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Report on Extreme Events for Insurers:

Correlation, Models and Mitigations

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Executive Summary

Insurers are exposed to a variety of extreme events that may have a significant impact on their financial condition and solvency position. An extreme event could be a system-wide event such as a credit crisis, an industry-wide event such as a natural disaster, or a company-specific event such as a failure of management. The reliability of risk models is of paramount importance for insurers who operate with a target confidence level of remaining solvent. However, for the assumed confidence level and underlying models to be meaningful, they must contemplate an appropriate range of outcomes, including events that are extreme in their likelihood, impact, and/or duration.

The modeling and mitigation of extreme events is without doubt a challenging task for insurers. The scarcity of experience data, the evolution of historical risks, the emergence of new risks, and the complex dependence among risk drivers are common issues when analyzing extreme events. Traditional statistical techniques are insufficient to address these issues. A more advanced toolkit, including Extreme Value Theory, is required. Leveraging these advanced techniques, the practitioner is better able to extract extreme event behavior from historical experience, including the detection of fat tails, temporal clustering, and periods of high correlation or dependence. By way of root cause analysis, catastrophe modeling, and network modeling, the practitioner is also better able to understand the drivers underlying extreme events and in turn identify possible future stress scenarios.

Equipped with better tools for modeling both historical and potential future extreme events, the practitioner can more accurately determine an insurance company's risk exposure. Risk managers can then design risk mitigation plans and hedging programs to limit or transfer undesired extreme event exposure. Additional benefits of thorough extreme event modeling include the potential for improved management action once the adverse event is underway and enhanced strategic planning since decision makers can proactively assess the financial impact of an extreme event under various business strategies.

This paper is intended to be a primer on extreme event modeling and mitigation for practitioners. It covers the complete cycle of extreme event risk management, from identifying experience data sources, to extreme event detection, to modeling and dependence analysis, and finally to monitoring and mitigation. This paper explores extreme events from the perspective of a global insurance company, with an emphasis on market risk, credit risk, insurance risk, liquidity risk, and business risk. Extreme risk concepts and models are introduced from a practitioner's perspective without unnecessary theoretical details. This report is not intended to be all-encompassing, but rather to focus on the key aspects of extreme risk analysis and its application. The input data and R programs underlying the numerical examples herein are available in the supporting files accompanying this report.

Contents

Please note that a glossary of the key technical terms used in this report is provided in Appendix A. Furthermore, the input data and R programs underlying the numerical examples herein are available in the supporting files accompanying this report.

1. Introduction

Contemporary history demonstrates that risk modeling and mitigation approaches based on normality and traditional statistical relationships fail to adequately capture extreme events. From the 1987 stock market crash, to the 2008 financial crisis, to the 2011 Fukushima Daiichi nuclear disaster, it is evident that extreme events are more common, and therefore of more relevance to insurers, than conventional theory would predict. For example, under the assumption of normality, one would expect to see a 12 percent one-day drop in the U.S. Dow Jones Industrial Average (DJIA), such as the decline observed on October 29, 1929, once only every 10^{30} years (Giesecke and Goldberg 2005). Yet a mere 58 years later, on October 19, 1987, the DJIA again crashed, that time by a staggering 23 percent.

No universal definition exists of what constitutes an extreme event. The concept of "extremeness" is relative and depends on context; what is extreme in one context may not be in another. For example, whereas a global financial crisis would be an extreme event for most, if not all, insurance companies (systemic risk–based extreme event), a major hurricane is less likely to be an extreme event for a life insurer than it is for a property and casualty (P&C) insurer (idiosyncratic risk-based extreme event). Although putting forward a precise definition is difficult, it is helpful to think about extreme events along the dimensions of *Rarity*, *Severity*, and *Rapidity* (Stephenson 2008). *Rarity* refers to the probability of the event; the smaller the probability of the event, the rarer and typically more extreme it is. *Severity* refers to the impact of the event, financial or otherwise; the greater the impact, the more extreme the event. Finally, *Rapidity* refers to the duration of the event; acute events are generally more extreme than chronic events because they provide less opportunity for management action once the event is underway.

Extreme events tend to have a significant impact on our environment, economy, and life. Insurance companies are subject to big losses caused by extreme events and consequently may choose to define an event as extreme if it has a material impact on their financials. For example, events that would result in a 20 percent reduction in available capital or a 50 percent reduction in expected earnings may be deemed extreme and managed accordingly. Key thresholds and criteria should be set in accordance with the company's risk appetite, strategy, and regulatory regime(s).

