1998 VALUATION ACTUARY SYMPOSIUM PROCEEDINGS

SESSION 18TS

EQUITY-INDEXED MODELING

Scott D. Houghton, Moderator

Alan R. Downey

Duc Xuan Ho

EQUITY-INDEXED MODELING

MR. SCOTT D. HOUGHTON: This session is in a teaching session format. Our instructors are Mr. Alan Downey and Mr. Duc Ho.

Alan Downey will speak first. Alan is the appointed Actuary of Keyport Life Insurance Company in Boston. Keyport has pioneered the equity-indexed annuity and today is the industry's leading seller of equity-indexed annuities.

MR. ALAN R. DOWNEY: The topic of this session is equity-indexed modeling and as Scott had previously mentioned, it's in the format of a teaching session. Having said that, my particular presentation is going to pertain to general topics and I will not get into the specifics of how calculations are done or very technical aspects in many respects. Duc will discuss that aspect. However, what I would like to discuss are the various things that you really need to consider in the context of modeling equity-indexed annuities.

I'm actually going to begin the session by speaking about practical considerations of modeling equity-indexed annuities. Next, I'll discuss constructing in-force liability cells in populations. After that, I'll address modeling assets, including hedging instruments, although my discussion will be more limited in scope. I think Duc's discussion will be considerably more detailed, especially in terms of the theoretical mathematical aspects. Next, I'll discuss setting assumptions, including dynamic behavior. After that, I'll briefly touch on economic scenario generators, particularly ones that we've used. Finally, I'll discuss crediting and investment strategies and give a brief summary which lists things and readings that you may want to look at in the context of modeling equity-indexed annuities as they pertain to products.

Among the practical considerations of modeling equity-indexed annuities is the use of the model. What do you want to use it for? Do you want to use it for cash-flow testing or some other application? Another consideration would be modeling capabilities and staff expertise. What models do you use, what models can you use, and what expertise do you have on your staff? The relative size of the equity-indexed block is a very important consideration as is the product design itself.

There are many varying uses of an equity-indexed model. The first that comes to my mind, and because of the fact that I'm an appointed actuary of the company, is the asset-adequacy analysis. Another very important use of the model will be to derive statutory reserves under Actuarial Guideline ZZZ. This topic will be discussed in much more detail at Session 43.

Another use of the model would be an ongoing evaluation of your hedging strategy. This would be from an internal perspective or in terms of looking at the "hedged-as-required" criterion in Guideline ZZZ if you do decide to use the Enhanced Discounted Intrinsic Value Method (EDIM) reserves. With regard to the valuation of a hedging strategy, the SEC market risk disclosures are an important consideration. Obviously, pricing, product development, and product design are other important uses of the model.

Another use of the model pertains to the GAAP valuation of derivatives under *FAS 133*. Under this pronouncement, which was released in June 1998 and becomes effective no later than January 1, 2000, asset options will be valued at fair value. We at Keyport believe liability options will be considered derivatives and will also be valued at fair value, while the guaranteed portion of the product will be valued on a book-value basis.

Another use of the model would be to derive GAAP deferred acquisition cost. Tax reserves may also be derived using the model. I have made sort of a misstatement here. When Guideline ZZZ is adopted by the NAIC, it will become the tax reserve tax method. It does not need to be adopted by the 26 states. The 26 states refer to the adoption of experience tables such as mortality tables. Mr. Bud Friedstat made that statement this morning in the general session, and I'm sure he is correct.

With respect to tax reserves, it's interesting that the tax reserve interest rate is defined from market-value adjusted contracts that have an equity component. IRS Notice 97-32 defines the tax interest rate to be used to calculate reserves for market-value adjusted contracts, but the method is not yet defined, so it seems rather useless at this point. Obviously, you can use the model for corporate level profits, surplus, and risk-based capital (RBC) projections as well.

Another consideration, from a practical standpoint, is your modeling capabilities and staff expertise. You not only have to consider your actuarial staff; you also have to consider your investment department and your accounting department staff. One option would be, obviously, to develop this modeling capability in-house. In order to do that, you need to have staff who can understand and implement investment and hedging strategies who are good programmers, who understand the modeling concepts of an equity-indexed product, and who understand the accounting issues associated with these products. If you don't have those capabilities or expertise and you don't want to bring them in-house, then your options will be to go to commercial packages.

The liability-side modeling for these packages is relatively new because the product has only been on the market for slightly more than three years. One thing with the commercial packages that you'll want to be sure of is that they have the capability of handling Guidelines ZZZ and *FAS 133*. Another option that's available, in terms of modeling, is you can go to the various big five or six accounting firms or the various Wall Street investment firms or actuarial consultants.

Another practical consideration is the relative size of the equity-indexed block. The complexity that you want to use in your modeling will be dictated by the relative importance of the product. If you're only selling a very small portion of it or a small proportion of equity-indexed products relative to your total in-force business, then you might not want to get into very sophisticated modeling. Having said that, you're still going to have to comply with Actuarial Guideline ZZZ for statutory reserves and eventually, for tax reserves. For example, if you plan on using the EDIM method for Guideline ZZZ, you also have to meet the "hedged-as-required" criterion. If the size of your block is very small, you're probably better off using the Commissioners Annuity Reserve Valuation Method with Updated Market Values (CARVM-UMV). Obviously, you're going to be required to report reserves under *FAS 97* and *FAS 133* for GAAP as well.

Another important consideration is the product design. Look-back, Asian-end, and ratchet options are more difficult, whereas, a point-to-point, in many cases, is relatively easy to model using the Black-Scholes option pricing model. However, you do want to consider the assumption limitations of Black-Scholes, and I believe Duc will address those later in his section. Other features of the product design that you may want to consider would be any market-value adjustment features and flexible premium contractual provisions you might have.

Next, I'm going to talk about constructing liability cells and populations. When constructing cells, you want to consider the product design, itself, as one of the cell criteria. There are four basic product designs. Clearly, you can have a combination of any of the product designs listed below. That's one of the main considerations you have in terms of developing liability cells.

- Look-back
- European
- Asian
- Ratchet

The equity-indexed basis, itself, is another consideration. The Standard & Poor's index is not the only index that's used. It is the predominant index that's used in equity-indexed products that are sold, but there are other indices that are coming into the marketplace now. The current and renewal term length, minimum and maximum guaranteed participation rates, total return, caps and floors, the minimum guaranteed values, and vesting schedule are the types of information that you'll want to include with your liability cells.

