Market Consistent Valuation and Funding of Cash Balance Pensions

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Executive Summary

Introduction

Cash Balance (CB) pension plans became popular in the late 1990s, when a large number of final salary Defined Benefit (DB) pension plans converted to cash balance plans, taking advantage of the opportunity to transition to a plan that, apparently, mirrored a Defined Contribution (DC) arrangement, whilst staying in the defined benefit category for regulatory purposes. Plan sponsors expected the CB plans to offer the same stable, predictable contribution rates as a DC plan, with the potential advantage of being cheaper than the equivalent DC plan. In the words of the consulting firm Kwasha Lipton (Gold, 2001), "A 5% of pay plan might only cost 4% of pay".

In this paper we have analyzed the CB pension and contributions using the methods and principles of financial economics, to give a market consistent evaluation of the costs of the CB liability. The market consistent valuation of the liabilities can be compared with a market valuation of the assets to give an objective assessment of the surplus or deficit of the plan. It gives a liability value that is appropriate for market consistent financial reporting, with objective assumptions that are derived from the markets, without requiring any subjective assessment of future interest rates or salary growth. We have ignored exits other than retirement, and we assume the retirement date is fixed and known. It is straightforward to adapt the results for demographic uncertainty.

Our analysis demonstrates that the early promises of stable contributions and discounted benefits (compared with a similar DC plan) are not supported when the benefits are analysed using the tools of modern financial theory.

Market Valuation of Projected Accrued Benefit

In Sections 2 and 3 of the paper, we develop valuation formulae for a cash balance benefit for a hypothetical participant. We define the accrued benefit to be the projected value at retirement of all past contributions, allowing fully for projected future interest credits, but with no allowance for future compensation credits, as at the valuation date.

Market valuation of financial liabilities can be determined by finding (or constructing) market instruments which replicate the liability. For the CB liability, this method can be applied in the case where the crediting interest rate is a fixed rate, such as 5% per year. In this case, if the participant fund at the valuation date is F, and the retirement date is assumed to be in, say, 20 years, then the projected benefit at retirement is $B = F(1.05)^{20}$. The market value of this is found by considering the price of a zero-coupon bond with face value B, which matures in 20 years. Using the US government bond yield curve as at 1 April 2013, we find the market value of the CB liability is $1.59F$. That is, the market value of the participant's fund is \$1.59 for every \$1 in the notional fund, and, further, the market value of a contribution rate of, say, 6% of salary is $0.06 \times 1.59 = 9.5\%$.

For crediting rates based on market interest rates, such as the yield to maturity on a 30-year government bond, we must select a model for future interest rates that is risk neutral, and that matches the market yield curve at the valuation date. In our analysis in Section 3, we have used the Hull-White model, matched to US interest rates at 1 April 2013. We use the Hull-White model both to project the ultimate CB benefit, and to discount the benefit. The market consistent valuation is the expected discounted value of the projected benefit. Note that, even though we value the liabilities using expected present values, the underlying theory is not really probabilistic, it is market-based.

Using a few simplifying assumptions, we derive an analytic form for the valuation formula for a CB benefit. We also show that the results are reasonably robust to loosening the assumptions, and available with far less complex computation. The important simplifying assumptions are

- The crediting rate is based on spot rates (in practice par-yields would be used)
- Crediting is continuous (in practice, typically crediting is annual or semi-annual)
- Interest rates follow the single factor Hull-White model.

The key valuation equations are reproduced here. The valuation factor, $V(0,T)$ is defined as the market value of the CB liability per \$1 of participant account at the valuation date (time 0), for a participant who will retire in T years. In equation (6) we show the valuation factor in terms of the two random process, $r^c(t)$ is the crediting rate assumed (loosely) to apply in the infinitesimal interval $(t, t+dt)$; $r(t)$ is the risk free rate of interest assumed (again, loosely) to apply in the infinitesimal interval $(t, t+dt)$;

$$V(0,T) = E^Q\left[e^{\int_0^T (r^c(t)-r(t))dt}\right].$$

Note that $r^c(t)$ and $r(t)$ are correlated stochastic processes, in general.

In the case where the crediting rate is fixed, at r^c, say, then this equation simplifies to

$$V(0,T) = e^{Tr^c}\, E^Q \left[e^{-\int_0^T r(t)\,dt} \right] = e^{Tr^c}\, p(0,T)$$

where $p(0,T)$ is the price at the valuation date of a T-year zero coupon bond with \$1 face value. This is the result used above for the fixed 5% example.

In the case where the crediting rate is equal to the instantaneous rate plus a fixed margin, m, so that $r^c(t) = r(t) + m$, the valuation formula simplifies to

$$V(0,T) = E^Q \left[e^{\int_0^T (r(t)+m-r(t))dt} \right] = e^{mT}$$

In the most common case, where the crediting rate is based on, say, a $k-$year market rate, we use the properties of the interest rate model, with parameters a and σ, to derive the valuation formula

$$V(0,T) = \exp(mT)\, \exp\left(\int_0^T -\frac{A(t,t+k)}{k}\, dt \right)\, E_0^Q \left[\exp\left\{ -\int_0^T \gamma\, r(t)\, dt \right\} \right]$$

where $A(t,t+k)$ is a deterministic function involving the interest rate parameters (a and σ) and the spot rates from the initial yield curve. It can be integrated numerically. The term $E_0^Q \left[\exp\left\{ -\int_0^T \gamma\, r(t)\, dt \right\} \right]$ is similar to the expression for valuing a zero coupon bond under the interest rate model, but with the extra γ term (a function of k and a). We have derived a closed form expression for the expectation (details in Appendix B) enabling us to calculate valuation factors efficiently, with significant speed advantages over simulation.

In Table 1, which is excerpted here for convenience, we show the resulting valuation factors for a range of crediting interest rates, and assuming the time to retirement is 5, 10 or 20 years.

Crediting Rate	Time T to exit		
(Spot Rates)	5-Yrs	10-Yrs	20-Yrs
30-yr rate	1.168	1.235	1.380
5-yr rate +0.25%	1.073	1.091	1.177
1-yr rate +1%	1.062	1.120	1.250
5% fixed rate	1.229	1.340	1.562

Liability per \$1 of account balance;
Excerpt from Table 1

The table shows that all the valuation factors are above 1.0, and in some cases well above. This indicates that the market value of the projected accrued liabilities of the CB plan are substantially greater than the aggregate of the participant accounts.

In Figure 1 we show how these values have changed over the past 16 years, based on the changing yield curves.

Funding Methods

The projected accrued benefit which we value in Sections 2 and 3 is not the only way to specify the CB liability. In Section 4 we compare three different specifications of the accrued benefit. The first is the projected accrued benefit; the second is an unprojected accrued benefit, where the liability is defined to be the participant account at the valuation date, with no allowance for future interest credits. The third approach is an approximation to a method in current use, where the full benefit is projected to retirement, including future compensation and interest credits. This benefit is assumed to accrue linearly, so that the accrued benefit is the estimated final benefit, multiplied by the ratio of past to projected total service.

Using three sample participants, with short, medium and long service, we illustrate how the valuation factors and contribution rates are impacted by the participant's salary, notional fund and time to retirement. The simple examples illustrate that it is not possible to have both stable contribution rates and low valuation factors.

Conclusions

The market risk associated with a CB plan is considerable, especially where crediting rates are based on long bond rates, or are fixed. We have shown that using stable crediting rates does not give stable valuation factors, because the valuation depends on the relationship between crediting and short rates. From Figure 1 we see that the Normal Contribution for a 30-year crediting rate, 20-year horizon increased by around 33% between 2000 and 2010. For a 10-year horizon, the cost increased by around 15% over the same period.

It is possible to achieve stable costs by using shorter rates for crediting. In Figures 1 to 3, we show that the valuation factors based on 1-year and 6-month treasury rates are almost flat over the 16 years covered. However, the additional margin makes these rates quite costly. With a 20-year horizon to retirement, and with a crediting rate equal to the 1-year government rate plus a margin of 100 basis points, every $1 of notional contribution is valued at $1.25.

The analysis in this paper indicates that some current valuation approaches (such as Method 3 in Section 4) which generate valuation factors less than $1 of liability value per $1 of participant account are particularly problematic. The implication of these methods is that an apparently fully-funded plan could be reduced to a significantly underfunded plan following a number of withdrawals, for example on restructuring of the firm. This situation is unlikely to arise in a fully-funded traditional DB plan, where typically, the value of a participant's withdrawal benefit is less than the accrued value of their continuing benefits, so that withdrawals do not inherently create deficits. In a CB plan, allowing valuation factors of less than 1.0 leads to problems of solvency and security that may be hidden behind the valuation method. It also generates unnecessary inequity between (vested) leavers and stayers. Leavers receive 100% of account value, but reduce the security of stayers who may not, ultimately, receive the full account value if the sponsor is unable to meet the shortfall – a shortfall that is not apparent from the valuation.

Contents

1 Introduction

Cash Balance (CB) pension plans play a significant role in US employer-sponsored pension provision. According to the US Department of Labor Employee Benefits Security Administration (EBSA, 2012), in 2010 there were over 12 million participants in Cash Balance pension plans, representing around 9% of the total number of participants in any employer-sponsored plan. The Cash Balance design started to become popular in the US during the late 1990s, when several large employers transferred their final salary Defined Benefit (DB) plans to Cash Balance plans.

From the early promotional literature on CB plans (for example, Kwasha Lipton, 1985), it is clear that the initial attraction of CB plans for employers was the apparent similarity to Defined Contribution plans (DC). Both ostensibly involve the payment of employer/employee contributions into a fund for each member; the fund accumulates over time to create a lump sum benefit at the member's retirement. The benefit may be annuitized, but the rates for conversion are determined at retirement. However, it is now recognized that there are important differences between the CB and DC plans. The first and most critical is that DC contributions are directly invested in assets, and the interest on the assets is passed back to the employee. The DC fund is a medium for investment, similar to a mutual fund. The participant's assets are identifiable within the overall fund. The total of the fund is equal to the total of the participants' account values. In the CB plan, the contributions (referred to as compensation credits) are notional; the amounts paid into the pension plan asset portfolio need not match the notional contribution rate. The interest rate credited to the (notional) funds is fixed in advance, either in absolute terms, or in terms of a market rate applying at each crediting date. So, for example, the employer may specify that all member accounts will earn 5% per year until retirement, or that the interest rate applied at each year end will be the published yield on 5-year government bonds, with an additional margin of, say, 0.25%. The interest credits are notional, in the sense that the actual yield on the plan assets may be greater or less than the specified crediting rate. Hence, the total pension plan assets may be more than or less than the total of the participants' notional funds. The assets are not allocated to individual participants.

This specification of the CB pension results in a benefit that is classified under US regulation as a Defined Benefit plan, not a Defined Contribution plan. This regulatory situation has been a motivating factor in some transitions from traditional DB to CB, as the requirements for changing from one type of DB plan to another are much less onerous, than switching from DB to DC, with, apparently, similar benefits to the employer. Because CB plans combine features of DB and DC benefit design, they are often referred to as hybrid plans, in the same category as floor-offset plans, which feature a guaranteed minimum benefit within a DC plan.

