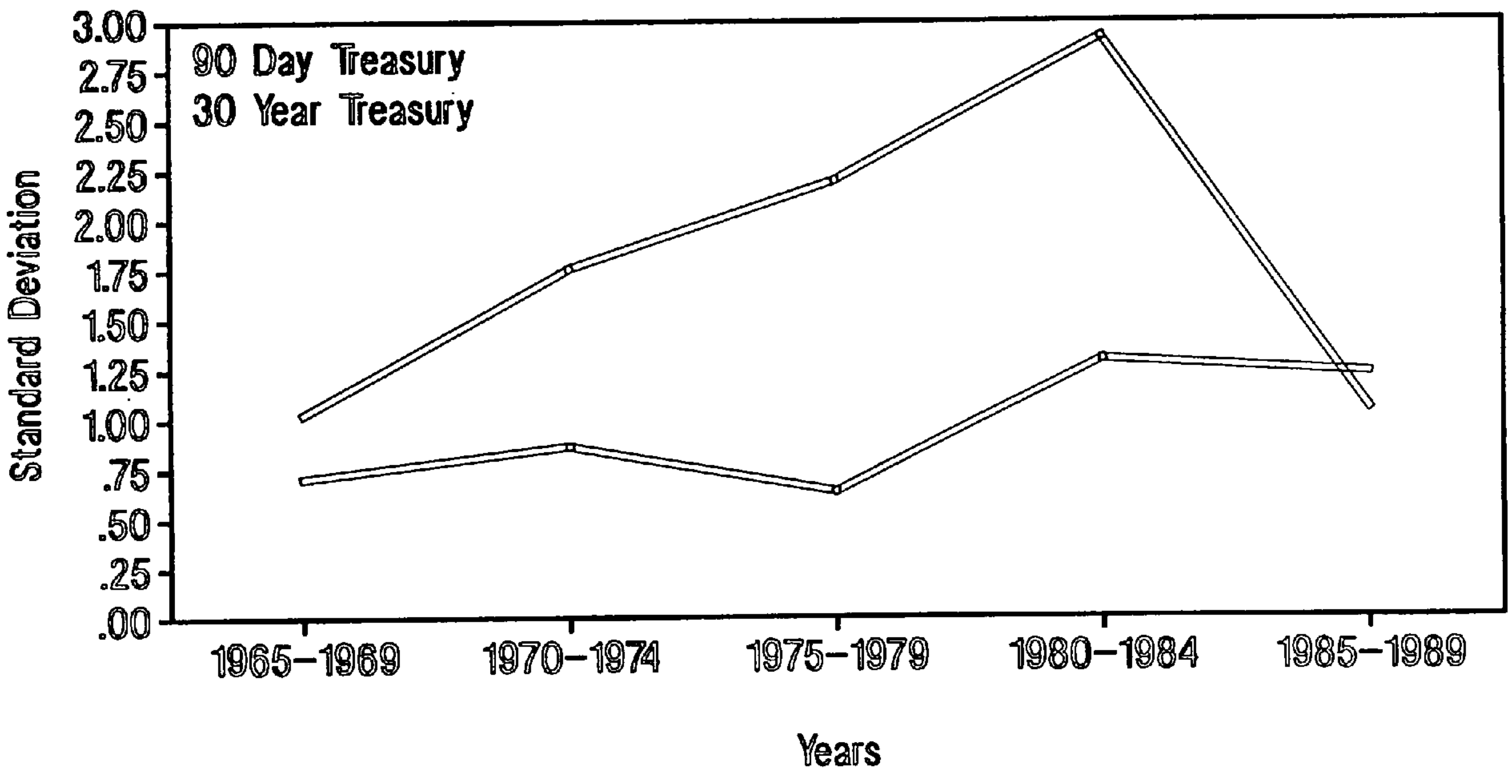


never gotten lower than about .85. If you get some results that are outside those boundaries, something is a little awry with your projection.

Chart 9 shows the standard deviation of Treasuries in five-year brackets. Again, we've taken the data from the Federal Reserve Statistical Bulletin and come up with five-year averages or five-year means and standard deviations over that five-year period. The graph shows both the 30-year and the 90-day Treasuries. The top line shows 30-day Treasuries. Up until the most recent five-year period, 1985-89, it was almost predictable. It's almost a straight line, and it's been getting more volatile. Big standard deviations imply volatility, risk, and concern. The good news is the long-term Treasuries haven't moved that much. There's a gradual increase, but they haven't gone up a tremendous amount. Interestingly enough, in the last five-year period, for the first time in history that I can find, even prior to 1965, the volatility on long-term Treasuries is more than the volatility on short-term Treasuries. Given that the insurance industry is generally a long-term investment vehicle, or that's where we've searched for our investments, that is potentially profound. Maybe it shows that the Treasury doesn't have anything to do with market. The point I want to make with this sort of an illustration is that, when you are doing your projections of yield curves, I believe it's important for you to recognize that there appears to be a fundamental change in volatility here. Is this just a happenstance?

CHART 9

STANDARD DEVIATION OF TREASURIES (30-YEAR AND 90-DAY)



I don't know, but I've always structured my transition probabilities and my yield curve universes with much more volatility in short-term rates than the long-term rates.

I want to leave you two things. One is that this is our approach, and there's not one answer. Any number of M&R actuaries or any number of you with these same data could probably set up different transition probabilities and different yield curve universes. This is simply one approach. There are a lot of different approaches.

The second thing I would like to leave with you is something I've already said. There's really no right answer. Projecting yield curves seems to have a fair amount of science involved with it, but I am struck by the art that's involved much more than the science. If you use this for anything other than a macro look at what happens to cash-flow testing, I believe you are woefully out of line and putting way too much creditability in something that is in fact a macro process as opposed to a micro process.