There are four statutory reserve methods you want to include: (1) CARVM-UMV, (2) Black-Scholes, (3) Market Value Reserve Method (MVRM), (4) EDIM. There are three defined methods in Guideline ZZZ. The Black-Scholes method is really a special case of CARVM-UMV method, but I included it here because it has been considered a method in some references.

The GAAP reserve method that you might use could be one of two approaches; a modified EDIM approach where the initial reserve is equal the premium—as opposed to the initial reserve being equal to the CARVM-UMV reserve under statutory reserve reporting using Guideline ZZZ. There's a bifurcated approach, which is used under *FAS 133*, where you, essentially, value the fixed guarantees at book value and the options at market. The initial reserve for the fixed guarantee portion under the bifurcated method would be equal to the contract premium less the market value of the option at issue.

The tax reserve method is another thing you'd want to consider with your liability cells. I believe it should be consistent with the statutory reserve no matter what you're using, but as of today the required method is not clearly defined. It's not really defined at all, because there will really be no statutory method for an equity-indexed product until Guideline ZZZ is formally adopted, which will probably happen in December. In spite of this, as I mentioned before, the interest rate for market-value adjusted contracts has been defined.

Let's move on to liability populations. The in-force summaries that you would need or want are: premiums, account value, surrender value, statutory reserve, GAAP reserve, and tax reserve. You'll clearly want to validate your model against each of these at the time of your initial valuation.

Other population data would include what the current participation rate is, duly considering minimum and maximum participation rates. I would think you'd want to consider any historical participation rates to the extent that your reset strategy, or crediting, or participation rate is linked to those prior participation rates. You'll obviously want to know the average current level of the equity index you're using. Included in that level would be: what is the current high watermark, what is the current average or Asian average, what is the implied volatility of the equity index, and what is the current implied dividend yield of the equity index?

Another thing you'll need to know is what the market value of the liability option is at the beginning of the term. You need to know this value in order to calculate GAAP reserves under newly promulgated *FAS 133*. You'll need to have as of the beginning of the term, the account value, the level of the equity index, measures of volatility, dividend yield, the Treasury yield curve, termination assumptions, and the vesting schedule. Duc will get into a little more detail about methods to actually derive the market value of an option later in the session.

Finally, you'll need to have the CARVM-UMV reserve at issue if you use the EDIM method. I believe the assumptions you would use would be much the same assumptions that you would use in deriving the market value of a liability option under *FAS 133*. You would want to include valuation interest rates, which typically would be type C rates unless you have a market value adjustment feature in your contract. The tax reserve interest rate would be the greater of the applicable federal rate and the prevailing state rate, at least at this point, for equity-indexed products (however you define the method). For market-value adjusted contracts, Notice 97-32 stipulates that, essentially, you would use 110% of Moody's corporate bond average as of December of the year of issue. It's the same reference as the rates in the Standard Valuation law, but you're only picking off that average for that one month of the issue year, unlike the Standard Valuation law which is a 12-month or 36-month average. You may want to have issue and attained ages brought into your model as well.

There are many different ways you can do this. You might want to determine the size of the block in relation to your total in-force business. If you're selling almost none of the product at all, you can possibly choose not to model it or just model it as one product. If you're selling a number of different product designs within your company, you have to make the decision as to whether or not you want a combined design if some of the designs are immaterial in size. If they're not immaterial in size, then you probably can't do that.

Relative to the issue period, if the block is material, you might want to model this on a weekly basis. If you want to use EDIM and the option replication provisions of the "hedged-as-required" criterion, you have to do it weekly; you don't have a choice, because I believe it's defined that way within

Guideline ZZZ. The size of the block would also depend on what you're modeling purpose is. If you're using it for corporate or other modeling purposes, you might make a different decision than to use weekly blocks.

I'd like to get into a brief discussion of asset modeling by first talking about equity-indexed call options. I'm referring to European options, the option where the sole exercise date is the date at maturity. I'll also very briefly talk about equity-indexed futures and forwards and modeling other traditional assets.

With respect to equity-indexed call options, I think that you probably should model each option individually. My own experience has shown that there are many fewer options that you have to model than there are policies that you have to model. You might combine them, but I would only combine them when you have a very homogenous set of parameters such as a consistent or identical equity-indexed option type, purchase date, maturity date, and exercise price.

Your in-force population data should include items such as the purchase date, the maturity date of the option, the exercise prices, and the current market value and the actual market value of the option. When I refer to the current market value, I'm not referring to the theoretical one that you would necessarily calculate, but the actual market value that you might get from a Wall Street firm.

Other uses of data in validating market values include the current level of the index, the current Treasury yield curve, and implied volatility and implied dividend yield. Implied volatility and dividend yield are derived from that "street market value." It's very useful to validate the other data that you've input to make sure that you're not getting some crazy results.

I don't have a lot to say about futures and forwards. I think they're fairly straightforward to model. You just mark them to market. There's really not a whole lot to be said in terms of modeling them. You want to make sure that if you're using futures and forwards as part of your investment strategy, you include them in your model.

For other traditional assets you do what you've been doing to date. You model the same way as you have for traditional annuities. You'll have less assets to invest because of the fact that you purchased the options to hedge the liabilities or the equity-indexed portion of the liability. You would include the following traditional general asset classes of fixed-income and other securities: Treasuries, corporate bonds, CMOs, asset-backed securities, fixed-income derivatives.

I'd like to take a look at setting assumptions. First, I'll take a look at setting economic assumptions, and then look at aspects of policyholder behavior. Finally, I'll touch on other types of decrements. With regard to economic assumptions, it's very important that these assumptions be consistent across assets and liabilities. Otherwise, the results you're going to gather are totally meaningless. You have to make sure that what you're using on one side of the balance sheet, you'll be using on the other side. Your bases should be consistent with your actual crediting and investment strategies. In other words, if your investment strategy or your crediting strategy is keying off of spot rates, you don't want to go out and use par coupon rates to do your modeling. You want to make sure those are consistent. You'd want to use your current Treasury yield curve as one of the assumptions. For the average dividend yield of the equity index, you could use the current implied dividend yield, or you could use a historical yield. It really depends on the purpose of your modeling. There's one other assumption you would want to use. Obviously, it is the average volatility of the equity index, which I'll discuss in a little more detail.