Much has been written about CB plans, in the financial and accounting literature, and in actuarial notes and monographs. In the 1990s and early 2000s, the literature focused on the impact on employers and plan members of the decision to move from traditional DB to CB. Kopp and Sher (1998) compared the benefits from traditional DB and CB plans, and concluded that CB was more favorable to younger participants with shorter service, while traditional DB provided better benefits for the older members. Clark and Munzenmaier (2001) subsequently reported on a number of case studies of early traditional DB to CB transitions. Coronado and Copeland (2003), who are very positive about the CB design, reviewed the case for switching from traditional DB to CB or DC, and proposed that the tax regulations would encourage overfunded plans to switch to CB, and underfunded plans to switch to DC. However, there have been CB transitions from underfunded plans, so this does not fully explain the phenomenon. Thomas and Williams (2009) review the costs and benefits to plan members, with a critique of the accounting approach to the pension decision, which, in their view, fails to recognize employees as stakeholders. Clark and Schieber (2000) also review the 'winners and losers' in the transition to CB, addressing some controversial techniques used for transferring benefits.

Gold (2001) considers the CB design from the perspective of maximizing shareholder value. His conclusion, that the optimal design should be based on equity yields (and that the optimal asset allocation should include no equities) is derived from a tax arbitrage between the fund and the sponsor. Gold's paper is not concerned with valuation or risk management, although he does make some pertinent observations on valuation; notably, that a vested compensation credit of 5% of pay cannot cost less than 5% of pay, although it may appear to do so through the filter of a subjective valuation.

Actuarial valuation of CB plans is addressed by McMonagle (2001), Lowman (2001) and Murphy (2001), with different conclusions. Both Murphy and McMonagle comment on the problems arising from applying valuation techniques designed for traditional DB plans to the CB design. Murphy, for example, shows that traditional actuarial methods generate a loss on termination – that is, the account value, which is the termination benefit for a vested participant, is generally greater than the actuarial valuation. This is consistent with our analysis of current valuation techniques in Section 4. Lowman (in his Figure 2.1) demonstrates that the actuarial value of the cash balance benefits would typically be less than the total account balance before retirement, recognizing that this leads to a loss for each vested pre-retirement exit. From the perspective of financial risk management, this situation is not supportable. In Section 5 we summarize the case for the assets to have higher value than the aggregate participant's funds. Brown *et al.* (2001) use a similar approach to ours, coming from the perspective of financial analysis. Their results are less dramatic than ours, as they considered the value based on the economic conditions of the time, and did not look at the sensitivity to time varying factors in the economy, or at funding implications.

Our approach will not take the corporate finance approach of Gold (2001), nor the traditional actuarial approach of McMonagle, Lowman and Murphy. We view the CB benefit as a financial liability of the sponsoring employer. This benefit can be analyzed using the models and paradigms of financial economics and risk management. In this context, the valuation of the liability would be the same whether the plan sponsor retains the assets/liabilities, or whether all payments are transferred to a third party, apart from expenses and any difference in default risk. The result is therefore a market consistent valuation of the CB liability. It is objective, in the sense that the parameters relating to valuation and crediting interest rates are taken from market data, not from subjective judgement. There is judgment involved in the choice of an interest rate model, but as long as each model is fitted to the same market data, different market models produce very similar results. Market consistent valuation is not, currently, consistent with the actuarial valuation techniques in use, particularly for US public sector pension plans, where very high valuation interest rates are commonly used but we contend that it is a better measure of cost and risk than traditional actuarial valuation, and pertinent with respect to developments in market consistent accounting. It is also worth noting that actual buy-out costs for CB liabilities, for example, on transfer to an insurance company, may differ from the valuation in this paper. There are several possible practical reasons why the 'market values' that we calculate are different to a buy-out valuation. For example, as there is not a liquid market in the liabilities, the insurer may increase their profit load. Also, there may be other complications, such as annuitization features, that we have not accounted for. One contribution of this paper is to give a method for insurers to benchmark the pricing of CB buy-outs.

Our approach is similar to Chen and Hardy (2009), where financial valuation principles were applied to the valuation and funding of floor-offset hybrid plans.

2 Framework, Assumptions, Notation

2.1 The CB benefit

We develop valuation formulae, and illustrate the method with numerical results for a simple but representative CB benefit. We value the pension obligation of a hypothetical participant whose benefit is fully vested. We will ignore (for the moment) mortality and other demographic considerations.

To value a contingent payment using a market consistent approach, we evaluate the expected discounted value of the payment, using a market consistent, arbitrage-free model for interest rates. The objective is a valuation that is equal to the cost of replicating the payouts using market instruments. Even though we value the liabilities using expected discounted values, the underlying theory is not really probabilistic, it is market-based. In particular, the sponsor's intention with respect to the investment of the plan assets is not pertinent to a market-based valuation.

We take an accruals approach to the valuation liability, as follows. We define the accrued benefit at t, assuming a retirement date at $T > t$, to be the benefit based on the participant's notional account value at t, with interest at the guaranteed crediting rate up to T. We make no adjustment for future salary increases, as the benefit funded by past contributions is not affected by future salary increases – that will only change the future contributions, which are not yet part of the plan liabilities. In Section 4 we discuss this assumption, and consider alternatives.

Let F_t denote the participant's notional account value at t. The account value is the total of the past contributions, accumulated to t at the appropriate credited interest rates.

The valuation date is $t = 0$, at which time the notional account value is F_0.

Assuming, first, annual interest rate crediting, let $i^c(t)$ denote the credited interest rate declared at t for the year t to $t+1$. The final lump sum benefit for the plan participant, at her retirement date T, say, is

$$F_T = F_0 \prod_{t=0}^{T-1} (1 + i^c(t))$$

Assuming continuous crediting, at a rate at t of $r^c(t)$ per year, compounded continuously, we have

$$F_T = F_0\, e^{\int_0^T r^c(t)dt}\,. \tag{1}$$

The frequency of interest crediting varies between plans, with annual crediting being most common (58% of plans, Hill *et al.* (2010)), and with monthly, quarterly and daily crediting also commonly utilized. It is convenient for our illustrative calculations in the following sections to work with continuous crediting, but it is straightforward to adapt the results for less frequent updating. For the rest of this section, we use continuously compounded rates for crediting and discount, unless specified otherwise.

The specification of the crediting rate varies in practice. According to Hill *et al.* (2010), 7% of CB plans offer a fixed crediting rate, with values of between 3% and 8% per year. More typically, r^c_t is a market dependent random variable. The most common design is to use the quoted par yield on 30-year Treasury Bonds (used by 41% of CB plans, Hill *et al.*, 2010). The next most common is to set $r^c(t)$ at the 1-year treasury bond yield at t, with an additional margin of (typically) 1% (used by 19% of CB plans). Treasury bond yields at other durations, usually with additional margins, account for a further 20% of CB plans. The remainder may use a CPI rate, or an equity-based rate, or a discretionary rate. Most rates are based on the 'safe harbor' list from IRS Report 96-08. In this paper we consider fixed and treasury-bond crediting rates.

2.2 The yield curve

We denote by $r(t)$ the continuously compounded, risk-free short rate of interest at t, (this could be interpreted as the annualized rate on overnight government securities) so $\{r(t)\}_{t>0}$ is a stochastic process. At the valuation date, time $t = 0$ we know $r(0)$, and we also observe a yield curve of spot rates for different durations. Let $r_k(t)$ denote the $k-$year spot rate observed at t (continuously compounded). Then, from the initial yield curve, we observe $r_k(0)$ for different durations k, up to the maximum period in the yield curve, and $r(0) = \lim_{k\to 0} r_k(0)$.

We will also utilize zero coupon bond prices. Let $p(t, T)$ denote the price at t of \$1 due at T. These prices are observable at t, and are used to construct the yield curve. Recall (see, for example, Dickson *et al.* (2013)) that for $t \le T$,

$$p(t, t + k) = e^{-k\, r_k(t)}\,.$$

The zero coupon bond price, $p(t, t+k)$ gives the market consistent discount factor at t for a k-year investment. That is, the present value at t of \$1 due at $t+k$ is $p(t, t+k)$.

We will assume an arbitrage free model for the future term structure of interest rates. For any such model, the expected discounted value at time $t=0$ of a fixed payment due at $t=T$, say, must match the known market value of that payment. That is, from the zero coupon bond rates, at time $t=0$ we observe that the present value of \$1 due at T is $p(0,T)$ by replication – that is, an investment in the zero coupon bond of $\$p(0,T)$ will exactly meet the \$1 due at T. By matching this to the expected discounted value of \$1 due at T, allowing for the unknown path for the future force of interest $r(s)$, we have

$$p(0,T) = E_0^Q\left[e^{-\int_0^T r(s)ds}\right]. \tag{2}$$

We use the Q superscript to indicate that this is a risk neutral valuation, and the 0 subscript to indicate that we are taking values at time 0, given all the information available at that time.

Forward rates at t can be derived from the curve of zero coupon bond prices or the spot rate curve. The continuously compounded instantaneous forward rate[1] contracted at t applying at time $t+k$ is $f(t, t+k)$, where

$$p(t, t+k) = e^{-\int_t^{t+k} f(t,s)\, ds} \tag{3}$$

$$\Rightarrow f(t, t+k) = -\frac{d}{dk}\log p(t, t+k) = -\frac{1}{p(t, t+k)}\frac{d}{dk}p(t, t+k). \tag{4}$$

At any time t, the market yield curve is known, and may be expressed in terms of the zero coupon bond prices, the spot rates or the forward rates. That is, at t, for all $k>0$, we observe any of $p(t, t+k)$, $r_k(t)$ and $f(t, t+k)$, and

$$p(t, t+k) = e^{-k\, r_k(t)} = e^{-\int_0^k f(t,t+u)du} = E_t^Q\left[e^{-\int_t^{t+k} r(s)ds}\right]$$

Note also that $r(t) = f(t,t)$.

2.3 The valuation formula

The value at $t=0$ of the (random) payoff of F_T at T is

$${}_0V = E^Q\left[F_T\, e^{-\int_0^T r(t)dt}\right].$$

[1]This is the ultra-short rate applying at $t+k$.

Both F_T and $r(t)$ are random variables in this expectation, and they are dependent, in general. Assuming continuous interest crediting, using equation (1) we can write the valuation formula as

$$ {}_0V = F_0 E^Q \left[e^{\int_0^T (r^c(t) - r(t))dt} \right] . \tag{5} $$

This demonstrates that the key factor in the cost of the CB benefit is the relationship between the crediting rate and the risk-free discount rate. In the general case we will need a joint model for $r^c(t)$ and $r(t)$ to determine the market consistent benefit value. However, there are two cases where we are able to determine the value of the cash benefit without specifying a model, and we consider these in the first part of Section 3.