ARBITRAGE-FREE INTEREST RATE MODELS

MR. STEVEN P. MILLER: At the annual meeting of the Society of Actuaries, held in Orlando in October 1990, there was a very well-attended workshop entitled "Interest Rate Scenarios." Even though I was one of those many unfortunate members who registered too late to be a confirmed attendee, I decided to be a gate-crasher. I justified my lawlessness by telling myself that I needed to prepare for this panel. Apparently, I wasn't the only person that felt a dire need to discuss this subject because there were 43 actuaries in a room furnished to seat about 20. I think it was the second biggest attraction at the meeting. The first had large, round ears.

One of the hottest topics of discussion in this workshop was the topic of arbitrage-free interest rate scenarios, which incidentally, is the topic of my discussion. At the time, I wasn't very sure about what I was going to say, so I took some notes about the questions people were asking, and then built my presentation around those.

One question that wasn't asked, but that seems like a good place to start was, "What is arbitrage, anyhow?" The usual definition of arbitrage is a risk-free profit that can be made by simultaneously buying and selling identical items in different markets. The efficient markets hypothesis of arbitrage pricing theory states that arbitrage cannot long exist because the buying and selling would soon drive prices to an equilibrium, where there was no profit to be made. The strongest form of the hypothesis holds that the prices change so quickly that arbitrage does not exist at all. Or, as a friend of mine is fond of saying, "No one can create arbitrage, because everyone does."

I would like to point out, though, that the use of arbitrage-free interest rate models does require you to believe this hypothesis in its strongest form. In order to illustrate the value of these models, I have created my own definition that I use when discussing interest rate models: "Arbitrage is the state of affairs whereby a clever computer can create money out of thin air." This definition points out the essence of arbitrage, (a profit without investment) and the danger to proper modeling.

An interest rate model can contain arbitrage either because some points on the yield curve are inconsistent with other parts of the same yield curve, or because the current yield curve is inconsistent with the range of future yield curves. One example of a curve that is inconsistent with itself is a yield curve that contains implicit negative spot prices (Chart 1).

Another is a curve that contains negative forward rates (Chart 2). The first allows arbitrage because it allows a clever computer to receive positive cash flows in the future for a negative "price." We can do this by buying the long bonds and selling a portfolio that

CHART 1

A YIELD CURVE WITH UNDEFINED SPOT RATES
(From 20 years to 30 years)