Recently significant efforts have been made in both industry and academia aimed at modeling extreme events, quantifying their impact, and reflecting them in capital planning. Nowadays, regulators, rating agencies, senior management, and investors are interested in understanding the range of possible outcomes under various stress scenarios and in the preparation of corresponding contingency plans. However, even with this increased focus, various extreme event modeling and mitigation challenges remain for the typical practitioner:

- **Traditional Statistical Techniques are Insufficient**: Traditional tools and techniques based on the assumptions of linearity (i.e., relationships between variables are stable) and normality fail to adequately capture extreme events. A different toolkit, such as Extreme Value Theory, is required.

- **Experience Data Limitations**: By definition, an extreme event is rare, so historical data are unlikely to be available to estimate probabilities with a high level of confidence, if at all. In many cases, assigning a probability to an extreme event will require a great deal of subjectivity and professional judgment, especially if the event in question has not been observed before. These limitations extend to back testing as well, in that if only historical data are relied on, testing a model for a 1-in-200-year event requires a very long history of experience data. For example, there is a probability of $(1 - 0.005)^{200}$ = 36.7 percent that a "1-in-200-year" event will not be observed within a 200-year window, assuming events are identically and independently distributed.
- **Risk Evolution**: Risks can evolve over significant periods of time. Structural changes may make current extreme events quite different from what they were years ago. For example, the frequency and severity of today's economic crises are different from those of the last century, in part because of changes in the leverage ratios of both corporations and consumers. One must always ask whether the past remains a reasonable barometer for the future.
- **Multivariate Dependencies and Temporal Clustering**: Extreme events often involve highly correlated underlying risk drivers across space and time. There is a saying in finance that "in times of stress, all correlations go to one." To appropriately model and mitigate the impact of extreme events, it is important to understand the underlying risk driver dependency structure. Unless this dependency structure is understood, the correlation among risk drivers may be underestimated and the diversification benefit under stress scenarios may be overestimated.

This paper is intended to be a primer on extreme event modeling and mitigation for practitioners and to assist them with addressing the aforementioned challenges. This paper explores extreme events from the perspective of a global insurance company, with an emphasis on market risk, credit risk, insurance risk, liquidity risk, and business risk. Both historical and future extreme events, including interdependencies, are discussed. Based on empirical studies and cause-and-effect relationships, a holistic view of extreme events is presented. This paper also includes a thorough discussion of various methods used to model extreme events and their correlation and dependency. The remainder of this paper is structured as follows:

- Section 2 (Extreme Events) provides a survey of historical extreme events and highlights various potential future extreme events for consideration.
- Section 3 (Individual Tail Risk Modeling) discusses a number of models and approaches for modeling individual risk drivers.
- Section 4 (Correlation and Contagion) discusses modeling approaches for correlation and cause-and-effect relationships between risk drivers.
- Section 5 (Managing Tail Risk) discusses how to assess, monitor, hedge, and mitigate tail risk.
- Section 6 (Examples) illustrates the modeling and management of tail risk using applied examples.
- Section 7 (Conclusion) summarizes the key points of this paper.

2. Extreme Events

This section discusses both historical and potential future extreme events. Extreme events can be identified using various techniques, including statistical analysis of historical data to detect outliers, root cause analysis to isolate and study underlying event drivers, and thought- or model-based experimentation to answer prospective "what-if" questions.

The use of historical data for modeling current and future risk exposures makes the implicit assumption that history will repeat itself. However, some future extreme events are beyond our expectation. Nassim Taleb introduced the famous black swan theory in 2001. Black swan events are unexpected, have extreme impact, and become explainable and predictable only after their occurrence. Although some extreme events are true black swans that will remain beyond human expectation until their occurrence, it is important that insurers explicitly consider both "seen" (historical) and "unseen" (potential future) extreme events in their risk modeling. One can effectively manage only what has been modeled, so the more comprehensive the range of extreme outcomes, the better prepared one can be.

2.1 Historical Extreme Events

Historical extreme events are a good starting point for modeling future extreme events. For risks that are difficult to understand and predict, historical experience may be the only objective information on which we can rely. Root cause analysis and descriptive statistics such as tail quantiles, maxima and minima, conditional tail expectation, and correlation and autocorrelation are useful tools for identifying historical extreme events. The practitioner is advised against using the standard moment statistics (mean, standard deviation, skewness, kurtosis) to analyze historical extreme events because these measures fail to adequately describe their distribution and in some cases may be unreliable (Giesecke and Goldberg 2005). This subsection focuses on the identification of historical extreme events with selected examples; however, it does not provide an exhaustive list of historical extreme events.