In the numerical illustrations in the following sections we will report the valuation factors, which we define as

$$ V(0,T) = E^Q \left[e^{\int_0^T (r^c(t) - r(t))dt} \right] . \tag{6} $$

The valuation factor $V(u, u+T)$ is the market consistent value at u of a CB benefit due at $u+T$, expressed per \$1 in the participant's hypothetical account balance at u.

3 Valuation Results

3.1 Fixed crediting rate

Suppose that $r^c(t)$ is constant – for example, say $r^c(t) = 0.05$ for all t, compounded continuously, and that the retirement date for the participant whose benefit we are valuing is in 20 years. Then $F_{20} = F_0 e^{20\,(0.05)} = 2.7183\,F_0$ is not a random variable, and from equations (2) and (5),

$$ {}_0V = E^Q \left[2.7183\,F_0\, e^{-\int_0^{20} r(t)dt} \right] = 2.7183\,F_0\, p(0,20) . $$

Recall that $p(0,20)$ is the price at the valuation date of a 20-year zero-coupon bond, which is available from published market data. For illustration, suppose that the yield on 20-year zero coupon bonds at the valuation date is, say, 2.5% per year compounded continuously, then $p(0,20) = 0.5985$, so that $V(0,20) = 2.7183 \times 0.5985 = 1.63$, which means that the valuation liability is 163% of the participant's account balance at the valuation date.

3.2 Crediting rate $r(s)$ plus a margin

A second case which allows valuation without specifying a model for future interest rates is where crediting is continuous (daily or weekly, in practical terms), and the crediting rate each day, before adding a fixed margin, is the same short term treasury rate that is used to discount the future cash flow. For example, suppose the crediting rate is

$$r^c(t) = r(t) + m$$

where $r(t)$, as defined above, is the instantaneous rate of interest on government bonds, and m is the pre-specified margin – around 1.75% would be consistent with the range of practice. In this case, the valuation formula (5) becomes

$$\begin{aligned} {}_0V &= E^Q\left[F_0\, e^{\int_0^T r(t)+m-r(t)dt}\right] \\ &= F_0\, e^{mT} \end{aligned}$$

Assuming a 20-year horizon to retirement, and a margin of $m = 1.75\%$ for the crediting rate over the government short-term rate, the valuation factor in this case is $V(0,T) = e^{mT} = 1.41$, which means that the value of the CB benefit in this case would be 141% of the participant's account balance at the valuation date.

In practice, this combination of daily crediting rates, based on daily government rates, does not appear to be a common system for the CB benefit. However, the results will be similar, for example, for quarterly crediting based on 3-month T-Bill rates, which is a combination that is utilized by some plans, with a typical margin of around 1.75%.

3.3 Crediting rate based on k-year spot rates : one-factor Hull-White model

In cases other than the two described above, we must specify a market consistent model of the term structure of interest rates to value the benefit using equation (5). In this section we use the one-factor Vasicek model for interest rates (Vasicek, 1977), as extended by Hull and White (1990). This model is reasonably tractable, and using the Hull-White extension, can be fitted to the term structure of interest rates at the valuation date to ensure that the process for $r(t)$ is consistent with the starting term structure, and therefore with market values. This model has been used in a number of actuarial applications, including those described in Boyle and Hardy (2003), Iyengar and Ma (2009) and Jørgenson and Linneman (2012).

We describe the one factor model briefly here. See, for example, Björk (2009) or Brigo and Mercurio (2006) for more details. We assume the following stochastic process for the future short term rates on government bonds, continuously compounded:

$$dr(t) = (\theta(t) - a\, r(t))\, dt + \sigma\, dW_t$$

where $a > 0$ and $\sigma > 0$ are model parameters, $\theta(t)$ is a deterministic function of t derived from the starting yield curve; W_t is a standard Wiener process.

Under this model the price at t of a zero coupon bond maturing at $t + k$ can be written as

$$p(t, t+k) = E_t^Q \left[e^{-\int_t^{t+k} r(s)\, ds} \right] = e^{A(t,t+k) - B(t,t+k)\, r(t)} \tag{7}$$

where $A(t, T)$ and $B(t, t+k)$ are the following deterministic functions:

$$B(t, t+k) = \frac{1 - e^{-a\,k}}{a}$$
$$A(t, t+k) = \log \frac{p(0, t+k)}{p(0, t)} + f(0, t)\, B(t, t+k) - \frac{\sigma^2}{4a}\, B(t, t+k)^2\, (1 - e^{-2at})$$

The values of $p(0, t)$, $p(0, t+k)$ and $f(0, t)$ are all taken from the yield curve information at $t = 0$. Note that $B(t, t+k)$ does not depend on t.

Equation (7) shows that the log of the zero coupon bond price can be expressed as a linear function of the short term rate. Models with this feature are called affine.

As we have demonstrated in equation (5), the value of the CB benefit depends on the relationship between the crediting interest rate process $r^c(t)$ and the instantaneous market rate process $r(t)$. The most common crediting rate is the par yield on 30-year treasury bonds, with no additional margin. Using stochastic simulation, we can simulate the joint process for $r(t)$ and for the 30-year par-yields, and we demonstrate this in Section 3.4 below. However, if the crediting rate is based on the k-year spot rate, rather than the k-year par yield, we can exploit the affine models to derive analytic results for the benefit valuation. This is valuable as an approximation to the par-yield case, and also as it offers some insight into the valuation results.

The k year spot rate at t, $r_k(t)$, is related to the k-year zero-coupon bond price at t as

$$\begin{aligned} p(t, t+k) &= e^{-k\, r_k(t)} \\ &\Rightarrow e^{A(t,t+k) - B(t,t+k) r(t)} = e^{-k\, r_k(t)} \\ &\Rightarrow r_k(t) = \frac{B(t, t+k)\, r(t) - A(t, t+k)}{k}\,. \end{aligned} \tag{8}$$

Now, suppose that the crediting rate is

$$r^c(t) = r_k(t) + m\,. \tag{9}$$

Then equation (5) gives the valuation formula, per \$1 of participant's fund at $t = 0$,

$$\begin{aligned} V(0,T) &= E_0^Q\left[\exp\left(\int_0^T r_k(t) + m - r(t)\,dt\right)\right] &(10)\\ &= E_0^Q\left[\exp\left(\int_0^T \frac{B(t,t+k)\,r(t) - A(t,t+k)}{k} + m - r(t)\,dt\right)\right] &(11)\end{aligned}$$

The only random terms in the expectation are those involving $r(t)$, so we can express the valuation formula as

$$V(0,T) = \exp(mT)\,\exp\left(\int_0^T -\frac{A(t,t+k)}{k}\,dt\right)\,E_0^Q\left[\exp\left\{-\int_0^T \gamma\,r(t)\,dt\right\}\right] \tag{12}$$

where

$$\gamma = \left(1 - \frac{B(t,t+k)}{k}\right) = 1 - \left(\frac{1-e^{-ak}}{ak}\right) \tag{13}$$

which, conveniently, is not a function of t.

For the second term in equation (12) we need

$$\begin{aligned}\int_0^T \frac{A(t,t+k)}{k}dt = \int_0^T f(0,t)B(t,t+k)\,dt + \int_0^T \log\frac{p(0,t+k)}{p(0,t)}\,dt\\ -\frac{\sigma^2}{4a}\int_0^T B(t,t+k)^2\left(1-e^{-2at}\right)\,dt\,.\end{aligned}$$

Now $B(t,t+k)$ depends on k, but not on t, and

$$\int_0^T f(0,t)\,dt = -\log p(0,T)$$

from equation (3). Also

$$\int_0^T \log\frac{p(0,t+k)}{p(0,t)}\,dt = -\int_0^T \left((t+k)r_{t+k}(0) - tr_t(0)\right)\,dt\,,$$

and this can be evaluated using numerical integration, given the initial spot rate curve. Finally,

$$\frac{\sigma^2}{4a}\int_0^T B(t,t+k)^2\left(1-e^{-2at}\right)\,dt = \frac{\sigma^2}{4a}B(t,t+k)^2\left(T - \left(\frac{1-e^{-2aT}}{2a}\right)\right)\,.$$

The third term in equation (12) is $E_0^Q\left[e^{-\int_0^T \gamma r(t)\,dt}\right]$, which can be determined by adapting the derivation of the bond equation above. The result is

$$E_0^Q\left[e^{-\int_0^T \gamma r(t)\,dt}\right] =$$
$$p(0,T)^\gamma \exp\left(\frac{\sigma^2\gamma}{2a^2}\left(\left(\frac{1-e^{-aT}}{a}\right)(1-2\gamma)+\frac{(1-e^{-aT})^2}{2a}+\frac{\gamma(1-e^{-2aT})}{2a}-T(1-\gamma)\right)\right) \tag{14}$$

See Appendix B for a derivation of this result.

We use this model fitted to the market yield curve on US government bonds as at 1 April 2013. Where necessary, we have interpolated to give spot rates at monthly intervals. We assume the spot rate curve is flat at durations greater than 30 years, which is the maximum recorded duration in the data.

We assume parameters $a = 0.02$ and $\sigma = 0.006$ for the interest rate process. These parameters indicate a long term, unconditional standard deviation for the short rate of 3% per year, and for the 30-year rate of around 2.3% per year. These are a little higher than recent experience, but somewhat smaller than the volatility experienced over the past 30-40 years.

We also consider three cases for the time horizon, $T = 5$ years, $T = 10$ years and $T = 20$ years to retirement. Exits are ignored.

We value the Cash Balance interest rate guarantee using the following crediting rates. The margins (given in basis points (bp) below), are consistent with the safe harbor recommendations in IRS report 96-08.

- A crediting rate equal to the 30-year spot rate: $r^c(t) = r_{30}(t)$.
- A crediting rate equal to the 20-year spot rate: $r^c(t) = r_{20}(t)$.
- A crediting rate equal to the 10-year spot rate: $r^c(t) = r_{10}(t)$.
- A crediting rate equal to the 5-year spot rate plus 25 bp: $r^c(t) = r_5(t) + 0.0025$.
- A crediting rate equal to the 1-year spot rate plus 100 bp: $r^c(t) = r_1(t) + 0.01$.
- A crediting rate equal to the 6-month spot rate plus 150 bp: $r^c(t) = r_{0.5}(t) + 0.015$.
- A fixed crediting rate of 5% per year compounded annually: $r^c(t) = \log(1.05)$.

Crediting Rate (Spot Rates)	Time T to exit 5-Yrs	10-Yrs	20-Yrs
30-yr rate	1.168	1.235	1.380
20-yr rate	1.130	1.189	1.361
10-yr rate	1.095	1.106	1.230
5-yr rate +0.25%	1.073	1.091	1.177
1-yr rate +1%	1.062	1.120	1.250
0.5-yr rate +1.5%	1.083	1.170	1.366
5% fixed rate	1.229	1.340	1.562

Table 1: Valuation factors, $V(0,t)$ per \$1 of account balance at 1 April 2013, using the Hull-White model, $a = 0.02$, $\sigma = 0.006$.