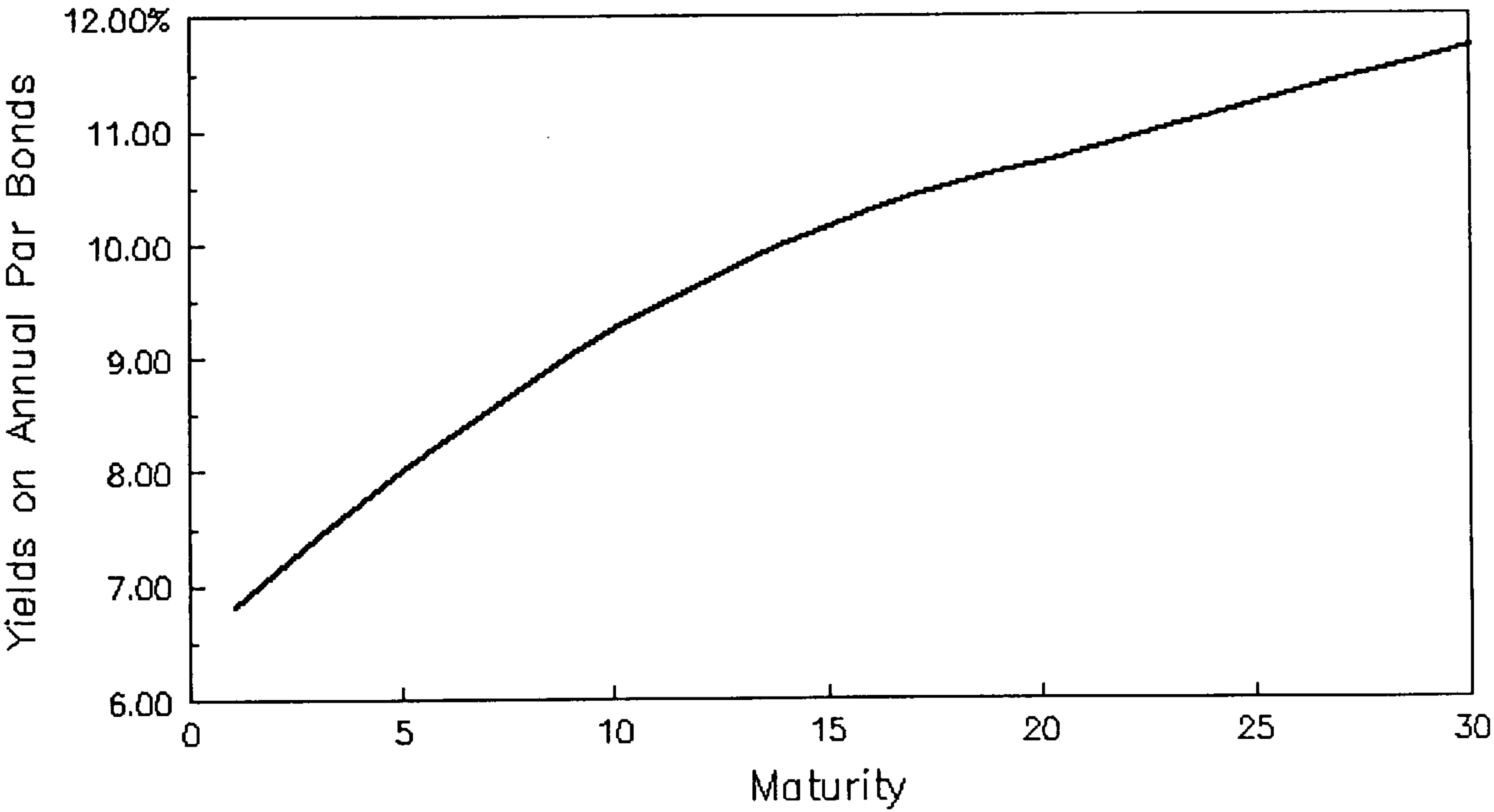
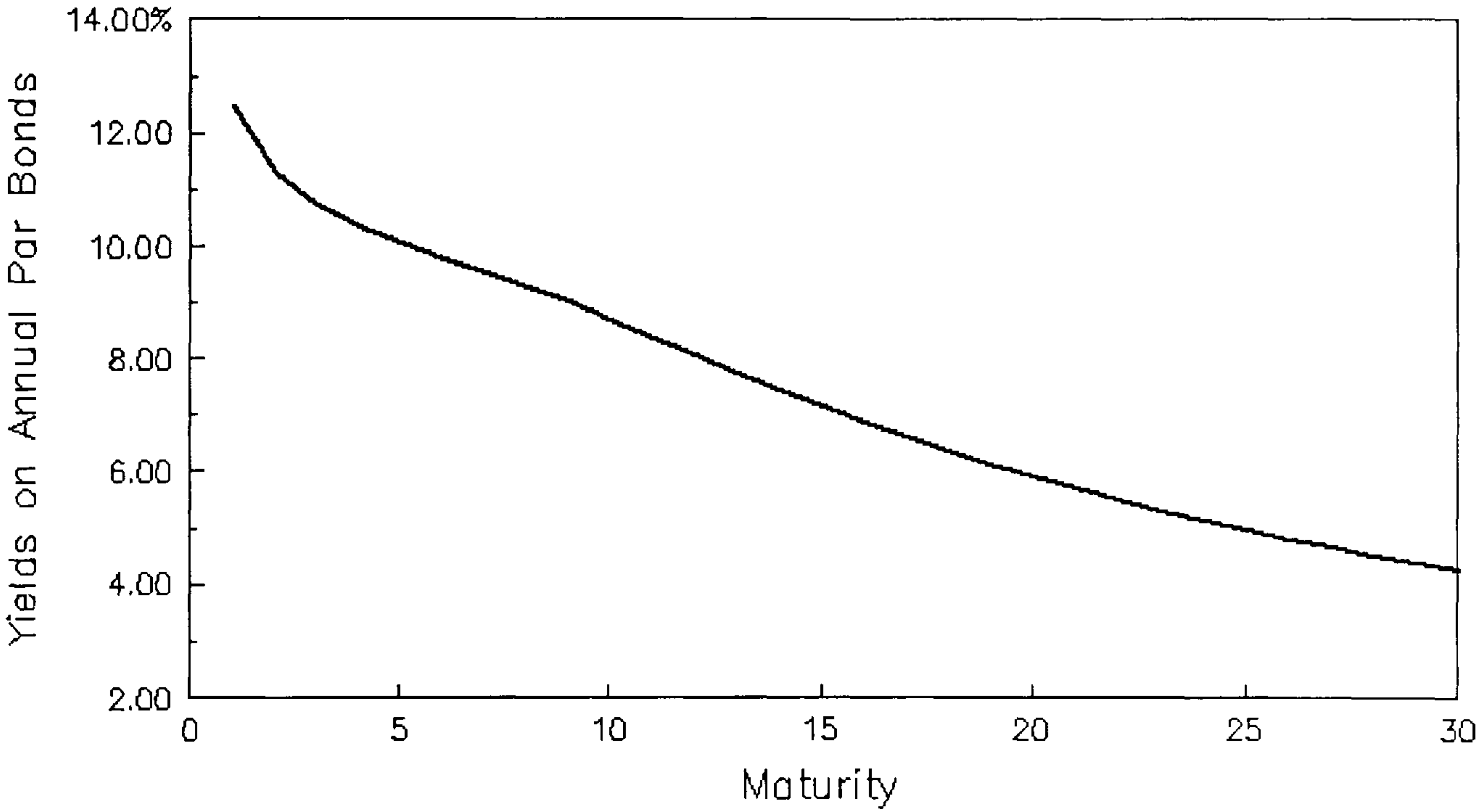


CHART 2

A YIELD CURVE WITH NEGATIVE FORWARD RATES
(From years 24 to 30)