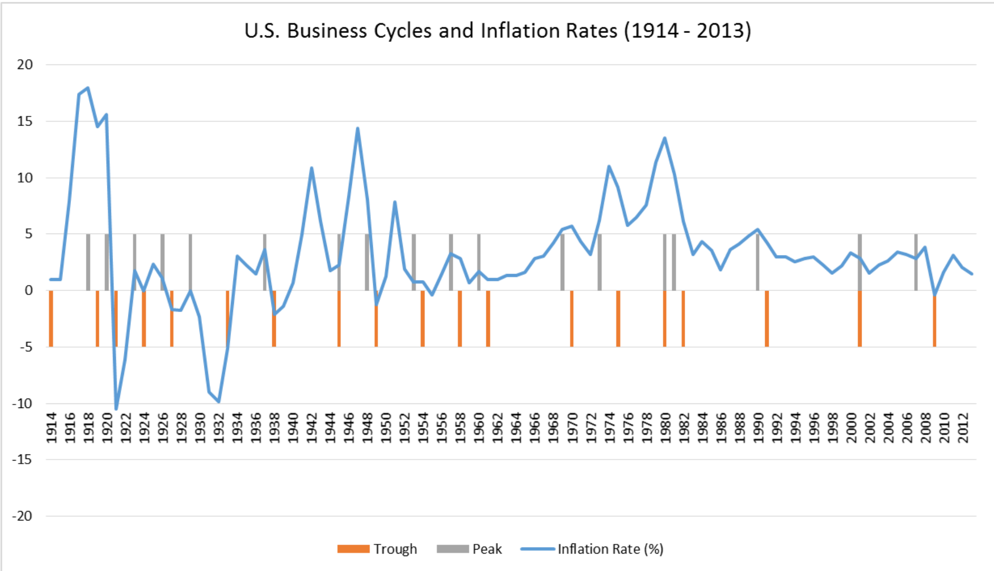
2.1.1 Economic Risk

Economic risk is the risk that the financial outcome of an investment or a financial institution is affected by macroeconomic conditions. It can be further partitioned into three categories: market risk, credit risk, and liquidity risk. Industry-wide extreme events caused by economic risk include extremely high or low inflation, extremely high or low interest rates, equity market crashes, real estate market crashes, high market volatility, rapid or extreme currency appreciation or depreciation, credit crises, credit rating downgrades, and liquidity crises.

Economic risk factors are affected by a common macroeconomic environment. Therefore, cause-and-effect relationships and high correlations, either positive or negative, exist among these risk factors in an extreme situation.

Historical extreme events for economic risk can be identified using business cycles. Most economies exhibit a cyclical pattern of development with a trough followed by a peak and vice versa. Figure 2.1 illustrates U.S. economic cycles and inflation rates since 1914. The U.S. business cycles determined by the Business Cycle Dating Committee of the National Bureau of Economic Research (NBER) are used here. The committee focuses on domestic production and employment as measures of economic activity. Both the magnitude and continuity of an increase or decrease in economic activity are critical for determining a trough or peak. Although the causes of economic recessions range from stock market crashes to structural changes, it is apparent that almost every trough is accompanied by a sharp decrease in the inflation rate. Other economic variables such as interest rate, unemployment rate, equity market value, exchange rate, and leverage ratio are also impacted by business cycles. The dependency is partly caused by practical macroeconomic policies, including monetary policies and fiscal policies. Such policies are often used to reverse the trend of economic development and return the economy to a state of equilibrium.

Figure 2.1 U.S. Business Cycles and Inflation Rates (1914–2013)

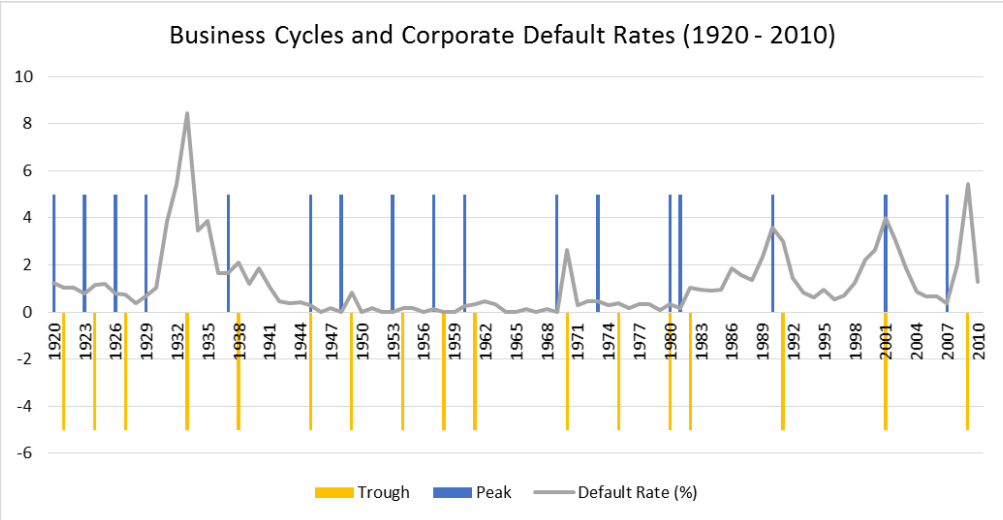


Data sources:

1. Inflation rates: CPI Detailed Report (table 24) of Bureau of Labor Statistics (2014).
2. Business cycles: National Bureau of Economic Research (2010).