The results are shown in Table 1. The values in the table represent a market-consistent valuation of the interest rate guarantee, per \$1 of each member's account balance, as at 1 April 2013. One interpretation of these figures is that if the sponsor transferred the liability, these values would represent a fair market value for the contract, based on the yield curve at 1 April 2013 (but without any margin for profits or expenses).

Another perspective is to consider the difference between the hypothetical fund account balance, and the value of the CB benefit with the guaranteed interest offered under the CB pension. For example, consider a member with 10-years to termination, and with a crediting rate based on 30-year yields. The valuation factor in Table 1 is \$1.235 per \$1 of fund. That means that if the member and/or her employer contribute, say, \$100 into the hypothetical account at the valuation date, the value of the additional liability is \$123.5. The premium of \$23.5 represents the cost of the interest guarantee.

The table shows that the interest rate guarantee has a significant market value under our valuation assumptions, particularly for members with longer horizons to retirement. We note that the most common crediting rate, the 30-year yield, generates the highest values other than the fixed 5% assumption, for all horizons. We also note that none of the crediting rates or time horizons give a valuation of less than 1.0, meaning that the interest rate guarantee in the CB plan has a positive, and in most cases very significant market value.

We could allow for exits. The common approach would be to apply deterministic exit rates, ignoring any dependency between exits and the future, random interest rates. In this case, the valuation would be a weighted average of the values for different T. For

$T = 0$, that is, for immediate terminations, the liability is \$1 for each \$1 of member account balance. So, even allowing for terminations cannot bring the valuation to or below \$1 per \$1 of fund.

3.4 Crediting rate based on k-year par-yield rates

We have used spot rates in the previous section, because the calculations are then analytic, with no sampling uncertainty. However, CB crediting rates are generally based on par-yields (at least where $k > 1$). We use Monte Carlo simulation to generate the valuation factors using par-yields, with the same model and assumptions as used in the section above. This will give more realistic valuation, and will also demonstrate how accurate the use of spot rates is as a proxy for par-yields.

Under this approach, we simulate the short rate of interest at the end of each month up to the horizon T. Using the simulated short rate at t, we can calculate all the s-year zero coupon bond prices at that time, $p(t, t+s)$ using equation (7). We then solve for the par-yield on a k-year bond, with $\frac{1}{2}$-yearly coupons, denoted $y_k(t)$, say, using the equation of value:

$$1 = \frac{1}{2}\, y_k(t) \left\{ p(t, t+\tfrac{1}{2}) + p(t, t+1) + p(t, t+1\tfrac{1}{2}) + \ldots + p(t, t+k) \right\} + p(t, t+k) \qquad (15)$$

The results are shown in Table 2, using the same parameters as for Table 1. As we might expect, given a rising yield curve, using par-yields generates slightly smaller values (in most cases) than the spot rates, particularly for larger values of k and T. However, the results are quite close, within 1.5% accuracy even for the long rates and long horizons. The standard errors for these estimates are small, at around 1.5×10^{-6}.

In a real-world application of market consistent valuation, the par-yield method could be used, with Monte Carlo simulation, but the number of projections required could be reduced by using the spot rate valuation as a control variate, as the values are very highly correlated. For more information on control variates, see, eg Hardy (2003). In practice, the results are sufficiently close that using the spot rates to approximate the par-yields may be adequate for valuation purposes.

Crediting Rate (Par-Yields)	Time T to exit 5-Yrs	10-Yrs	20-Yrs
30-yr rate	1.151	1.212	1.379
20-yr rate	1.124	1.177	1.354
10-yr rate	1.094	1.108	1.238
5-yr rate +0.25%	1.073	1.095	1.192

Table 2: Valuation factors $V(0,T)$ at 1 April 2013, using the Hull-White model with par-yields; 10,000 simulations

3.5 Impact of the initial yield curve

A feature of arbitrage-free interest rate models, such as the Hull-White model, is that the model is fitted to the initial yield curve at the valuation date, and the shape and level of that curve can have a significant impact on the valuation. We see from Tables 1 and 2 that the cost of the CB pension liability is very significant, in market consistent terms, based on the yield curve at 1 April 2013. It is interesting to repeat the valuation using yield curves from past dates, to evaluate the sensitivity of the results to the interest rate curves, and also to see how the valuation factors have evolved through the period since CB plans first became popular.

We have calculated valuation factors (that is, the market value at various dates per $1 balance in the member's account) applying yield curves as at 1 November for each calendar year from 1998 to 2012, and for 1 April 2013 (these are the numbers from Table 1 above). The parameters for the Hull-White model are unchanged at $a = 0.02$ and $\sigma = 0.006$. The results are summarized in Figures 1, 2 and 3. The rates used are spot rates, but we find very similar results using par-yields. To illustrate the impact of the time horizon, each graph is shown using the same scale.

We note several features from these figures.

First, the valuation factors can be very sensitive to the initial yield curve, but the impact is much more significant for the fixed rate guarantee, for longer horizons, T, and for longer duration of crediting rates k.

Consider Figure 1. It is very noticeable that the volatility for the 5% fixed rate is substantially greater than for any of the market based crediting rates. The curves derived from market rates (that is, excluding the 5% p.y. fixed rate curve) are decreasing in volatility

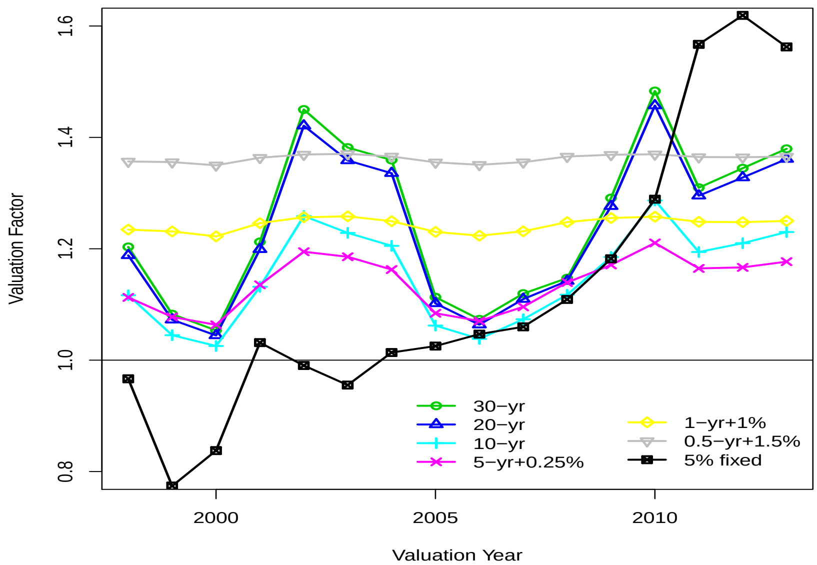

Figure 1: Valuation Factors for $T = 20$-year horizon, as at Nov 1998 to April 2013.

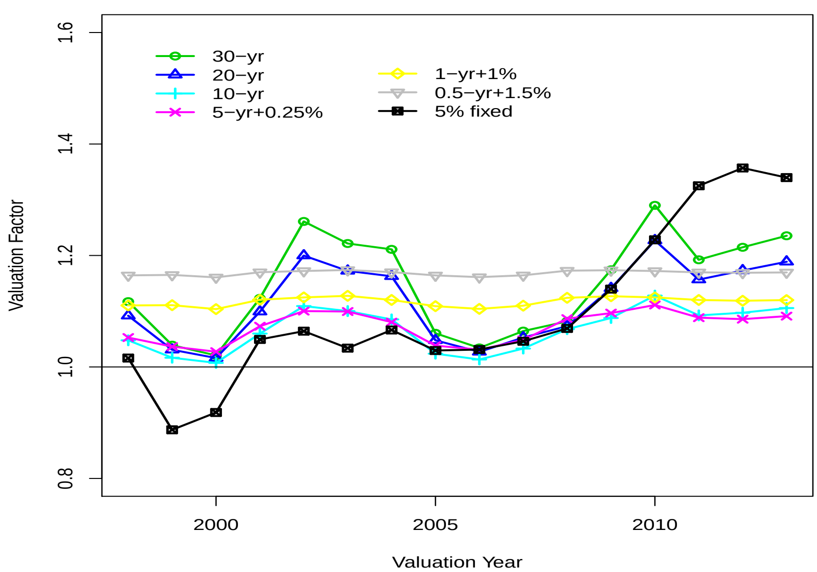

Figure 2: Valuation Factors for $T = 10$-year horizon, as at Nov 1998 to April 2013.

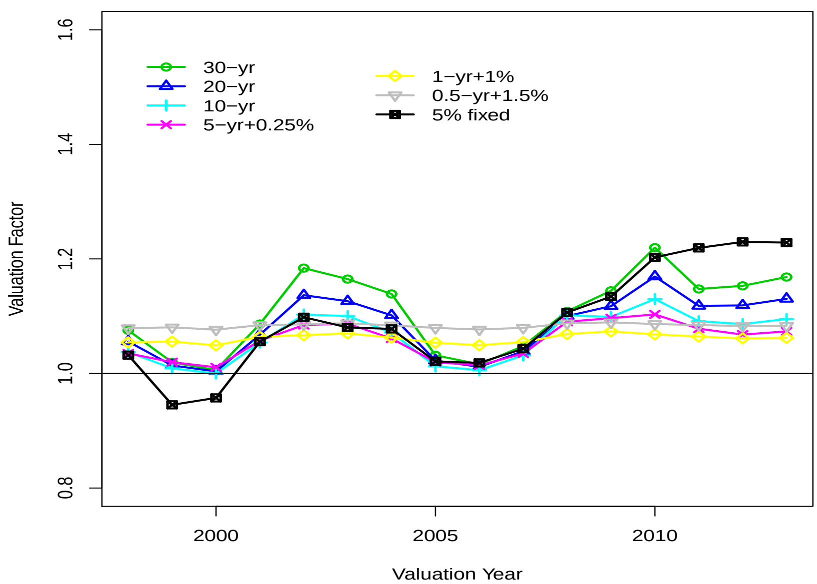

Figure 3: Valuation Factors for $T = 5$-year horizon, as at Nov 1998 to April 2013.

as k decreases, with the highest volatility arising from the 30-year and 20-year crediting rates (which are very close in value) which both have quite volatile valuation factors. At the other end, the valuation factors for the 6-month and 1-year crediting rates are almost flat across the different years. Recall that in Section 3.2 we showed that using a crediting rate equal to the short rate plus a fixed margin gives a fixed valuation, independent of the yield curve or any other market factors at the valuation date. In these graphs we see that the effect of using shorter rates is similar, in that the impact of the different yield curves is small.