The average volatility of the index can be broken down into four different categories. You could use the current implied volatility or you might use historical volatility. You might want to actually use a term structure of volatilities or you might want to assume a constant level of volatility. You may also want to utilize a strike structure (or volatility skew). A strike structure would be volatility, where the actual level of volatility varies by the strike price. In addition, do you want to use bid, ask, or mid-market volatilities? You use bid volatilities for liabilities and ask volatilities for assets, or you could use a mid-market assumption on both sides.

Again, what do you use for volatility? It really depends on what you want to model. For cash-flow testing, you might want to grade from a current level of implied volatility to some historical level. For current market values you might want to link it to the current term structure, or the strike structure of volatilities.

The next assumption is policyholder behavior. Many products have limited liquidity. Many equity-indexed products have limited liquidity and will have either a substantial loss of principal or a forfeiture of nonvested index credits. A limited liquidity should keep surrenders low, while more (or better) liquidity will (or should) result in higher surrender rates.

A very difficult aspect of modeling equity-indexed or, for that matter, any interest or equity product is what to do about the dynamic surrender assumption. I think there are some general statements which probably can be made with regard to that assumption. First of all, as interest rates rise, you would expect surrenders to increase. Second, a bull market should be a disincentive to surrender, especially if the policyholder has a substantial level of unvested index credits; that would be a substantial disincentive for the policyholder to surrender the contract. A bear market would be added incentive to surrender, particularly if the policyholder has not accrued any index credits since purchasing the policy. Finally, it seems to me, the worst case scenario would be if interest rates were to spike and the stock market were to crash, and you have a policy in which no index credits are accrued. At that point, there would be very little incentive for a policyholder to want to stick around.

In deriving a dynamic surrender algorithm, one needs to consider or recognize the equity index at various points in time. It can be at the start of the term, at any relevant look-back point, at averaging points, and on the current date. The fact is, there's little or no actual experience to date for equity-indexed annuities. One could impute experience from traditional Single Premium Deferred Annuity (SPDA) experience; however, if you've looked at the results of the most recent studies from the Society of Actuaries and the Life Institute and Marketing Research Association (LIMRA), you'd notice that there's very low correlation between the actual changes in the economy and surrenders. We feel that the dynamic surrender or an increased surrender activity might be driven more by major events, such as large insurance company failures or the threat of a proposed tax reform. In reality, you're making an educated guess as to what policyholder behavior is going to be, but, in any case, the sensitivity testing of assumptions is very important. In doing this, you should really have an understanding of what the potential risk exposure might be to the company under various scenarios.

Let's look at surrenders at the end of the term. There is little or no actual experience to date. You might impute it from SPDA experience. To the extent you have renewal commissions, it should impact your experience in a positive way by reducing your level of surrender rates. There is almost no actual experience because equity-indexed products have only been out there for three years at the most. I think it's very important that whatever assumptions you use, you sensitivity test them.

Looking at other decrements, I won't get into this very much in the interest of time. Regarding mortality, there's very little equity-indexed experience, but there's likely very little difference from SPDAs. I think annuitant mortality is probably appropriate. You can use your company experience, if it's credible. You want to consider annuitization, systematic withdrawals, nursing home waiver and perhaps other types of decrements as well.

I'm going to briefly touch now on economic scenario generators: first, the interest rate scenarios, then the equity index and, finally, fixed scenarios that could be used under asset adequacy analysis. The typical interest rate scenario generators are fairly well known to actuaries. A typical model would have a lognormal distribution with mean reversion; a correlation among short-term, medium-term, and long-term rates; and, higher interest rate volatility for short-term rates than for long-term rates.

You can use a lognormal distribution for equity-indexed scenarios. It's questionable in my mind as to whether mean reversion is appropriate here. I don't believe there is the general acceptance of mean reversion for equity-indexed scenarios that there might be for interest rate scenarios. Correlation between interest rates and bond prices makes the model somewhat more complex. While the correlation is relatively low, you may not want to ignore it.

In setting the equity-indexed scenarios, you'll want to look at the historical mean and standard deviation of the index. An observation period of 10 years is definitely too short. Fifty years might be too long. I believe 25 years is a fairly reasonable period to use and has brought us through enough economic cycles that I believe it's reasonable.

Let's discuss fixed scenarios as they pertain to asset-adequacy analysis. You still need to do the New York 7 interest scenarios. The question is, how do you integrate equity-indexed scenarios into the New York 7 framework? I don't think anything is clearly defined at this point. A few states have adopted some regulations, but nothing is very clearly defined at this point.

I'd like to talk about renewal participation rates. Participation rate is driven by factors that drive the option cost. Higher interest rates will generally result in a higher option cost, but that option cost should be more than offset by a greater spread on your fixed assets that you've purchased as well. Therefore, you'd end up with a higher participation rate. Higher implied volatility is something that has been rearing its ugly head with the stock markets all over the world bouncing everywhere. It would result in higher option costs and, thus, a lower participation rate. Your product design would also have an impact on the nominal level of the participation rate. In particular, a look-back design would tend to have lower participation rates than other designs because of the lock-in feature. You'll also want to consider minimum and maximum participation rates as well.

Investment strategies for fixed-income assets are probably no different than for your typical SPDA; however, there are fewer assets to invest due to cost of the options. You will have a higher basis point earned over credited spread because of the fact that you're only crediting 3%, in most designs, to the fixed portion. Assets allocated to the minimum guarantees are, generally, something like 75% to 85%, so you have to have that additional spread in order to have a profitable product.

With regard to equity-indexed derivative investment strategies, there are three general types of hedging. There may be other categories as well. First, you could model or you could hedge on an exact basis, which means you'd purchase an option that would exactly match the liability contract that you've sold.

Second, you could use notional hedging where the hedge is, essentially, based on the intrinsic value of the options. You'd need to periodically rebalance based on surrender experience and expiration of the options.

Third is dynamic macro hedging, in which the hedge is made based on the market value of the options and other factors regarding the change in the market value of the options. Again, you'd have to rebalance fairly frequently. An example is using the "Greeks," which are measures akin to those of duration and convexity for fixed-income assets. Duc will describe those in more detail.