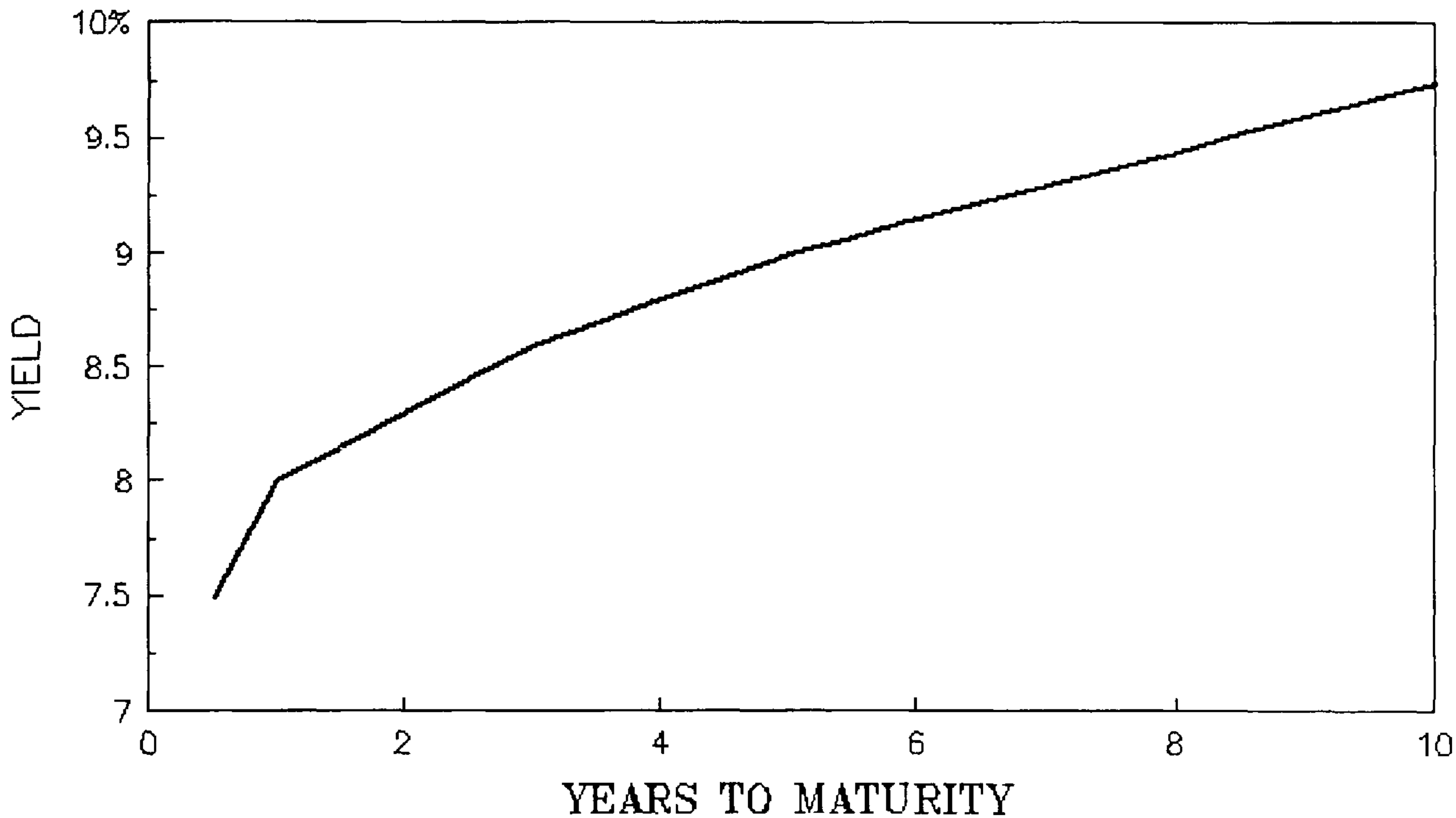
replicates the early coupons. The portfolio that we sold is more valuable than the bonds that we bought, even though the long bonds pay all the cash that our shorter portfolio does, and more. This type of arbitrage would only appear on models that projected yields on coupon bonds, rather than spot rates.

A second type of model arbitrage is probably more rare, only appearing in models that have a low correlation between long and short rate changes. Chart 2 is an example of a yield curve with negative implied forward rates. This creates arbitrage if we make the reasonable assumption that cash pays 0% interest. By borrowing long and investing short, we can lock in a future loan with a negative interest rate. Since we can hold the cash at 0%, we can repay the loan by holding the sum of all payments and still have money left from the proceeds of our loan. Once again, we have made money appear out of thin air.

The third type of arbitrage is the most common and also the hardest to understand. Here we will use the example of a typical yield curve (Chart 3). Like most real world problems, I have created this curve from incomplete data. The unknown rates were created by linear interpolation. Specifically, the curve is assumed to be linear between the five- and ten-year maturities. I am going to assume that the yield curve six months from now is going to be parallel to this one. In other words, if one rate goes up by 50 basis points, they all do. Now I am going to test the investment strategy of selling seven-year bonds and buying