Initially, these results are counterintuitive. The most volatile results are generated by the least volatile crediting rates – the 5% rate is fixed; the 30-year rate is much less volatile than the 6-month rate. One might expect the crediting rates that are less variable to generate valuations that are less variable. However, the absolute level of the crediting rate is not the major factor in the valuation; it is the variability in the difference between the crediting rate and the short rate, which generates variability in the valuation results. There is more volatility in the spread between 30-year yields and very short rates (say, overnight yields) than between 6-month yields and very short rates.

The second point is that there is no evidence that the cost of the CB benefit is generally increasing over time. With the exception of the 5% fixed rate, this is not a situation where a benefit which was initially affordable became unaffordable as markets moved. Costs for all the market-related crediting rates were high in several early years, then fell and rose again after 2008. If we look at the 30-year crediting rate, a common design choice, we see that the valuation factors were highest in 2002 and in 2010. In 1999, long term rates were high, with 30-year rates hitting 6.3%, but since short rates were also high, at around 5%, the valuation factors are considerably lower than in 2010, when long rates were around 4%, but short rates were close to 0%.

The fixed rate guarantee is quite different. The valuation in this case depends only on the spot rates corresponding to the T-year horizon. The cost will be low when market rates are high, and when risk free rates are low, as in recent years, the cost of funding a 5% guarantee is high.

We note also that the guaranteed rate is the only case where the valuation factor ever falls below 1.0. In 1999, the yield curve exceeded 5% for all durations, indicating that a 5% guarantee could be funded with an investment of less than \$1 per unit of participants' funds. In all other cases, where rates are related to market (plus margin as appropriate), the valuation factor is greater than 1.0, and often very significantly greater.

3.6 Parameter sensitivity

The Hull-White one-factor model implicitly sets the expected values for future spot rates equal to the time 0 values of the corresponding forward rates. The two parameters of the model, a and σ control the variability around those values, and the speed of the mean regression. In fact, a wide range of values for a and σ give similar results for the valuation factors. The most sensitive factors are for higher values of k and T. To illustrate this, in Table 3, we show the valuation factors for a 20-year term to retirement, using the April 2013 yield curve, for the following sets of a and σ parameters.

A: $a = 0.02$, $\sigma = 0.006$ as used in the previous sections.

B: $a = 0.022$, $\sigma = 0.009$, calibrated to US swap rates at 1 November 2012[2]

[2]Retrieved from http:quant.stackexchange.comquestions4708reasonable-hull-white-parameters, July 2013.

C: $a = 0.05$, $\sigma = 0.007$ calibrated by matching the model standard deviation for $r(t)$ and for $r_{30}(t)$ to the sample standard deviations from the 16 years of yield curve data, used in Section 3.5.

D: $a = 0.1$,$\sigma = 0.01$ from Letian Wang's analysis of LIBOR rates, 2011 (Wang, 2011)

E: $a = 0.15$, $\sigma = 0.029$, from Bolder (2001), using Canadian bond data.

F: $a = 0.15$,$\sigma = 0.01$, from Brown et al (2001).

Recall that the valuation factors for the fixed 5% crediting rates are not model dependent, so here we consider results only for the market based crediting rates.

We see that there is little impact on the valuation factors for crediting rates based on shorter market rates. There is more variability for the longest crediting rates, with highest values coming from the swap rate calibration, and with lower values coming from the values calibrated in the early 2000's. Nevertheless, the values are sufficiently close that the results and conclusions based on the parameter set used in the previous sections may be considered reasonably reliable, even for long term crediting rates and for long retirement horizons.

Crediting Rate	Model Parameters					
(Spot Rates)	**A**	**B**	**C**	**D**	**E**	**F**
30-yr rate	1.380	1.440	1.343	1.328	1.323	1.324
20-yr rate	1.361	1.411	1.336	1.323	1.322	1.318
10-yr rate	1.230	1.258	1.220	1.213	1.221	1.208
5-yr +0.25%	1.177	1.192	1.173	1.170	1.183	1.166
1-yr +1%	1.250	1.253	1.249	1.249	1.256	1.248
0.5-yr +1.5%	1.366	1.367	1.365	1.365	1.368	1.365

Table 3: Valuation factors, $V(0, 20)$ per \$1 of account balance at 1 April 2013, using the Hull-White model various parameter sets.

3.7 Crediting rate based on k-year spot rates : two-factor Hull-White model

The formula (8) above shows that at any time t, under the one-factor model, the k-year spot rate, $r_k(t)$ is a simple linear function of the short rate at that time, $r(t)$. The correlation between these variables is therefore constrained by the model to be 1.0 at all times (because any two random variables X, Y, which are related as $X = a + bY$ for constants a and $b > 0$ have correlation 1.0). Equation (5) illustrates that the relationship between the short rate and the crediting rate is a key factor in the benefit valuation. The spread risk, that is, the risk that the crediting rate and short rate spread changes over time, appears a potentially significant factor. We have therefore extended the model to allow for more stochastic variation in the relationship. To do this, we introduce a second stochastic process, which gives the two-factor version of the Hull-White model. The parameterization we use is the $G2{+}{+}$ model from Brigo and Mercurio (2006), which is equivalent to the model proposed by Hull and White (1994). Again, we describe the model briefly, and then present some formulae and results.

We now assume that the short rate process $r(t)$ combines two correlated stochastic processes, $x(t)$ and $y(t)$ as follows.

$$r(t) = x(t) + y(t) + \varphi(t) \tag{16}$$

$$dx(t) = -a_1 x(t)dt + \sigma_1 dW_1(t), \quad x(0) = 0 \tag{17}$$

$$dy(t) = -a_2 y(t)dt + \sigma_2 dW_2(t), \quad y(0) = 0 \tag{18}$$

where (W_1, W_2) is a two-dimensional Brownian motion with instantaneous correlation ρ.

The parameters $a_1, a_2, \sigma_1, \sigma_2$ are all positive constants, and $-1 \leq \rho \leq 1$. The starting value $r(0)$ is taken from the starting yield curve, and the deterministic function $\varphi(t)$ is determined by fitting the model to the starting yield curve.

Assuming, again, a crediting rate of $r^c(t) = r_k(t) + m$, continuously compounded for T years, the valuation factor per \$1 of fund value at $t = 0$ is

$$V(0,T) = \exp\left\{\int_0^T \frac{-A(t,t+k)}{k} dt\right\} \exp\{Tm\} \exp\{A^*(0,T)\} \tag{19}$$

$$\tag{20}$$

where

$$A(t,t+k) = \log \frac{p(0,t+k)}{p(0,t)} + \frac{1}{2}\left(\nu(k) + \nu(t) - \nu(t+k)\right) \tag{21}$$

$$A^*(t,t+k) = \log \frac{p(0,t+k)}{p(0,t)} + \frac{1}{2}\left(\nu^*(k) + \nu(t) - \nu(t+k)\right) \tag{22}$$

and where ν and ν^* are deterministic functions of the parameters, as follows. Let

$B_k(\alpha) = \frac{1-e^{-\alpha k}}{\alpha}$, then

$$\nu(k) = \frac{\sigma_1^2}{a_1^2}\Big(k - 2B_k(a_1) + B_k(2a_1)\Big) + \frac{\sigma_2^2}{a_2^2}\Big(k - 2B_k(a_2) + B_k(2a_2)\Big) + \frac{2\rho\sigma_1\sigma_2}{a_1 a_2}\Big(k - B_k(a_1) - B_k(a_2) + B_k(a_1+a_2)\Big) \tag{23}$$

and similarly, let $\gamma_j = 1 - B_k(a_j)/k$, for $j = 1, 2$, then

$$\nu^*(k) = \frac{\gamma_1^2\sigma_1^2}{a_1^2}\Big(k - 2B_k(a_1) + B_k(2a_1)\Big) + \frac{\gamma_2^2\sigma_2^2}{a_2^2}\Big(k - 2B_k(a_2) + B_k(2a_2)\Big) + \frac{2\rho\gamma_1\gamma_2\sigma_1\sigma_2}{a_1 a_2}\Big(k - B_k(a_1) - B_k(a_2) + B_k(a_1+a_2)\Big) \tag{24}$$

See the Appendix for the derivation of this formula.

Clearly, this is a more complex model, but the results, though not very elegant, are relatively simple to implement. We use parameters for the $G2$++ model which match the long term standard deviation of the short rate and of the 5-year, 10-year and 30-year spot rates from the one-factor model, with $a = 0.02$ and $\sigma = 0.006$, which are the parameters used for Table 1 and Figures 1, 2 and 3. We also generate a correlation of around 60% between the short rates and the 30-year rates, which is similar to the sample correlation from the past 15 years data. The parameter values are

$$a_1 = 0.055 \quad a_2 = 0.108 \quad \sigma_1 = 0.032 \quad \sigma_2 = 0.044 \quad \rho = -0.9999$$

Note that the σ and ρ parameters do not refer to the Hull-White model parameterization, they are from the $G2$++ version of the model, as described in equations (16)-(18). The Hull-White model parameters can be calculated from the $G2$++ parameters; see Brigo and Mercurio (2006) for details.

The results are shown in Table 4. Comparison with Table 1 shows that the impact of the additional factor is relatively small, at around 1.5% for the longer horizons and longer crediting rates, and rather less for shorter horizons and shorter crediting rates.

Crediting Rate (Spot Rates)	Time T to exit 5-Yrs	10-Yrs	20-Yrs
30-yr rate	1.166	1.237	1.405
20-yr rate	1.128	1.186	1.376
10-yr rate	1.093	1.100	1.225
5-yr rate +0.25%	1.072	1.087	1.169
1-yr rate +1%	1.062	1.119	1.247
0.5-yr rate +1.5%	1.083	1.169	1.364

Table 4: Valuation factors per $1 of account balance at 1 April 2013, using the Hull-White two-factor model.

4 Funding Methods for CB Plans

4.1 Defining the accrued benefit

The cash balance pension design is categorized as a defined benefit plan under US regulation. There is a range of acceptable funding methods for DB pensions, and in this section we will consider alternatives to the definition of the accrued liability which we have used so far in the paper, and derive the associated formulas for the Actuarial Liability, AL, and the Normal Contribution (or Normal Cost), NC, using different approaches to the accrued benefit. The Normal Contribution rate is the ratio of the Normal Contribution to the participant's salary. It is (generally) different from the notional contribution rate, which is the stated percent of salary notionally invested in participant funds.

Accruals-based valuation methods for traditional DB value the liability which relates to past service only. The two most common approaches for traditional final salary related benefits under the accruals method are

- Traditional unit credit (TUC), where the actuarial liability does not allow for future service or for indexation of past service benefits from future salary growth. The Normal Contributions fund the increase in the past service benefits arising from salary growth, and also fund the additional benefit arising from additional service during the contribution period.
- Projected unit credit (PUC), where the future benefit increases from salary growth are included (at least approximately) in the liability valuation. The Normal Con-

tributions fund the additional benefit arising from additional service during the contribution period.

In principle, if experience exactly matches the assumptions, the Normal Contributions together with the liability valuation will exactly meet the future benefit. Additional contributions will be paid in the case that experience is adverse, compared with the assumptions.