I've put together a list of what I feel is must reading. I avoided using the word required reading for those who are still Associates and still taking the exams. However, you definitely should read Guideline 33 and ZZZ. You should also look at *FAS 97* and the new *FAS 133*. You'll want to look for sure at the final report of the Equity-Indexed Products Task Force which describes in detail resolutions to a number of issues that the task force chaired by Donna Claire has addressed. You'll want to look at the various Actuarial Standards of Practice that are associated with modeling and asset adequacy analysis (Numbers 7, 10, 14, 22, and 23). Finally, look at the practice note on equity-indexed products.

You should also look at any references that you can get your hands on regarding option valuation, investments, and hedging strategies.

MR. HOUGHTON: Our next speaker is Mr. Duc Ho. Duc is director of asset and liability management for London Life Reinsurance in Blue Belle, Pennsylvania. He's in charge of pricing and management of investment-related products for London Life Reinsurance. Many of London Life's clients rely on Duc's work for their own risk management and regulatory requirements.

MR. DUC XUAN HO: As Alan said, equity-indexed products cover a very broad area of topics. What I'd like to do is focus more narrowly on the valuation of options underlying the equity-indexed products. I thought the topic would be appropriate, because in order to satisfy reserve requirements or perform asset adequacy analysis, one must be able to evaluate the options in the book.

Let me start by stating the obvious which is—an option is a derivative; therefore, the cash-flows or the values are derived from the underlying securities. In most cases, underlying securities are stocks. Because the options are derived from the underlying security, it would make sense to project the distribution of that underlying security to evaluate the option. Once you have that distribution of the underlying security, you would want to project the distribution of events that affect the option payoff, independent of the underlying security. Once you do that, you want to combine the distribution of the underlying security with the distribution of events that affect the option payoff. This enables you to arrive at the distribution of the option cash-flows themselves. Once you have the distribution of the option cash-flows, you would want to do an expected value calculation and then discount it and arrive at the option value. That is the overall approach in valuing any type of option, whether it's an equity option, interest rate option, or an option on buying a house.

Let me clarify that approach a little bit by giving a very general example. Suppose that you were asked to value a one-month call on a stock. You know that the current stock price is $110. You're given a strike price of $120. The interest rate is 1% per month. You're also given the distribution of the stock price in one month's time as shown in Table 1 below. It can be $80 at 40%, $125 at 30%, and $145 at 30%. You're also given the probability of exercise. Obviously, if the stock price is below the strike price, then the probability of exercise will be zero. If the stock price is pretty close to the strike price, then you may get a 50% exercise rate. If the stock price is very high above the strike price (say $130 or higher), then it's almost certain that you will exercise the option.

TABLE 1

Price Level	Probability	Price Level	Probability of Exercise
80	0.4	Below 120	0%
125	0.3	120–130	50%
145	0.3	Above 130	100%

When pricing an equity option, one would usually assume that once the stock price is above the strike price, it certainly will be immediately exercised (i.e., one assumes investor behavior is rational). However, there might be times where even if it makes sense to exercise the option, the investor may not do it (i.e., exhibit irrational behavior). For example, this would happen in single premium deferred annuity (SPDA) products, or in the exercise of a callable bond. Sometimes even though the price of the bond is above the call price, the companies still may not call the bond. So in valuing any type of option, the assumption of rational behavior is something you might want to question.

As can be seen in Table 2, if given a range of possible stock prices and their associated probabilities, and given the likelihood of exercise at each stock price level—combine the distribution of stock prices with the likelihood of exercise to project option payoff under each possible event. Next, I simply calculate the present value of each of those events, and weight them by the probability. I then arrive at the option value of a very generic option.

TABLE 2
Distribution of Option Cash-Flows

Stock Price	Probability of Stock Price	Probability of Exercise	Option Cash-Flow	Present Value	Expected Value
80	0.4	0	0	0	0
125	0.3	0.5	5	4.95	0.74
145	0.3	1.0	25	24.75	7.43
Total					8.17

If you go through that exercise, what do you see? One thing that I noticed is that the exercise of calculating the option value is very similar to calculating the premium of a life insurance product. When you calculate the premium, you actually calculate the expected value of the claim at death. You have to figure out the probability of surviving to t years and the probably of death, and the payoff (i.e., death benefit upon death). Then you go through the expected value calculation. When you calculate the value of the option, you go through exactly the same exercise. You figure out the distribution of the stock price. You figure out, within each of the events that make up the distribution, what the payoff would be under each of those events, and then you weight them by probabilities.

Having said that, what are the variables that affect the option price? There is the distribution of the stock prices and the discount rate. The discount rate that you apply to the various cash-flows represent the discount rate to a risky event. By that, I mean that the event is not 100% certain. It may or may not happen. The cash-flow may or may not be there. Therefore, one might place a different premium upon that discount rate for each individual investor. Given the same set of cash-flows, one person might put a 10% discount rate on that set of cash-flows, while another individual may put a 5% discount rate on the same set of cash-flows.

What does this mean? (Because different investors may have different views and different degrees of risk aversion), they may pay different prices for the same set of options. If that's the case, then the opportunity for riskless arbitrage exists. In other words, if one sees the same option being priced differently by two individuals, he would buy the option from the one with the cheaper price and sell it to the one that's willing to pay for a higher price. Therefore, he makes a profit without dispensing any cash.

In finance theory, you never want to have a riskless arbitrage. You never want to have that happen. Therefore, we need a way to value the option in such that, given a set of options, and a set of cash-flows, you can arrive at a unique price for it. It is in their path-breaking paper, that Black and Scholes solved that problem.

The Black-Scholes valuation is a method for valuing option cash-flow. In order to do so, Black and Scholes placed the following assumptions in valuing the option. One is that stock prices follow a lognormal process. What does that mean? That means that the stock price has no memory. What happened in the past has no impact on the future of the stock price. You don't know what the price is likely to be. All you know is that the return on the stock price will most likely be lognormally distributed. That's what they meant by lognormal distribution of the stock price.

Second, they assumed that the interest rate, the expected growth of stock prices, and volatility are constant. In reality, those assumptions are probably not realistic, but those assumptions are not central to the results of Black-Scholes analysis. They assume this to simplify the calculation.