CHART 3

A TYPICAL YIELD CURVE



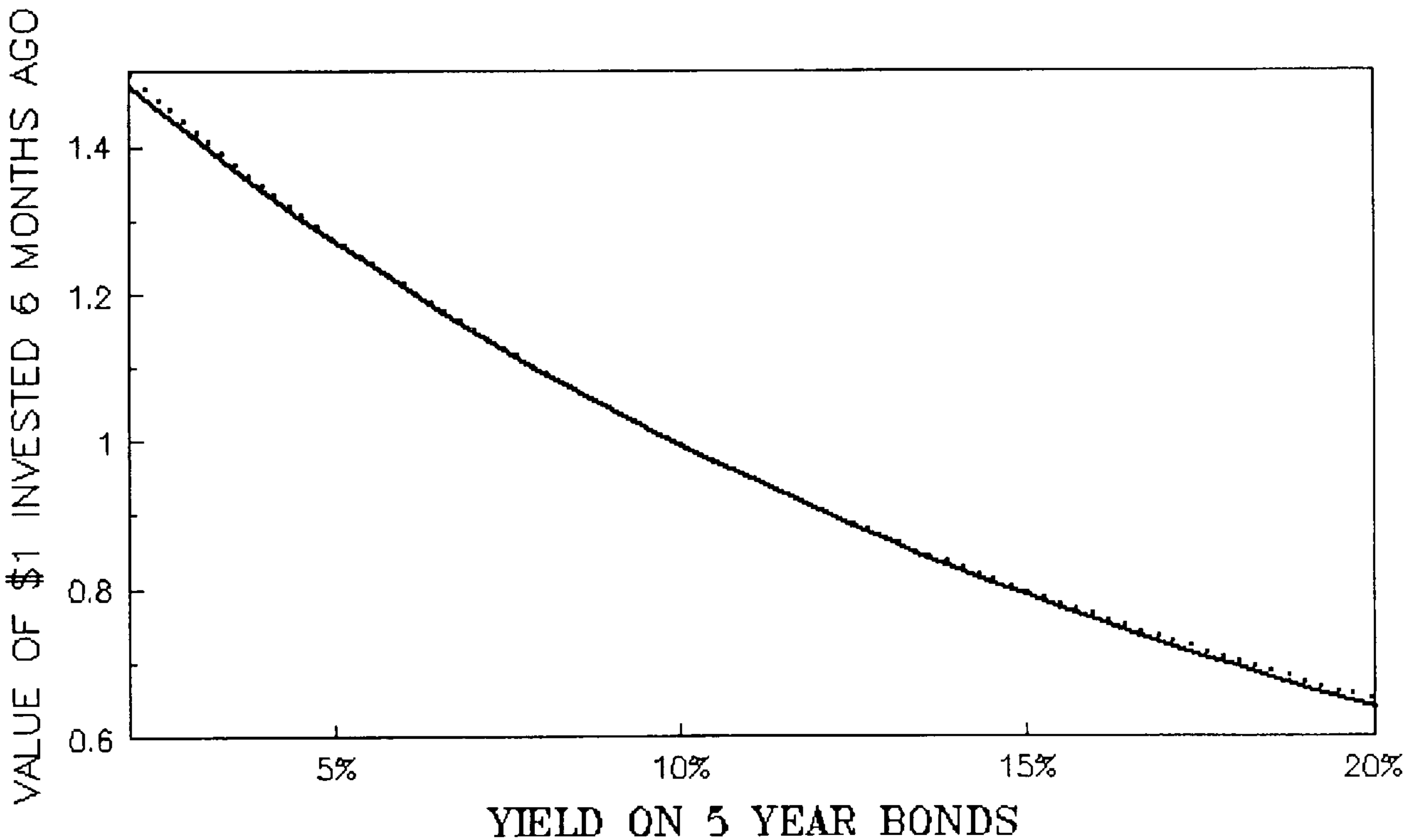
an equal amount of a barbell portfolio of five- and ten-year bonds. After six months, I am going to sell my portfolio. According to my model, this strategy will allow me to make money no matter what happens to interest rates (Chart 4). At the very minimum, I can make .12% of the amount that I invested in seven-year bonds. While this doesn't sound like much, remember that I didn't invest any of my own money. I imagine that anyone here would love to have .12% of their company's money.

In this particular example (Chart 5), it is easy to see what is happening. First, as all three securities age, they move down the yield curve. But the difference between the five (now 4.5) year and the rest of the bonds has increased by 2.5 basis points, allowing for a slight capital gain relative to the other bonds. The duration of the barbell is equal to the duration of the seven-year bullet, but the convexity is greater. This means that for small changes in interest rates, the two portfolios move the same amount in relation to interest rates, but for large changes, the barbell rises by more and falls by less.

Most models that actuaries would use would be more sophisticated than this one, but that doesn't mean that arbitrage won't exist in these models. I have looked at some interest rate models that seemed to be so complex that this couldn't possibly happen. Yet, by trial and error, I have found portfolios that cost nothing, yet produced positive results over thousands of scenarios.

CHART 4

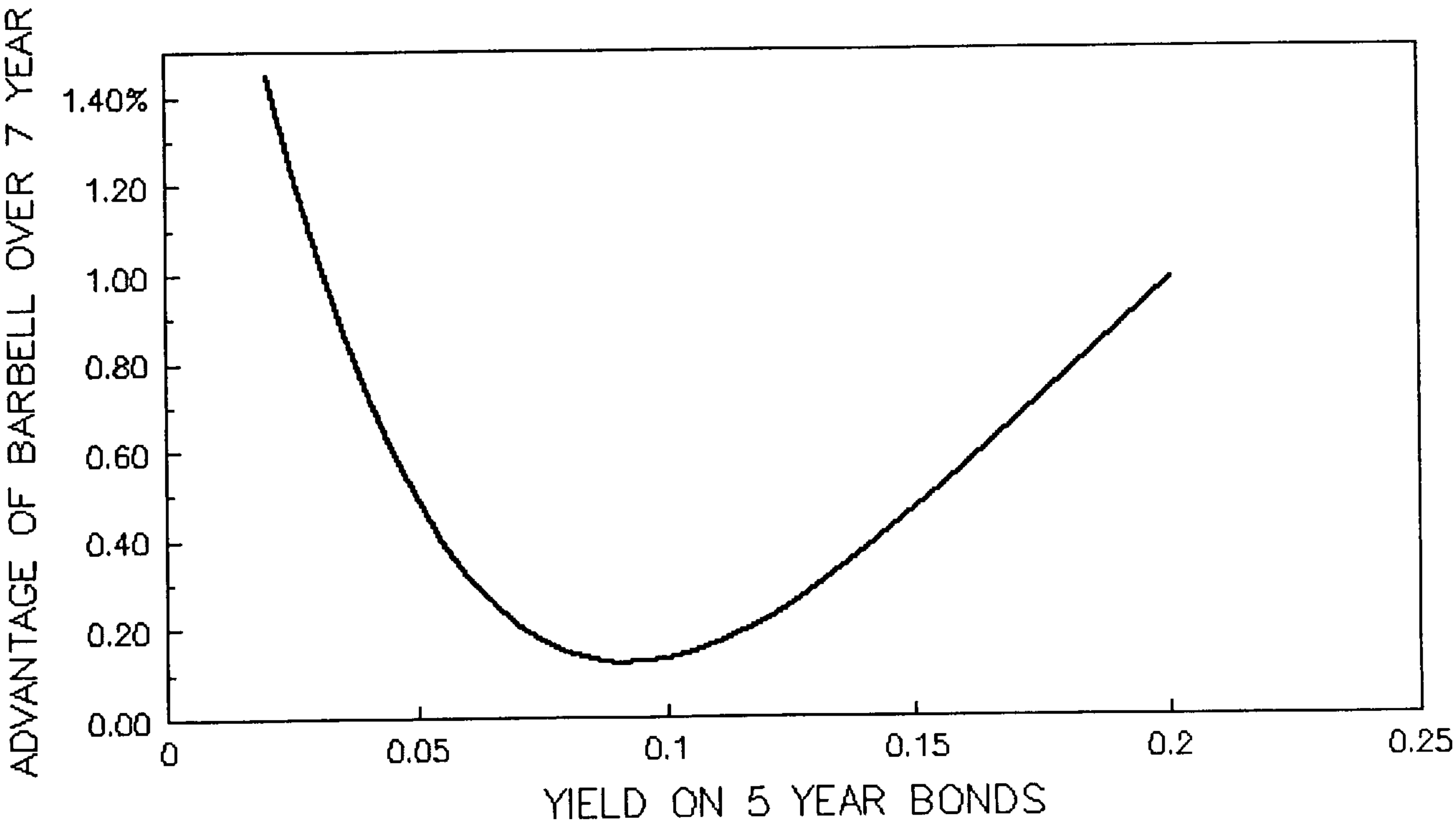
ARBITRAGE ON BARBELLS
(parallel yield curve shifts)