In practice, the PUC approach is most common amongst the accruals based methods for traditional final salary pensions. Enderle *et al.* (2006) make an economic case for the TUC approach. Part of their case is based on the fact that the employer that sponsors the pension plan has the ability to control the salary growth which indexes the pensions.

There are different ways to interpret the unit credit principles in the CB context. In this section, we describe the actuarial liability, AL_t, under each of three methods, two which are analogous to the TUC and PUC methods, and one which is, arguably, not an accruals-based method, but which is commonly applied in practice. We also briefly outline the associated Normal Contributions. As the PUC/TUC language is ambiguous here, we will use different descriptors.

Let F_t denote the notional amount of a participant's fund at the valuation date t, excluding any contribution paid at t; S_t denotes the salary at t and c denotes the notional contribution rate into the participant's fund, as a proportion of salary.

Where necessary, we assume a valuation interest rate such that $v_i(k)$ is the valuation discount factor from k back to the valuation date. For traditional valuation this will be v_i^k for a valuation interest rate i, but to compare the three approaches using market consistent assumptions, it is appropriate to discount using the prevailing yield curve. In this case $v_i(k)$ is the price of a k-year unit zero coupon bond, or, equivalently, $v_i(k) = e^{-kr(k)}$, where $r(k)$ is the k-year spot rate at t, continuously compounded.

The Normal Contribution at t is denoted NC_t, and is set to be the contribution required such that $NC_t + AL_t$ accumulates to AL_{t+1} at $t + 1$ under the valuation assumptions. We assume, for simplicity, that Normal Contributions are paid in full at the start of each year, and we ignore exits before T. Then, loosely,

$$AL_t + NC_t = v_i(1)E_t[AL_{t+1}]$$

for each of the methods described below. The expectation will be under Q-measure for market-consistent valuation. For Methods 2 and 3 we assume annual crediting, at rate

$i^c(t)$, in the AL and NC formulas, slightly different to the assumption of Method 1, which is the approach from the previous sections, where we used continuous crediting to derive the analytic valuation formulas.

Method 1. Past service, with credited interest to retirement
This is the liability which has been valued in the previous sections of this paper. Under this approach, the actuarial liability at t is the value of the member's fund at t, with allowance for interest at the specified crediting rate to retirement, discounted back to the valuation date.

If the valuation uses market consistent assumptions and methods, the actuarial liability represents (broadly) the cost at the valuation date of securing the retirement benefit through the capital markets. Using the notation of the previous sections, recalling that $V(t,T)$ is the value at t of the CB benefit due at T, per \$1 of notional fund at t, it follows that the actuarial liability at t for a participant with $T-t$ years to retirement is

$$AL_t = F_t V(t,T)$$

The associated Normal Contribution is the cost of the new notional contribution in the participant's account, which is

$$NC_t = cS_t\, V(t,T).$$

Method 2. Past service, no future credited interest
Under this approach the actuarial liability at t is

$$AL_t = F_t.$$

This is the wind-up liability at t, assuming benefits are vested.

The associated Normal Contribution is in two parts. The first part funds the increase in F_t from the new notional contribution. The second part funds the increase in F_t and the new contribution from the credited interest in the year t to $t+1$

$$NC_t = cS_t + (F_t + cS_t)\big((1+i^c(t))v_i(1) - 1\big)$$

Method 3. Full service, credited interest to retirement, accrued pro-rata to service
Under this approach the full final benefit is projected using a model for future salaries

and a deterministic assumption for future crediting rates. The projected final benefit is assumed to accrue linearly over the period of service of the plan participant.

That is, suppose the participant is projected to retire at T, and has n years service at the valuation date, t. The participant's account at t is F_t. Let $\tilde{i}^c_{t+k}$ denote the assumed crediting rate in year $t+k$. Then the projected final benefit is

$$\tilde{F}(T) = F_t(1+\tilde{i}^c_t)(1+\tilde{i}^c_{t+1})(1+\tilde{i}^c_{t+2})\cdots(1+\tilde{i}^c_{T-1}) \\ + \sum_{k=0}^{T-1} cS_{t+k}(1+\tilde{i}^c_{t+k})(1+\tilde{i}^c_{t+k+1})\cdots(1+\tilde{i}^c_{T-1})$$

The actuarial liability is

$$AL_t = \frac{n}{n+T-t}\,\tilde{F}(T)\,v_i(T-t)$$

and the Normal Contribution is

$$NC_t = \frac{1}{n+T-t}\,\tilde{F}(T)\,v_i(T-t)$$

In the first approach, as explained in previous sections, the AL values the benefit from past contributions as a financial liability with a $T-t$ year horizon, and the NC values the new (notional) contribution, also as a financial liability with a $T-t$ year horizon.

In the second approach, if applied with market rates and models, the AL and NC jointly value the liability under the plan over a 1-year horizon. It may be reasonable to take the second approach, which does not assume existence of the plan and benefits beyond the one-year horizon.

As the first method projects the benefit accumulation to retirement, and the second method only projects to the next valuation date, the first method is analogous to the PUC approach, and the second to the TUC approach, in the traditional DB setting. This is because the credited interest rate in a CB context takes on the role that salary growth has in the traditional DB valuation, in that it acts as the indexation process for the benefit while the participant is an active plan member. However, unlike the traditional DB situation, where the indexation variable is controllable by the sponsor, in the CB plan the indexation variable is exogenous.

The third approach is harder to explain, and requires many more assumptions. It explicitly incorporates future notional contributions, which are dependent on future service, which means that it is not, technically, an accruals based method. It has developed as a somewhat ad hoc adjustment of traditional final salary DB valuation. It is often referred to in practice as a projected unit method, because of the use of projected salaries.

Member	Past Service	Future Service	Salary at Valuation	Fund at Valuation
A	1	19	\$50 000	\$3000
B	10	10	\$60 000	\$55 000
C	19	1	\$75 000	\$100 000

Table 5: Sample plan participant information.

4.2 Examples

We illustrate the three funding methods described, using three sample plan participants. We use some simplifying assumptions for clarity. We assume a notional contribution rate of 6% of salary, and that all participants remain in the plan for 20 years. The crediting rate used is the 30-year spot rate, with no additional margin. Benefits are assumed vested immediately. The pertinent information for the sample plan members is given in Table 5.

We assume, first, a market consistent valuation, using the yield curve at 1 April 2013 (as used in Tables 1, 2 and 3 above). We set $t = 0$ at the valuation date.

Method 1

The Actuarial Liability allows for full credited interest to retirement on past contributions, but with no allowance for future contributions.

Participant A

Actuarial Liability $AL_0 = 3000\,V(0,19) = 3000 \times 1.365 = 4095$

Normal Contribution $NC_0 = 0.06(50000) \times V(0,19) = 3000 \times 1.365 = 4095$

Participant B

Actuarial Liability $AL_0 = 55\,000\,V(0,10) = 55\,000 \times 1.235 = 67925$

Normal Contribution $NC_0 = 0.06(60000) \times V(0,10) = 3600 \times 1.235 = 4446$

Participant C

Actuarial Liability $AL_t = 100\,000\,V(0,1) = 100\,000 \times 1.035 = 103\,500$

Normal Contribution $NC_t = 0.06(75000) \times V(0,1) = 4500 \times 1.035 = 4658$

Method 2

The Actuarial Liability allows for no future credited interest or future contributions. We need the crediting rate and the 1-year discount factor for the Normal Contributions. Using the 1 April 2013 yield curve for US government securities, we have $i^c(0) = 3.62\%$ (the 30-year spot rate) and $v_i(1) = 0.99854$ (the 1-year zero-coupon bond price).

Participant A

$$AL_t = 3000$$
$$NC_t = 3000 + (3000 + 3000)((1.0362)(0.99854) - 1) = 3208$$

Participant B

$$AL_t = 55\,000$$
$$NC_t = 3600 + (55000 + 3600)((1.0362)(0.99854) - 1) = 5633$$

Participant C

$$AL_t = 100\,000$$
$$NC_t = 4500 + (100000 + 4500)((1.0362)(0.99854) - 1) = 8125$$

Method 3

The Actuarial Liability allows for full service and full credited interest to retirement, accrued pro-rata to service. For projecting the final benefit, we assume future crediting rates constant at 3.62% per year, with salary growth assumed constant at 3% per year. The market discount factors are $v_i(1) = 0.99854$, $v_i(10) = 0.82163$ and $v_i(19) = 0.61203$.

Participant A

$$\text{Estimated final benefit } \tilde{F}(19) = (3000)(1.0362)^{19} + 3000\left(\frac{(1.0362)^{19} - (1.03)^{19}}{1 - (1.03)/(1.0362)}\right)$$
$$= 112121$$
$$AL_t = \frac{1}{20}\, 112121\, v_i(19) = 3430$$
$$NC_t = \frac{1}{20}\, 112121\, v_i(19) = 3430$$

Participant B

$$\text{Estimated final benefit } \tilde{F}(10) = 55000(1.0362)^{10} + 3600\left(\frac{(1.0362)^{10} - (1.03)^{10}}{1 - (1.03)/(1.0362)}\right)$$
$$= 128499$$
$$AL_t = \frac{10}{20}\, 128499\, v_i(10) = 52789$$
$$NC_t = \frac{1}{20}\, 128499\, v_i(10) = 5279$$

Participant C

$$\text{Estimated final benefit } \tilde{F}(1) = 100000(1.0362) + 4500(1.0362) = 108283$$
$$AL_t = \frac{19}{20}\, 108283\, v_i(1) = 102\,719$$
$$NC_t = \frac{1}{20}\, 108283\, v_i(1) = 5\,406$$

To assess the differences between these approaches, we consider the valuation factors, defined as the ratio of the actuarial liability, AL_t to the notional participant fund, F_t, immediately before payment of the contribution at t, and also the contribution rates, defined as the ratio of the Normal Contribution, NC_t to the participant's salary, S_t. In Table 6 we show the valuation factors and the Normal Contribution rates for each method, for each participant, at the valuation date.

The valuation factors for Method 1 correspond to the $V(0,T)$ factors calculated in Section 3. We see, as before, that the factors are higher at earlier ages. Intuitively, $V(0,T)$ represents the cost of pre-paying for the difference between the crediting rate and the risk free discounting rate up to time T. As the crediting rate is greater than the short rate most of the time, the cost of this pre-payment is higher for longer periods to retirement. The valuation factors will be greater than 1.00, unless the crediting rate is systematically lower than the risk free rate through the time to retirement. In order to build up the larger fund at early durations, the Normal Contribution rates for Method 1 are also higher for members further from retirement. It would be possible to allow for early exits in the method 1 valuation, where the valuation factors from different exit dates would be weighted by the exit probabilities. For immediate exits, $V(0,0) = 1.0$, so the result would be a weighted average of valuation factors from $V(0,0) = 1.0$ up to $V(0,T_{\max})$.