The third assumption Black and Scholes made was that short selling of securities are permitted. This is actually happening in the marketplace. No transaction costs or taxes are assumed. Again, these assumptions are used solely to simplify the calculation.

Armed with those assumptions, Black and Scholes tried to find the relationship between the derivative price of the option price and the price of the underlying securities. They arrived at a formula called the Black-Scholes differential equation. This Black and Scholes differential equation is one that every derivative security, regardless of its payoff, has to satisfy. If you take a close look at this equation, you see that the price of the derivative (f) depends on interest rate (r), time to expiration (t), security price (s), and volatility (σ).

$$r*f = \frac{\partial f}{\partial t} + r*S*\frac{\partial f}{\partial S} + 1/2\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}$$

One thing you might notice is that there's no expected rate of return of the stock in this equation. What does that mean? That means if you have an option on security and one expects a 10% return on the stock and another expects a 5% return on the stock, they would arrive at the same price. What does that mean? That means that, regardless of the risk preference of the investor, one would arrive at the same price for the option whether one is risk-averse, risk-seeking, or risk-neutral. Because the option price is the same regardless of risk aversion or the degree of risk aversion of the investor, we can choose the assumption that the investor is risk-neutral, and that would not affect the option price. Now, what does that mean? When you choose the assumption of being risk-neutral, it means that you do not pay for risk-taking. There's no premium for risk, so all the discount rates now become the risk-free rate, and the expected return on the stock is also the risk-free rate.

What Black and Scholes has done is basically imposed a restriction on the distribution of the stock prices that you use to value the option. In the risk-neutral world, the stock now follows a lognormal process which is one of the simplifying assumptions. Assuming you live in a risk-neutral world, stock grows at a risk-free rate, (which is like 4.5% or 5% today) and the cash-flows have to be discounted at the risk-free rate. When you make that risk- neutral assumption, all the options based on the same security with the same payoff would have exactly the same price. Thus, Black and Scholes eliminate the possibility for riskless arbitrage. The importance of the Black-Scholes paper is not in the formula itself, but in the fact that one can assume risk-neutrality in valuing the option. Therefore, in projecting the distribution of the stock and by discounting the option or stock cash-flow, one can project it forward at the risk-free rate and discount at the risk-free rate. When Black and Scholes apply the differential equation to a European call, they were able to solve for an analytical solution for the European call.

What you see below is that the formula for the European call is a special case to the Black-Scholes differential equation. In that special case, the payoff is the excess of the stock price over the strike price at expiration.

$$c = e^{-rt}E[\max(S_t - X, 0)]$$
$$= S * e^{-qt} * N(d1) - X * e^{-rt} * N(d2) \text{ where}$$

N(x) is the standard normal cumulative function

$$d1 = \frac{\ln(S/X) + (r - q + \sigma^2/2)\, t}{\sigma\sqrt{t}}$$

$$d2 = d1 - \sigma\sqrt{t}$$

For other types of options (e.g., look-back options, Asian options, or exotic options), you will most likely not be able to find an analytical solution to the value of the option. Because it's pretty difficult to solve for an analytical solution for the option value, you need to come up with an approach that allows you to uniquely value any option. One of the tools that can be used to value any type of option—whether it's look-back, average, ratchet, American, or European—is binomial valuation. That's one of the two based on Black-Scholes valuation to calculate the value of any type of option.

Binomial valuation is a technique for projecting out the stock prices. If you are given a stock price at time t, for example, then at time t + (delta t) which is let's say one day or one week from the time t, there are only two possible stock prices—stocks that move up or stocks that move down. There is no other possibility. Given the stock price is S at time t, the stock can be moved up to S times u, or moved down to S times d. The probability of moving up is p, and the probability of moving down is 1 – p. The formulas below show how u, d, and p can be calculated:

> Given stock price of S at time t, stock prices can be S*u with probability p and S*d with probability 1 – p at time t + Δt where:
>
> $u = e^{\sigma\sqrt{\Delta t}}$
>
> $d = 1 / u$
>
> $a = e^{r\Delta t}$
>
> $p = (a - d)/(u - d)$

One might ask, how does one arrive at these formulas? Basically, armed with the assumption that stock has to grow at a risk-free rate in a risk-neutral world, you know that the expectation of return is equal to r, the risk-free rate. You also know that the standard deviation, i.e., the volatility of the stock movement, is given as Sigma. Now, if you're given a binomial distribution you can also calculate the expected value and the standard deviation in term of p and u. So you don't know p and u, but you know r and Sigma. You now have a system of two simultaneous equations with two unknowns, and the formulas for u, d, p are the solution to that system of simultaneous equations.

Let's work with an example to demonstrate how we can use binomial valuation to calculate the option price. Let's calculate the price of a three-month European call, given that the current stock price is \$100, the strike price is also \$100, the volatility is 20%, and the interest rate is 5%. You apply the formulas I gave previously to calculate u, which is the amount that stock has to move up; d, which is the amount that the stock has to move down; p, which is the probability of stock moving up, and then 1 minus p, which is the probability of stock going down.

$$u = e^{0.2\sqrt{1/12}} = 1.05943$$
$$d = 1/1.05943 = 0.94340$$
$$a = e^{0.05/12} = 1.004175$$
$$p = (1.004175 - 0.94340) / (1.05943 - 0.94340)$$
$$= 0.52171$$

Given those results, you can project a tree of stock prices (Table 2). You start with \$100, which is your current stock price. In month one, it can move up to \$105 or move down to \$94. There's no other possibility.

Let's say that if the stock price is \$105 at month one, then it can move back down to \$100 again, or move up to \$112 and so on and so forth. So at the end of month three, there are four stock price possibilities. It can be \$118.90, \$105.90, \$94.40, and \$84.10. Given those four possibilities, you can calculate the payoff of the option under each of those possibilities, and then you can apply those payoffs with the probability of each of the payoffs to come up with the expected value of the payoff. Then you can discount that expected value at the risk-free rate to arrive at the option value.

The methodology for calculating this option is very similar to that used in the very first example that I pointed out. The only difference is that now the process of the stock price movement is lognormal, and the stock prices grow at the risk-free rate; however, the mechanism itself does not change.