BARBELL IS ALWAYS GREATER
MINIMUM GAIN: 12 BASIS POINTS

CHART 5

ARBITRAGE ON BARBELLS
(parallel yield curve shifts)



Now that we know what arbitrage is, it is an easy step to define an arbitrage-free model as one that does not allow this to happen. It is not so easy to actually create one, which leads to the first question: "Is it really necessary?"

This is an especially valid question since all of my examples have involved the buying and selling of bonds, while most insurance companies tend to have a buy and hold strategy. Since we don't plan on active trading, do we need to go through the trouble? The answer is, "It depends." Remember that even if we don't short bonds or borrow money, we do credit interest, which is very similar. If you were given the task of doing cash-flow testing on a product where you were told that the credited rate is the portfolio rate, and that the investment strategy was five-year bonds, I doubt that you would have problems. Your computer wouldn't be clever enough since it is given both the crediting strategy and the investment strategy. You may have a problem, though, if you found the results unacceptable and were then told to find a better crediting or investment strategy. The more talent you had toward solving that problem, the more likely you would be to find the arbitrage in the model.

For example, in our parallel shift model, you may try a five- and ten-year barbell, instead of a seven-year investment. This would give you at least a 12 basis point improvement, all of which is attributable to arbitrage in the model, rather than to any real world economics.

Worse, even though this difference is large enough to make a significant difference on a leveraged product, it is probably too small to attract notice. It is very possible that this new investment strategy would be hailed as a significant improvement, even if it wasn't. The situation can get worse if you start to apply advanced optimization techniques in your analysis, because then you have a very clever computer and are assured of finding arbitrage, unless your constraints eliminate it.

The recent Exposure Draft entitled "Actuarial Standards of Practice -- Performing Cash Flow Testing for Insurers" requires assumptions to be internally consistent. Arbitrage in an interest rate model is an internal inconsistency. The current yield curve is inconsistent with the assumed range of future yield curves. I think that an actuary ought to make certain that this inconsistency does not distort his or her results. On the other hand, there are many problems where the extra effort would not provide additional value, such as where both the crediting strategy and investment strategy are simple and given in advance, and the information sought is a range of possible results. In all circumstances, one should be on the lookout for results that are biased by arbitrage. A good rule of thumb is that anything that looks too good to be true, probably is.

Another question that is often asked is, What good is an arbitrage-free model? One answer is that it helps an actuary avoid the types of errors that we are talking about. He is then free to worry about the thousands of other types of errors he might make. Another is that arbitrage-free models are the essential ingredient of option pricing.

It can be shown that, if there exists a probability distribution function where the expected prices of bonds in the future are equal to the forward prices of those same bonds, then the model is arbitrage free. Notice that I merely referred to the existence of such a distribution. It is not necessary for this distribution to be used to generate random interest scenarios. Arbitrage is a function of the range of possible future yield curves. It is not a function of the probabilities of those curves. In a positive yield curve environment, most people would probably say that this "arbitrage distribution" is excessively biased toward rising interest rates. However, there is one big advantage to calculating expected values using this distribution.

Imagine that you are given a product where the cash flows are a function of future interest rates. You would like to know how much it would cost to exactly fund that obligation. You are going to have the perfect investment strategy so that no matter what path interest rates take, you have exactly the amount of cash needed to fund the liability. No more, no less. It can be shown that the amount of money needed to execute this strategy is equal to the expected present value of the cash flows, where the expectations are taken using the "arbitrage distribution" and the present values are calculated using the short-term rates along each interest rate path. Notice that in this perfect world, the actual probabilities are irrelevant because the investment strategy works in every path, not just "on the average."

Since this is a valuation actuary symposium, we should answer the question, "What good is option pricing to a valuation actuary?" The answer is that valuation is the perfect place to use it. An option-based valuation can tell you whether you have enough assets to pay for your obligations, what risk there is due to changes in interest rates, and what steps could be taken to reduce that risk. The main disadvantage of an option-based valuation is the large amount of effort and computer resources required to perform one. I personally believe that this is a temporary problem and that both computer hardware and solution methods will be improved to the point where option-based valuations are practical for a large number of insurance companies.