Participant	Valuation Factors			Normal Cont Rates		
	Method 1	Method 2	Method 3	Method 1	Method 2	Method 3
A	1.365	1.00	1.143	0.082	0.064	0.069
B	1.235	1.00	0.960	0.074	0.094	0.088
C	1.035	1.00	1.083	0.062	0.108	0.072

Table 6: Valuation factors, AL_t/F_t, and Normal Contribution rates, NC_t/S_t for three funding methods, for short, medium and long service participants.

For Method 2 the valuation factors are 1.00 by construction. This method therefore values the termination liability. The Normal Contributions pay for the increase in the fund F_t from the notional 6% of salary contributions, and also funds the cost of applying the crediting rate to the existing fund and the new notional contribution. Clearly, this cost will increase with service, as the fund grows. We see this in the Method 2 contribution rates, which are close to the notional contribution rate of 6% at early durations, but become much greater, at over 10% of salary, closer to retirement. This is analogous to the TUC funding method for traditional DB benefits. It has the advantage that the actuarial liability makes no assumption that the pension plan continues indefinitely into the future, and also gives costs that are low for new participants, reducing the pension costs of recruiting new employees, but if the demographic profile of the plan ages, the increasing contribution rates could be a problem. It is possible that the crediting rate could be less than the valuation rate of interest, that is, when long rates are lower than short rates, for example, in which case the second part of the Normal Contribution formula would be negative, as the balance will be funded from the excess of the one-year rate over the crediting rate. More often, the crediting rate will be greater than the one-year rate, as in the example shown here, in which case the second term can be substantial.

For Method 3 the valuation factors depend on the actuarial assumptions about salaries, and about future crediting rates. The results are difficult to interpret. There is no predictable pattern for longer and shorter durations to retirement. For participants with a long horizon to retirement, the results are very sensitive to the salary and crediting rate assumptions. As the assumptions are subjective, the results are less reliable for comparison of funding security between schemes, or for assessment of risk, for example for PBGC premium calculations. There is no mechanism that ensures that the valuation is greater than the termination liability. In the example, for participant B the valuation factor is 0.960, which means that the actuarial liability is 4% less than the participant's immediate termination benefit. The contribution rates are also difficult to interpret. The Normal Contribution rate is a function of the ratio of the fund value to salary, and of

Participant	Valuation Factors			Normal Cont Rates		
	Method 1	Method 2	Method 3	Method 1	Method 2	Method 3
A	1.135	1.00	0.954	0.068	0.064	0.057
B	1.142	1.00	0.888	0.068	0.094	0.081
C	1.035	1.00	1.027	0.062	0.108	0.072

Table 7: Valuation factors and Normal Contribution rates, after adjustment for AA corporate bond discounting

the assumptions about crediting rates and salary growth, and of the past and potential future service.

In the preceding results we have assumed that the funding valuation uses risk-free rates for discounting, consistent with the financial principles used in Section 3. However, it is more common for pension liabilities to be discounted at corporate bond rates. We can make some simple adjustments to the results for the three methods. Corporate Bonds with a rating of AA or similar tend to have spreads over risk free rates that run from around 25 basis points for short durations, to around 100 bps at longer durations. For Method 2 (Normal Contributions), and for Participant C (all methods), the liabilities are discounted for 1 year only, so the difference between the market valuation and a bond valuation is small. For Methods 1 and 3, the change from market to corporate bond rates would decrease the Actuarial Liability and the Normal Contribution by a factor of around 7.5% for Participant B with 10-years to retirement (assuming 80bp spread) and by around 17% for Participant A, with 19 years to retirement (assuming 100bp spread). The effect on Participant C is less than 1%. The approximate valuation factors using corporate bond discounting are given in Table 7. Using corporate bond rates, we still get valuation factors greater than 1.0 for Method 1, because the crediting rates, on average, exceed the corporate bond rates, based on the model and yield curve at the valuation date. For Method 3, we see that the valuation factors for participants A and B are now significantly below 1.0.

5 Conclusions

The Cash Balance plan was designed to capture the advantages of the DC design, within a DB legal framework. It is likely that many participants believe that the CB and DC plans are essentially the same, or, at most, that the only difference is that in the CB plan the interest rate is specified in advance, without the need for the participant to select asset funds or risk asset values declining. Plan sponsors also assume that CB plans will be similar to DC plans, in particular, in respect to predictable contributions and minimal investment risk. These assumptions are neatly summarized in the explanatory materials provided by Kwasha Lipton, an actuarial firm that was instrumental in promoting CB plans. In Kwasha Lipton (1985) (cited in Gold (2001)), it is claimed that in the CB plan "*The company's contribution is clear-cut and easily understood*", that "*A 5% of pay plan might require a contribution of only 4% of pay, after a realistic investment differential is taken into account",* and that (in Gold's precis) "*tangible, comprehensive benefits mirror Defined Contribution plans*".

In fact, all of these assumptions are wrong, in important ways.

The investment risk associated with a CB plan is considerable, especially where crediting rates are based on long bond rates, or are fixed. We have shown that using stable crediting rates does not give stable valuations, because the valuation depends on the relationship between crediting and short rates. From Figure 1 we see that the Normal Contribution for a 30-year crediting rate, 20-year horizon increased by around 33% between 2000 and 2010. For a 10-year horizon, the cost increased by around 15% over the same period.

It is possible to achieve stable costs, by using shorter rates for crediting. In Figures 1 to 3, we see that the valuation factors based on 1-year and 6-month treasury rates are almost flat over the 16 years covered. However, the additional margin makes these rates quite costly. With a 20-year horizon, using a crediting rate equal to the 1-year government rate plus a margin of 100 basis points, every $1 of notional contribution costs the employer $1.25. If the participants pay a portion of the notional contribution, then there is a leveraging effect on the employer. That is, suppose the notional contribution is 6% of salary, then the required Normal Contribution is 7.5% of salary. Assuming the employees' part is fixed at 3%, the employer must pay 4.5% to cover the balance of cost, considerably more than the notional 3% employer rate.

The statement that the plan might cost less than the notional contribution relies on a valuation which uses a high rate for discounting benefits. It is not true, as we have demonstrated, on a market value basis, nor even using corporate bond rates for discounting, if the accrued benefit is defined as the cost generated by accumulated past contributions. In the 1990's (and later), the belief was that equity investment would inevitably outperform the crediting rate instrument, and the equity premium could be anticipated to reduce costs. More recent understanding of financial economics and financial markets (as well as more recent experience of equity under-performance) has led actuaries and regulators to move away from this 'financial illusion' (Gold, 2001), which claims, in essence, that \$1 of stock has more actuarial value than \$1 of bonds.

From the participant perspective, there is a significant difference between the CB and DC plans, arising from the fact that in the CB plan their fund is only notional. Consistent with the views in Lowman (2001), it appears to be common practice to hold less in assets than the total of the participant funds, usually because of a non-accruals based valuation method (such as Method 3 above), together with high discount rates. This significantly impacts the security of the benefit promise. A DC fund is, by construction, always 100% funded, which means that the participant accounts are equal to the fund assets. A CB plan, using the more arcane actuarial funding approach of Method 3, could be '100% actuarially funded', and yet only hold assets equal to, perhaps 85% of the participants' funds. Even worse, as a DB plan, the CB plan could be underfunded for periods, without significant consequence, resulting in a plan where the assets comprised, perhaps, only 75% of the participants' funds. The justification is that not all participants will exit at the same time – but, in fact, a large proportion may do so, if a substantial part of the business is restructured, for example, resulting in potentially catastrophic underfunding of the remaining participants' benefits. If each exiting participant reduces the funding level of an 'actuarially fully-funded' plan, then either the actuarial liability, or the valuation assumptions, or both, are flawed. Funding less than the participants' accounts leads to inadequate assessment of premiums and risks with respect to the Pension Benefit Guaranty Corporation. It is assumed in the PBGC calculation that a fully funded plan can afford termination benefits for all members (at least, before allowing for termination expenses). Using Method 3 to determine the funding level is inconsistent with this principle. Inglis and Macdonald (2011) propose that the objective of CB pension valuation is to find a method that gives a valuation equal to the total account balance. In fact, Method 2 does this directly, by defining the actuarial liability to be the account balance, but the fact is that Cash Balance plans offer guarantees that can be costly. Using Method 2, the AL does not account for the future interest credits, which means that they must be funded from the Normal Contribution, and, as the examples above demonstrate, that will generally lead to contributions that increase steeply with the participant's service, to values that are significantly above the notional contribution rate. The Normal Contribution rates will also be very sensitive to the emerging crediting rates, and therefore very volatile – exactly the pattern of contributions that the plan sponsors seek to avoid when moving to CB from traditional DB.

The analysis in this paper leads us to the strong conviction that Method 3 should be reviewed as it is not suited to the CB design, in particular with respect to the nature of the withdrawal liability. It should not be acceptable for a fully funded plan to hold less than the aggregate vested participants' account balances. Method 2 may lead to volatile contributions, but it is transparent, and avoids pre-funding many years of interest credit. Method 1 is transparent, and consistent with the principles of projected unit credit funding, and in our view, provides a better approach to valuation and funding, with a long-term perspective. It might be appropriate to adapt Method 1 to be partially-projected, for example by setting a maximum horizon of 5-years. This would determine the capital required to continue crediting the participant's funds for up to 5-years, implicitly acknowledging the sponsor's right to discontinue the plan in the future.

We have not discussed how the plan sponsor can mitigate the investment risks described. For crediting rates based on long bonds, this is a complex question. It is much simpler for short bonds as the employer can match the assets and liabilities by investing in the crediting rate assets. See Inglis and Sparling (2011) for more discussion of this from a consultant perspective. In future work we hope to explore effective risk management strategies for the long crediting rate plans.

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Appendices

A Step through calculation of the valuation factors using spot rates

We illustrate the calculation assuming the retirement horizon is $T = 20$ years, the crediting rate is based on the 5-year spot rate, $r_5(t)$, so $k = 5$, with an additional margin of $m = 0.0025$.

The instantaneous spot rate, $r(t)$ may be interpreted as the very short risk-free rate earned on cash deposits. This is a stochastic process, which we model in this paper using the Hull-White model. The interpretation is that \$1 invested at the risk free short rate from $t = 0$ to T, say, will accumulate to the random amount $e^{\int_0^T r(t)dt}$.

We use the inverse of this, $e^{-\int_0^T r(t)dt}$ as a stochastic discount factor in the following equations.

The Hull-White parameters are $a = 0.02$, $\sigma = 0.006$.