TABLE 2
Tree of Stock Prices

Month				Option Payoff	Probability
0	**1**	**2**	**3**		
			118.911	18.911	0.142
		112.240			
	105.943		105.943	5.943	0.391
100.000		100.000			
	94.390		94.390	0	0.358
		89.095			
			84.097	0	0.109
Expected Payoff				5.007	
Option Value				4.944	

One thing I should also mention is that the probability for each event is the product of the probabilities along each path. For example, if stock keeps moving up and up and arrive at $118.90, then the probability of 0.142 is equal to the probability of moving up from time zero to time one, times the probability of moving from time one to time two, times the probability of moving up from time two to time three and so on and so forth. If you have many paths that arrive at the same stock price, you simply calculate the probability of each path and sum them together for the probability of arriving at that particular stock price.

To verify whether this calculation is at least approximate to the Black-Scholes analytical formula, we apply that formula to the variables that are given, and we arrive at a similar price. The price calculated using the binomial valuation is roughly $4.9, and if you apply the Black-Scholes formula, you arrive at something like $4.6 which is fairly close, given the fact that you have only three steps. Each step is one month in a three-month option. If you increase the number of steps, then the price of the option calculated using the binomial valuation will approach the Black and Scholes method.

Let's do something a little bit more complicated. Let's go back and calculate the price of a look-back option. A look-back option is an option whose payoff at expiration equals the maximum stock price over the period, minus the strike price. The payoff of the option depends on the path that the stock price has moved from the time of the inception of the option until expiration of the option. If you want to value this using the binomial method, you have to project the value along every individual path of the stock price. It is very important. If you want to calculate the value of any path-dependent option, you have to trace every single individual path that the stock has taken. Then you have to calculate the payoff and the probability of that particular path. Once you have a table of different paths and the payoff of each individual path, you weight them together to arrive at the expected payoff, and then discount it to arrive at the option value.

Let's look at Table 3. The very upper most path, UUU, represents up, up, up. The stock goes from 100 to 105.9 to 112.24 to 118.9. Out of those values, the maximum is 118.9 and then you can arrive at the payoff.

TABLE 3
Calculation of a European Call Look-back

	Month						
Path	**0**	**1**	**2**	**3**	**Smax**	**Option Payoff**	**Path Probability**
UUU	100.000	105.943	112.240	118.911	118.911	18.911	0.142
UUD	100.000	105.943	112.240	105.943	112.240	12.240	0.130
UDU	100.000	105.943	100.000	105.943	105.943	5.943	0.130
UDD	100.000	105.943	100.000	94.390	105.943	5.943	0.119
DUU	100.000	94.390	100.000	105.943	105.943	5.943	0.130
DUD	100.000	94.390	100.000	94.390	100.000	0.000	0.119
DDU	100.000	94.390	89.095	94.390	100.000	0.000	0.119
DDD	100.000	94.390	89.095	84.097	100.000	0.000	0.109
					Expected Payoff	6.536	
					Option Value	6.454	

Let's look at the next one, UUD. The stock i moves up and up and then down. Even though the stock ends up at $105.9, the maximum price of the stock was $112.24 in month two. You have to basically record that number in order to calculate the option payoff. The only problem with this is that the number of paths that you have to travel in order to calculate the expected value of the option increases exponentially as the number of steps increases. For example, in this case, we have three steps, so the number of paths would be 2V3 which is eight. If you have, say, 16 steps, then the number of paths that you have to take is 2^{16} power which is roughly about 65,000. The number of paths and the amount of computation power required would increase exponentially. However, the one thing you can see from the example is that once you've developed the table of different paths, it's pretty easy to calculate the value of the option.

Let me demonstrate that point again by using the same option, and just changing this from a look-back call to an Asian call (Table 4). If you changed this from a look-back call into an Asian call, the process repeats. The only difference is that when you calculate the payoff instead of taking the maximum of the stock price over each path, you calculate the average over each path. Then, you apply the same probability factor and do the same expected value calculation.

TABLE 4
Calculation of an Asian Call Look-back

Path	Month				$S_{average}$	Option Payoff	Path Probability
	0	1	2	3			
UUU	100.000	105.943	112.240	118.911	109.274	9.274	0.142
UUD	100.000	105.943	112.240	105.943	106.032	6 032	0.130
UDU	100.000	105.943	100.000	105.943	102.972	2.972	0.130
UDD	100.000	105.943	100.000	94.390	100.083	0.083	0.119
DUU	100.000	94.390	100.000	105.943	100.083	0.083	0.130
DUD	100.000	94.390	100.000	94 390	97.195	0.000	0.119
DDU	100.000	94.390	89.095	94.390	94.469	0.000	0.119
DDD	100.000	94.390	89.095	84.097	91.895	0.000	0.109
					Expected Payoff	2 510	
					Option Value	2.479	

Same as Table 3 except the payoff is Max(0, $S_{average}$ - 0).

If you change the option to an Asian-end call you repeat the same process (Table 5). The difference in this case is that you're only taking the average over the last month of the stock rather than for the entire period of three months. Therefore, the price of the Asian-end option is a little bit higher than the price of the Asian option.

TABLE 5
Asian-End Call

Path	Month				Smax	Option Payoff	Path Probability
	0	1	2	3			
UUU	100.000	105.943	112.240	118.911	115.576	15.576	0.142
UUD	100.000	105.943	112.240	105.943	109.092	9.092	0.130
UDU	100.000	105.943	100.000	105.943	102.972	2.972	0.130
UDD	100.000	105.943	100.000	94.390	97.195	0.000	0.119
DUU	100.000	94.390	100.000	105.943	102.972	2.972	0.130
DUD	100.000	94.390	100.000	94.390	97.195	0.000	0.119
DDU	100.000	94.390	89.095	94.390	91.742	0.000	0.119
DDD	100.000	94.390	89.095	84.097	86.596	0.000	0.109
					Expected Payoff	4.169	
					Option Value	4.117	

Same as Table 4 except the averaging period is over the last month of the option.

As I mentioned previously, the amount of time or the number of paths taken increases exponentially by the number of steps. If you increase the number of steps, the price of the option calculated using the binomial method would approach that of using the Black-Scholes method, which is the true analytical price of the option. One would have to wonder if there's anything we can do to improve the price of the option using some other method rather than just simply increasing the number of steps.