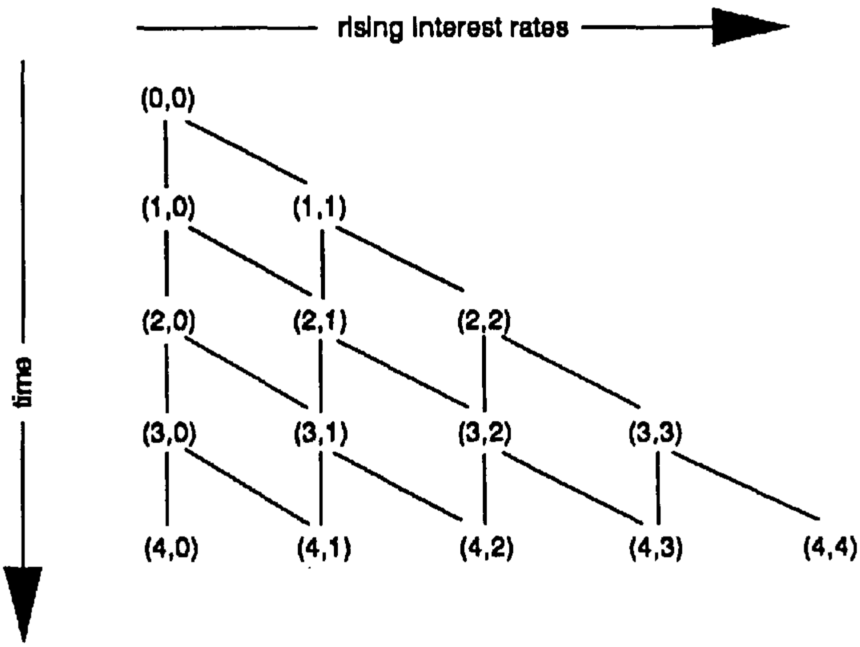
An option-based valuation is similar to a market valuation in that it estimates the value of an enterprise under current conditions. The values of both the assets and liabilities are calculated using current interest rates and volatility assumptions. These values are normally estimated by running random scenarios from an arbitrage free interest model, although there are more efficient calculation methods for some types of securities. The difference between the assets and the liabilities is the current surplus.

It is also important to understand how sensitive the surplus is to interest rates. This can be done by simply calculating the values with different starting yield curves. If you create parallel shifts in your yield curves, and estimate the negative derivative of the natural logarithm of the option value, with respect to the shift, then you have an option adjusted, or effective, duration. You can calculate one measurement of convexity by taking the second derivative of this quantity. You can calculate these values for both the assets and the liabilities, and for the surplus. If the duration of the surplus is positive, then you are exposed to the risk of rising interest rates. If the convexity of surplus is negative, you are exposed to any changes in interest rates. If the durations are matched and the convexity is positive, you are immunized against changes in the level of interest rates. Now you can worry about different yield curve shifts.

One other question that was asked at the workshop was, How does one create an arbitrage-free model? At first I decided not to tackle that one, because a panel discussion is a difficult medium for that kind of information. Then I noticed that the program had promised "Practical 'how-to' project yield curves," and since I haven't said anything practical yet, I figured that I had better try. I am going to try to give a general description of how to create a binomial lattice of interest rates, which is easy. The hard part is to make it price the current yield curve.

I like to think of a binomial lattice as more of a triangular matrix, than as a tree. As time progresses, you go down one row per period. You can either go straight down, where interest rates are lower, or down and to the right, where interest rates are higher. Now, let me define some functions.

Binomial Lattice
(Matrix Representation)



$L(t,u)$ is the point on the lattice that is at time t, if interest rates have increased u times. Our starting point is at $L(0,0)$.

$i(t,u,m)$ is the interest rate on m year zero coupon bonds, at $L(t,u)$.

$S(t,u,m) = (1+i(t,u,m))^{-m}$ is the price of an m year zero coupon bond at $L(t,u)$.

$p(t,u)$ is the probability of going from $L(t,u)$ to $L(t+1,u+1)$.

$O(t,u)$ is the value of an option paying $1 at $L(t,u)$ and nothing at every other node.

In an arbitrage-free lattice, the expected present value of the price of a one-period bond in one period is the same as the value of a two-period bond now.

$$S(t,u,2)=S(t,u,1)\bullet(p(t,u)\bullet S(t+1,u+1,1)+(1-p(t,u))\bullet S(t+1,u,1))$$

so

$$p(t,u)=\frac{S(t,u,2)/S(t,u,1)-S(t+1,u,1)}{S(t+1,u+1,1)-S(t+1,u,1)}$$

We can create our entire lattice with any two of one- and two-period bonds, and the arbitrage possibilities.

Now we show how to create values of O(t,u) in terms of our probabilities and spot prices.

For one-period options:

$O(1,0) = S(0,0,1) \bullet (1-p(0,0))$

$O(1,1) = S(0,0,1) \bullet p(0,0,)$

For options of greater than one period:

$O(t,0) = O(t-1,0) \bullet S(t-1,0,1) \bullet (1-p(t-1,0))$

$O(t,u) = O(t-1,u) \bullet S(t-1,u,1) \bullet (1-p(t-1,u)) + O(t-1,u-1) \bullet S(t-1,u-1,1) \bullet p(t-1,u-1) \ (0 < u < t)$

$O(t,t) = O(t-1,t-1) \bullet S(t-1,t-1,1) \bullet p(t-1,t-1)$

Now we can specify some parameters so that we can reprice the yield curve. The first thing we are going to do is specify the short-term interest rates by multiplying or dividing the forward short-term rates by a volatility multiplier, c. The one-period rates at time 1 will be the forward rates times c^{-1} and c, for u=0 and u=1. For time 2, the multipliers will be

c^{-2}, c^0, and c^2, and so on. Assume that we are given S(0,0,m) for all m. We now have enough information to calculate p(0,0,), O(1,0) and O(1,1), but we need more information to calculate the rest of the numbers.