We assume that the yield curve is available in the form of spot rates at (say) monthly intervals, so that we have $r_n(0)$ for $n = 0, \frac{1}{12}, \frac{2}{12}, ...30$. These rates are available from market data providers, but may need interpolation for monthly values. The rates are compounded continuously. From these spot rates we can calculate the price of a $k-$year zero coupon bond, $p(0, k)$, as

$$p(0, k) = e^{-kr_k(0)}.$$

We can also calculate $k-$year forward rates, compounded continuously, fixed at time 0 and applying from t to $t + k$,

$$f(0, t, t + k) = \frac{(t + k)r_{t+k}(0) - tr_t(0)}{k}$$

The valuation formula, from equation (12), is

$$V(0, T) = \exp(mT)\ \exp\left(\int_0^T -\frac{A(t, t + k)}{k}\, dt\right)\ E_0^Q\left[\exp\left\{-\int_0^T \gamma\, r(t)\, dt\right\}\right]$$

where

$$
\begin{aligned}
B(t, t+k) &= \frac{1 - e^{-a\,k}}{a} \\
A(t, t+k) &= \log \frac{p(0, t+k)}{p(0, t)} + f(0, t)\, B(t, t+k) - \frac{\sigma^2}{4a}\, B(t, t+k)^2\, (1 - e^{-2at}) \\
\gamma &= \left(1 - \frac{B(t, t+k)}{k}\right)
\end{aligned}
$$

Consider the three terms of $V(0, T)$, separately, that is, let

$$V(0, T) = C1 \times C2 \times C3$$

where

$$C1 = e^{mT} = e^{0.0025 \times 20} = 1.0513$$

$$C2 = \exp\left(\int_0^T -\frac{A(t, t+k)}{k}\, dt\right),$$

and

$$
\begin{aligned}
&\int_0^T -\frac{A(t, t+k)}{k}\, dt \\
&\quad = \int_0^T -\log \frac{p(0, t+k)}{p(0, t)} - f(0, t)\, B(t, t+k) + \frac{\sigma^2}{4a}\, B(t, t+k)^2\, (1 - e^{-2at}) dt \\
&\quad = \int_0^T -\log \frac{p(0, t+k)}{p(0, t)} dt - \int_0^T f(0, t)\, B(t, t+k) dt + \int_0^T \frac{\sigma^2}{4a}\, B(t, t+k)^2\, (1 - e^{-2at}) dt
\end{aligned}
$$

which we write as

$$D1 + D2 + D3\,.$$

$$
\begin{aligned}
D1 &= \int_0^T -\log \frac{p(0, t+k)}{p(0, t)} dt = \int_0^T -\log \left(e^{(-(t+k) r_{t+k}(0)) - (-t r_t(0))}\right) dt \\
&= \int_0^T \left((t+k) r_{t+k}(0) - t r_t(0)\right) dt\,.
\end{aligned}
$$

The integrand here is k times the continuously compounded k-year forward rate at time 0, from t to $t+k$, which we denote $f(0, t, t+k)$.

We integrate this numerically, using repeated Simpson's rule (see Dickson *et al*, 2013) with monthly steps. Simpson's rule is a numerical integration method similar to the trapezium rule, but instead of fitting a straight line between two consecutive points of a curve, Simpson's rule fits a quadratic through three consecutive points. It is a far more efficient method than the trapezium rule. The repeated Simpson's method applies Simpson's rule to consecutive sets of three points, with the resulting general equation for a function $f(t)$, using n steps of length h,

$$\int_0^{hn} f(t)dt \approx \frac{h}{3}\Big(f(0) + 4f(h) + 2f(2h) + 4f(3h) + \cdots + 4f((n-1)h) + f(nh)\Big)$$

so integrating the forward rates, with $k = 5$, $T = 20$, $h = 1/12$, using repeated Simpson's rule gives

$$\begin{aligned}
&\int_0^{20} k\, f(0,t,t+k)dt = \int_0^{20} 5\, f(0,t,t+5)dt \\
&\approx 5\left(\frac{1/12}{3}\right)\Big(f(0,0,5) + 4f(0,\tfrac{1}{12},5\tfrac{1}{12}) + 2f(0,\tfrac{2}{12},5\tfrac{2}{12}) + 4f(0,\tfrac{3}{12},5\tfrac{3}{12}) + \cdots \\
&\qquad\qquad + 4f(0,19\tfrac{11}{12}) + f(0,20)\Big)
\end{aligned}$$

Based on the spot rates from the April 2013 yield curve, we find

$$D1 = 3.1568\,.$$

Next,

$$D2 = -\int_0^T f(0,t)\, B(t,t+k)dt$$

Now $B(t,t+k)$ does not depend on t, and for $k = 5$

$$B(t,t+5) = \frac{1-e^{-a\,k}}{a} = \frac{1-e^{-(0.02)(5)}}{0.02} = 4.7581\,.$$

Also

$$\exp\left(-\int_0^T f(0,t)dt\right) = p(0,T) \Rightarrow \int_0^T f(0,t)dt = -\log p(0,T) = T\, r_T(0)\,.$$

For $T = 20$ we use the 20-year spot rate from the initial yield curve, $r_{20}(0) = 0.026476$ so we have

$$D2 = -4.7581 \times 20 \times 0.026476 = -2.5195\,.$$

Finally,

$$
\begin{aligned}
D3 &= \frac{\sigma^2}{4a}\int_0^T B(t,t+k)^2\left(1-e^{-2at}\right)\,dt \\
&= \frac{\sigma^2}{4a}\,B(t,t+k)^2\left(T-\left(\frac{1-e^{-2aT}}{2a}\right)\right)\,.
\end{aligned}
$$

Substituting $B(t,t+5)=4.7581$, $T=20$,$a=0.02$, $\sigma=0.006$, we find

$$
D3 = 0.0635\,.
$$

Then

$$
C2 = \exp\left(\int_0^T -\frac{A(t,t+k)}{k}\,dt\right) = \exp\left(\frac{3.1568-2.5195+0.0635}{5}\right) = 1.15044\,.
$$

Next, we use equation (14), which is derived in Appendix B

$$
\begin{aligned}
C3 &= E_0^Q\left[e^{-\int_0^T \gamma\, r(t)\,dt}\right] \\
&= p(0,T)^\gamma\,\exp\left(\frac{\sigma^2\gamma}{2a^2}\left(\left(\frac{1-e^{-aT}}{a}\right)(1-2\gamma)+\frac{(1-e^{-aT})^2}{2a}+\frac{\gamma(1-e^{-2aT})}{2a}-T(1-\gamma)\right)\right)
\end{aligned}
$$

And in the case $k=5$, $T=20$, we have

$$
\begin{aligned}
&\gamma = 1-\tfrac{4.7581}{5} = .04837 \qquad p(0,20) = e^{-20r_{20}(0)} = 0.588888 \\
&\quad \Rightarrow C3 = 0.97310
\end{aligned}
$$

So

$$
V(0,20) = 1.0513\times 1.15044\times 0.97310 = 1.177
$$

which corresponds to the value in Table 1 for $T=20$, $k=5$.

B Hull-White and G2++ Results

B.1 Hull-White Model Derivation

Let $Z = \int_0^T r(t)\,dt$. According to Brigo and Mercurio, (page 75) Z is normally distributed with a mean μ (which will not concern us), and variance:

$$s^2 = \frac{\sigma^2}{a^2}\left(T + \frac{2}{a}e^{-aT} - \frac{1}{2a}e^{-2aT} - \frac{3}{2a}\right)$$

Now, using the notation $M_\lambda(Z) = \exp(\mu\lambda + \frac{\lambda^2\sigma^2}{2})$ for the moment generating function we have:

$$\frac{E_0^Q\left[e^{-\int_0^T \gamma r(t)\,dt}\right]}{p(0,T)^\gamma} = \frac{M_{-\gamma}(Z)}{(M_{-1}(Z))^\gamma} = e^{\gamma(\gamma-1)\frac{s^2}{2}}$$

A straightforward calculation then yields:

$$\begin{aligned} e^{\gamma(\gamma-1)\frac{s^2}{2}} &= \exp\left(\gamma(\gamma-1)\frac{\sigma^2}{2a^2}\left(T - \frac{1-e^{-aT}}{a} - \frac{(1-e^{-aT})^2}{2a}\right)\right) \\ &= \exp\left(\frac{\sigma^2\gamma}{2a^2}\left(\left(\frac{1-e^{-aT}}{a}\right)(1-2\gamma) + \frac{(1-e^{-aT})^2}{2a} + \frac{\gamma(1-e^{-2aT})}{2a} - T(1-\gamma)\right)\right) \end{aligned}$$

That is

$$\begin{aligned} &E_0^Q\left[e^{-\int_0^T \gamma\, r(t)\,dt}\right] \\ &= p(0,T)^\gamma \exp\left(\frac{\sigma^2\gamma}{2a^2}\left(\left(\frac{1-e^{-aT}}{a}\right)(1-2\gamma) + \frac{(1-e^{-aT})^2}{2a} + \frac{\gamma(1-e^{-2aT})}{2a} - T(1-\gamma)\right)\right) \end{aligned}$$

as required.

B.2 G2++ Model Derivation

We have that:

$$V(0,T) = E_0^Q\left[\exp\left(\int_0^T r_c(t) - r(t)\,dt\right)\right] = e^{mT}\left[\exp\left(\int_0^T r_k(t) - r(t)\,dt\right)\right]$$

Now $r_k(t) = k^{-1}\log(P(t,t+k))$, so Brigo and Mercurio, in their equation (4.11) give:

$$r_k(t) = k^{-1}\left(\int_t^{t+k} \varphi(u)\,du + B_k(a_1)x(t) + B_k(a_2)y(t) - \tfrac{1}{2}\nu(k)\right)$$

Substituting this into the expression for $V(0,T)$ above, using the definition of $r(t)$ and some straightforward manipulations then yields:

$$V(0,T) = e^{mT} \exp\left(-\int_0^T \frac{A(t,t+k)}{k}\,dt\right) \exp\left(-\int_0^T \varphi(t),\ dt\right) E_0^Q\left[\exp\left(-\int_0^T \gamma_1 x(t) + \gamma_2 y(t)\,dt\right)\right]$$

Since we are matching the current term structure, (Brigo and Mercurio, page 146, 4.13) gives that

$$\exp\left(-\int_0^T \varphi(t)\,dt\right) = \exp(A^*(0,T))\exp(-\tfrac{1}{2}\nu^*(k))$$

So we are done if we can show that:

$$E_0^Q\left[\exp\left(-\int_0^T \gamma_1 x(t) + \gamma_2 y(t)\,dt\right)\right] = \exp(\tfrac{1}{2}\nu^*(k))$$

Define

$$\tilde{x}(t) = \gamma_1 x(t) \quad \text{and} \quad \tilde{y}(t) = \gamma_2 y(t)$$

so that

$$\begin{aligned} d\tilde{x}(t) &= -a_1\tilde{x}(t)dt + \gamma_1\sigma_1 dW_1(t) \\ d\tilde{y}(t) &= -a_2\tilde{y}(t)dt + \gamma_2\sigma_2 dW_2(t) \\ \tilde{x}(t) &= \tilde{y}(t) = 0. \end{aligned}$$

Then, using Brigo and Mercurio's Lemma 4.21, we have that

$$\tilde{Z} = \int_0^T \tilde{x}(t) + \tilde{y}(t)\,dt \sim \mathrm{N}(0, \nu^*(k))$$

from whence the result follows since the expectation is just $M_{-1}(\tilde{Z})$.