One method I propose is as follows. If you assume that the errors that are introduced by using the binomial method rather than by the Black-Scholes analytical formula are similar for different options, then we can use the following formula in order to improve the option price using the binomial method.

$f^E = f^E_{Bi} - f^v_{Bi} + f^v_{BS}$ where

f^E = value of the exotic option (to be calculated)
f^E_{Bi} = value of that exotic option using the Binomial method
f^v_{Bi} = value of a similar vanilla option using the Binomial method
f^v_{BS} = value of a similar vanilla option using the Black-Scholes formula

Let me talk about this in detail. Let's say that you calculate the option value using the Black-Scholes method for a generic European call and come up with $5.00. Then you calculate the same option using the binomial method and come up with $6.00. That means the error introduced by the binomial method is $1.00. If you assume that the error introduced by the binomial method for the vanilla European call is similar to a look-back option using the binomial method, then you can basically back out to the true option value using the Black-Scholes method.

For example, let's assume the option value calculated using the Black-Scholes method is $5.00, and the option using the binomial method is $6.00, and the option using the binomial method for a look-back call is $10.00. Then if you were to use the Black-Scholes formula to calculate the price for the look-back option, it would be something like $9.00. If you assume that the error is consistent, then you can basically use and apply the same error to the option price using the binomial method. Let me give you a clearer example.

Let's assume that you want to improve the value of the look-back option that has a value of $6.45. When I tried to calculate the value of a generic option using the binomial method I come up with 4.9, and if I use the Black-Scholes method I come up with 4.6. That means that by applying the binomial method, I overstate the value of the option by 0.3. If I use the binomial method for the look-back and come up with 6.4, it is likely that the true value of the option would be in the neighborhood of 6.1, or it exceeds the true value by 0.3. That's how I arrived at the "improved" estimation for the option price using the binomial method.

$$f^{E}_{Bi} = 6.454$$
$$f^{V}_{Bi} = 4.944 \text{ (Table 2)}$$
$$f^{V}_{BS} = 4.615 \text{ (Table 2)}$$

$$\begin{aligned} f^{E} &= 6.454 - 4.944 + 4.615 \\ &= 6.125 \end{aligned}$$

FROM THE FLOOR: Why wouldn't you use a percentage error as opposed to this method?

MR. HO: We can use that, too. There's no reason why we can't use that. This is simply a suggestion.

FROM THE FLOOR: Wouldn't that make more sense to deal with a relative error in asking price?

MR. HO: Yes. I went through the process of calculating the option for basically any type of payoff using the binomial method.

One of the things that Alan talked about in his presentation was hedging. Let's go back and ask ourselves, what do we mean by hedging? Let's say you have written a basket of options. What do you mean by hedging it? If you're selling an equity-indexed product and you want to hedge that product, what does it mean? I think it means you try to come up with an asset or a liability that will offset your position. If you sell an equity-indexed product and you have a liability, which is essentially an option, you want to come up with something that offsets against the liability—offset that against that option. Therefore, you are left with the same positions regardless of stock price movement. That's what I think we should mean when we say we hedge for something.

In order to hedge an option, the very first thing we need to do is to measure the sensitivity of that option to various variables that affect the option prices. We have Delta which is the change of the derivative price over the underlying security. Gamma is the change of the Delta over the change in the security price. Vega is the change of the derivative price over the change in volatility. Theta is the measure of change in the derivative price over the passage of time. Rho is the change of the derivative price over the change in interest rate.

Δ = $\partial f/\partial S$ measures change in option price as stock price changes
Γ = $\partial \Delta/\partial S$ measures change in Delta as stock price changes
V = $\partial f/\partial \sigma$ measures change in option price as volatility changes
Φ = $\partial f/\partial t$ measures change in option price as a function of time
Θ = $\partial f/\partial r$ measures change in option price as interest rate changes

Of those measures of sensitivity of the derivative price to various variables, I'd like to focus on Delta and Gamma because I think they have the most impact on the option prices. You would think that the price of the security would have a very direct impact on the price of the derivative. Therefore, you would want to look at Delta.

If you define Delta as the change in the derivative price over the change in the security price, then you can estimate Delta by applying the definition. Basically, this formula is the change in the derivative price divided by the change in the security price. If you look at the definition of Gamma, which is the change in the Delta over the change in the security price, it is simply a second derivative. So you can use the same formula but apply it to the Delta factor rather the derivative price factor. You can go one step further and arrive at a formula for measuring Gamma based on the derivative price and the stock price.

Calculation and Uses of Delta and Gamma

$$\Delta \approx (f^{+} - F^{-}) / (S^{+} - S^{-})$$
$$\Gamma \approx (\Delta^{+} - \Delta^{-}) / (S^{+} - S^{-}) = (f^{+} - 2f^{0} + f^{-}) / (0.5*(S^{+} - S^{-}))^{2}$$

What can the Gamma and the Delta be used for? It can be used for estimating the option price given a change in the security price, or it can be used to hedge a portfolio. I will go into how we hedge a portfolio using Delta and Gamma in more detail, but first let me demonstrate how we estimate the option price from the security price given Delta and Gamma.

First, let's say that we want to estimate the option price of that look-back call that we calculated. In order to estimate the Delta, we simply pick two starting stock prices. For two different starting stock prices, we calculate two different option prices, and then we can calculate the ratio. We can calculate Delta by picking a stock price at $105 and another one at $95; then we can simply calculate the ratio of the change in the derivative price versus the change in stock price. In this case, the ratio came out to be 0.88. What does that mean? That means that if the stock price changed by $1.00, then the option price changed by 88 cents. If you use Delta to estimate the change in the option price, then you can basically apply Taylor approximation to estimate the option price. The option price can be derived given the option price at another point, given the Delta and the change in the stock price. Or, if you want to also use Gamma, then you would add another factor for Gamma which is one-half of gamma times the change in the stock price squared.

If you look at this formula, you should see a striking similarity to the formula that you use for fixed-income securities. If you know duration and convexity of d bond, then given a change in interest rates, you can estimate the price of that bond. You will be able to do the same given the Delta and Gamma of the option. As shown below, I went through the exercise of estimating the price of the option given the Delta which is 0.88 and the Gamma which is 0.06 of the option given a change in the stock price. If I verify them using the binomial method, then I see a slight difference that's likely because of the number of steps in the binomial method, which is only three steps, in this case. So the result is not precise enough.