For times 1 and greater, estimate the two-period spot rates in a similar fashion the way we created the one-period rates, call our estimate r(t,u,2). We are going to solve for k such that

$$\sum_{u=0}^{t}\{O(t,u)/(1+k\bullet r(t,u,2))^2\} = S(0,0,t+2)$$

This says that the value of an option that gives a two-period bond at time t, is worth the same as a t+2-period bond, where the two-period bond yields a constant, k, times our estimate. The above equation is easily differentiable and can be solved using the Newton-Raphson method. We now substitute $i(t,u,2)=k\bullet r(t,u,2)$, which allows us to calculate S(t,u,2), p(t,u), and O(t+1,u) for all u. Then we can calculate i(t+1,u,2) and start the process again. After we have finished with our lattice, we can calculate the remaining spot rates by backward induction.

You probably noticed that there were many other ways that I could have forced my lattice to price the curve, and I won't claim that this way is the best. I will say that I think it is rather efficient, and has some other pleasing features. If the two-period volatility multiplier is slightly less than the one-period multiplier, then the lattice will be mean-reverting. You have to be careful, though, because if there is too much of a difference in the multipliers, then the volatility of the long-term rates becomes very small.

This demonstration was not complete enough for you to go back and immediately build an arbitrage-free interest rate model. You would have to go back to work and fill in some holes, and I hope make some improvements. I do hope that I have given you some guidance, and some insight into why I think it is worth the effort.

Appendix A -- Numerical Example of Binomial Lattice

Assumptions:

Spot Curve

Maturity	Yield
1	7.00%
2	7.50
3	7.75
4	7.90
5	8.00

Volatility Multiplier for 1-year rates $= e^{.15} = 1.161834$
Volatility Multiplier for 2-year rates $= e^{.148} = 1.159513$

Step 1:

Calculate spot prices, S(0,0,m)

Maturity	Price
1	.935479
2	.865332
3	.799370
4	.737758
5	.680583

Calculate forward rates

Years Forward	One-Year Price	One-Year Rate
1	.925905	8.0023%
2	.923772	8.2517
3	.922924	8.3251
4	.922501	8.4009

Years Forward	Two-Year Price	Two-Year Rate
1	.855326	8.1269%
2	.852572	8.3014
3	.851398	8.3760

Step 2

Specify short-term rates on the lattice, using the one-year forward rates and the one-year volatility multiplier.

Example: $i(1,1,0) = 8.0023\% \div 1.161834 = 6.8876\%$
$i(1,1,1) = 8.0023\% \bullet 1.161834 = 9.2973\%$

Lattice of One-Year Interest Rates, i(t,u,1)

	u				
t	0	1	2	3	4
0	7.0000%				
1	6.8876	9.2973%			
2	6.1130	8.2517	11.1386%		
3	5.3249	7.1879	9.7027	13.0973%	
4	4.6105	6.2235	8.4009	11.3400	15.3074%

Step 3:

Calculate the probability p(0,0)

$$p(0,0) = \frac{S(0,0,2)/S(0,0,1) - S(1,0,1)}{s(1,1,1) - S(1,0,1)} = .468111$$

Where
$S(0,0,2) = (1.075000)^{-2} = .865332$
$S(0,0,1) = (1.070000)^{-1} = .934579$
$S(1,0,1) = (1.068876)^{-1} = .935562$
$S(1,1,1) = (1.092973)^{-1} = .914936$

Step 4:

Calculate the option values O(1,0) and O(1,1)

$O(1,0) = S(0,0,1) \bullet (1-p(0,0)) = .934579 \bullet (1-.468111) = .497092$
$O(1,1) = S(0,0,1) \bullet p(0,0) = .934579 \bullet .468111 = .437487$

Step 5:

Estimate two-year rates r(1,0,2) and r(1,1,2) using two-year forward rates and two-year volatility multiplier.

$r(1,0,2) = .081269 \div 1.159513 = .070089$
$r(1,1,2) = .081269 \bullet 1.159513 = .094232$

Solve for k, in the equation

$$\frac{O(1,0)}{(1+k\bullet r(1,0,2))^2} + \frac{O(1,1)}{(1+k\bullet r(1,1,2))^2} = S(0,0,3)$$

or

$$\frac{.497092}{(1+k\bullet .070089)^2} + \frac{.437487}{(1+k\bullet .094232)^2} = .799370$$

using iterative techniques, $k = 1.000985$

$i(1,0,2) = r(1,0,2)\bullet k = .070089 \bullet 1.000985 = .070158$
$i(1,1,2) = r(1,1,2)\bullet k = .094232 \bullet 1.000985 = .094325$

Step 6:

Calculate p(1,0) and p(1,1) using i(1,0,1), i(1,1,1), i(1,0,2), i(1,1,2), i(2,0,1), i(2,1,1), and i(2,2,1).

$p(1,0) = .487124$
$p(1,1) = .462446$

Step 7:

Calculate option values O(2,0), O(2,1), and O(2,2)

$$\begin{aligned} O(2,0) &= O(1,0) \bullet S(1,0,1)\bullet(1-p(1,0)) \\ &= .497092 \bullet .935562\bullet(1-.487124) = .238518 \end{aligned}$$

$$\begin{aligned} O(2,1) &= O(1,1) \bullet S(1,1,1)\bullet(1-p(1,1)) + O(1,0) \bullet S(1,0,1) \bullet p(1,0) \\ &= .437487 \bullet .914936\bullet(1-.462446) + .497092 \bullet .935562 \bullet .487124 \\ &= .441710 \end{aligned}$$

$$\begin{aligned} O(2,2) &= O(1,1) \bullet S(1,1,1) \bullet p(1,1) \\ &= .437487 \bullet .914936 \bullet .462446 \\ &= .185104 \end{aligned}$$

As a check, notice that $O(2,0)+O(2,1)+O(2,2) = .865332$, which is equal to $S(0,0,2)$.

Repeat steps 5 through 7 with the new values of $O(t,u)$.