Hence, Delta of the look-back call is:

$$\Delta \approx (11.715 - 2.913) / (105 - 95) = 0.88$$

Gamma of the call:

$$\Gamma \approx (11.715 - 2*6.454 + 2.913) / 25 = 0.0688$$

Delta and Gamma can be used to estimate option price given change in stock price as follow:

$$f' = f_0 + \Delta (S' - S_0) + 1/2\, \Gamma (S' - S_0)^2$$

Estimate the price of the above look-back call if the stock price is 102:

$$f' = 6.454 + 0.88 (102 - 100) + 1/2 * 0.0688 * (102 - 100)^2$$
$$= 8.352$$
$$f(102) = 8.559 \text{ using Binomial method}$$

Hence the concept of Delta and Gamma to option pricing is similar to that of duration and convexity to fixed-income pricing.

Given that we can estimate the change in the option price knowing Delta, we want to go one step further and see if we can hedge this using Delta and Gamma. What is Delta hedging? Delta hedging is a process where we replicate the option value by maintaining a position in the underlying security itself rather than in the option. In other words, if I have an option, I can replicate the value of that option by taking a position in the stock.

For example, let's assume you sold a look-back European call when the stock price was too low (as in Table 3) and collect $6.40. Now you have to hedge that call which ought to be 6.4. If the stock price moved to a $101.00, then the option price, by definition, would have to move by $6.40, plus the Delta, so it would be something like $7.30. Let's say you hold $6.40 in cash rather than buy the option, and you borrow $88.00 to buy 0.88 shares of stock. If the stock appreciates by $1.00, you will have 88 cents worth of gain, which is equal to the change in the option price. In effect, you just Delta hedged.

Delta hedging is in the process of replicating an option portfolio by maintaining a position in the underlying securities.

- Assume we hold a portfolio of one look-back call in Table 2
- If stock price is $100 a share, then the value of the portfolio is $6.454
- If stock price is $101 a share, then the value of the portfolio is approximately $6.454 + 0.88 = $7.355
- Alternatively, we can hold $6,454 in cash and borrow $88 to buy 0.88 share of stock
- If stock price is $100 a share, then the value of the portfolio is $6.454 + 0.88 * 100 – 88 = $6.454
- If stock price is $101 a share, then the value of the portfolio is $6.454 + 0.88 * 101 – 88 = $7.335
- We just replicate the option portfolio using cash and borrowed stock

- We can engage in stock borrowing by entering into a long stock index futures contract or a equity swap contract
- Most options associated with equity-indexed products have payoffs dependent on stock market indices, most notably the S&P 500 index
- Hence, Delta hedging for such options involves entering into S&P 500 futures contract.

Instead of borrowing individual stocks, you have long S&P 500 stock-index futures because most of the options that you would encounter in equity-indexed products are options on the S&P 500 index. We just went through an example of demonstrating how we can replicate or reproduce the option by simply holding cash and taking a position in the stock.

Now, what are the advantages of Delta hedging? One of the advantages is that since you can calculate Delta for any type of option, even if you hold many option types and contracts, you can combine the Deltas and hold a single position in the underlying stock. For example, you have one look-back call, a 10-year put, a 10-year call and five Asian-calls. You can come up with the Delta for each of them, add them together, and come up with a single Delta measure. Then you can take one single position in the underlying stock and replicate four different types of options.

The second advantage of Delta hedging is that the actual cost of hedging over time equals the actual volatility of the stock market, not the implied volatility of the option market.

If you sell equity-indexed products and go to Union Bank of Switzerland (UBS) or you go to Lehman Brothers and buy options, they're going to charge you something in the neighborhood of, say, 25% in terms of implied volatility. If you observe the market, you know that the market is roughly 18% volatile, so they make money on that. If you're going to hedge using futures, then the costs you incur will be based on that 18% volatility that you actually observe on the market rather than the 21% or 25% that they charge. I don't mean to bash the investment bankers, but that's actually what happens.

The other advantage is that the gain and loss on the Delta hedge are marked to market on a daily basis, because you're actually holding futures. Therefore, you're not exposed to the credit risk of the counterparty. In other words, if you Delta hedge using futures and if Lehman Brothers goes under, you would not go under with it.

If that's all there is to it, then it seems like Delta hedging is the greatest thing since sliced bread, right? There's a catch. If you are going to play with futures, you need to have very strict controls in place because you can bring down the house if you take a position in futures. I think that a couple of years ago there was a guy in Singapore who brought down Barings Bank because he took a position in the futures market. Thus, you need to have a very strict controls procedure in place.

Second, as Alan alluded to earlier, if you are going to hold cash and long-stock index futures in the amount of the Delta, then as the Delta changes, you have to adjust your position with the Delta. Sometimes you don't need to adjust very much, but at other times, especially when the market is extremely volatile, such as one we've observed over the last couple months, you have to constantly adjust the Delta position. Doing this takes a lot of time, effort, and money, and you need to have a strategy in place in order to adjust for your Delta. If you do so too frequently, then you would incur transaction costs, which turn out to be completely needless.

You will be exposed to the difference between the spot and the future price. Normally, the future prices and the spot prices are traded at a very fixed relationship. Basically, the future price is the spot price of the index, plus a cost of carry. But when the market is volatile, for example, in the day of the crash of 1987, the future price and the spot price practically had no relationship. If you were to trade or rebalance your position on that day, you are exposed to that differential in the spot and the futures price.

Another disadvantage of the Delta hedging is that you are exposed to the change in volatility and interest rates. If you buy the option from the bankers, then, basically, the bankers take on that risk. If you're going to hedge your option and allocate enough money to cover, say, 18% volatility, and the volatility shoots up to 22%, then you're in a deficit and you're exposed to that risk.

The futures contracts are traded at a multiple of the S&P 500. For example, I think it's currently trading at 250 times the index, which means that the minimum size of the contract has to be around $250,000. There might be some tracking error there. Because you can only trade in whole future contracts, your position has to move in $250,000 increments. Therefore, you would not be able to completely hedge your position if you were to Delta hedge using cash and positions in stock index futures.