In addition to market risk, the systemic portions of credit risk and liquidity risk are highly correlated to business cycle. Using corporate default rates as a combined measure of credit and liquidity risk, Figure 2.2 shows that the default rate is much higher during a recession.

Figure 2.2 Business Cycles and Corporate Default Rates (1920–2010)



Data sources:

1. Corporate Default rates from Moody's (2011). Uses global default experience.
2. Business cycles from National Bureau of Economic Research (2010). Here U.S. business cycles are assumed to represent global cycles.

With increasing globalization, it is useful for the practitioner to analyze extreme economic event experience from around the world, even if one's scope is a single country or economy. Table 2.1 summarizes the most extreme values (maxima and minima) of selected key economic variables from 1960 to 2013 (54 years). Not all countries have a complete record for 54 years. Countries with less than 11 years of data have been excluded. The purpose of this adjustment is to maintain a certain level of volatility in the raw data. The measures covered here are far from complete but help illustrate the volatility among countries.

When constructing stress scenarios for a given economy, the practitioner should analyze available experience from comparable economies. Measures such as *GDP per capita* and the United Nations' *Human Development Index* can assist with identifying countries that are at similar levels of development. For economies that are at different levels of development (e.g., G7 member vs. BRICS member), the practitioner may consider applying experience from the more developed economy to the less developed economy under the assumption that the latter will continue to evolve and converge to the former over a sufficiently long time horizon.

Extreme economic events are not universal and their severity will depend on a company's specific business and assumed risks. Furthermore, not all extreme events for an economy are necessarily extreme events for an insurance company. For example, whereas a 10 percent one-day drop in the S&P 500 would be considered extreme from a macro- (external) viewpoint, an insurer that has hedged against that size of drop would not necessarily view it as extreme from a micro- (internal) standpoint. Each company needs to develop its own view of extreme economic events.

Table 2.1 Explanation of Measures for Extreme Economic Events

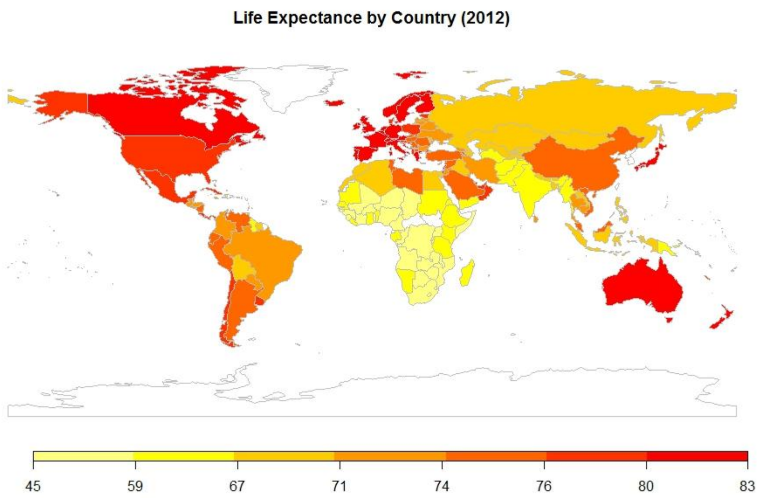
Appendix Reference	*Economic Variable*	*Description*	*Range across Countries*	*Mean and Volatility across Countries*[3]
Figure B.1	Minimum GDP Annual Growth Rate	Key measure of economic activities	[−62%, 3%]	−10.7%, 10.2%
Figure B.2	Minimum Annual Inflation Rate	Derived from Consumer Price Index (CPI)	[−33%, 9%]	−1.2%, 4.9%
Figure B.3	Minimum Real Interest Rate	Lending interest rate adjusted for inflation. It reflects the real economic growth rate.	[−98%, 18%]	−15.4%, 22.3%
Figure B.4	Maximum Lending Rate	The bank rate that usually meets the short- and medium-term financing needs of the private sector.	[7%, 121,906%]	827.5%, 9,462.9%
Figure B.5	Bank Capitalization Ratio	The ratio of bank capital and reserves to total assets. It reflects the stability and leverage ratio of the financial system.	[−8%, 19%]	7.5%, 3.5%
Figure B.6	Maximum Unemployment Rate	Reflects the level of economic development, either expansion or contraction.	[5%, 28%]	11.8%, 5.5%
Figure B.7	Minimum Population Growth Rate	Population change affects economic development, sustainability, and the labor force.	[−11%, 3%]	−0.0%, 1.9%

[3]For reference only; as per Subsection 2.1, mean and volatility are generally unreliable descriptors of extreme value distributions.

2.1.2 Insurance Risk

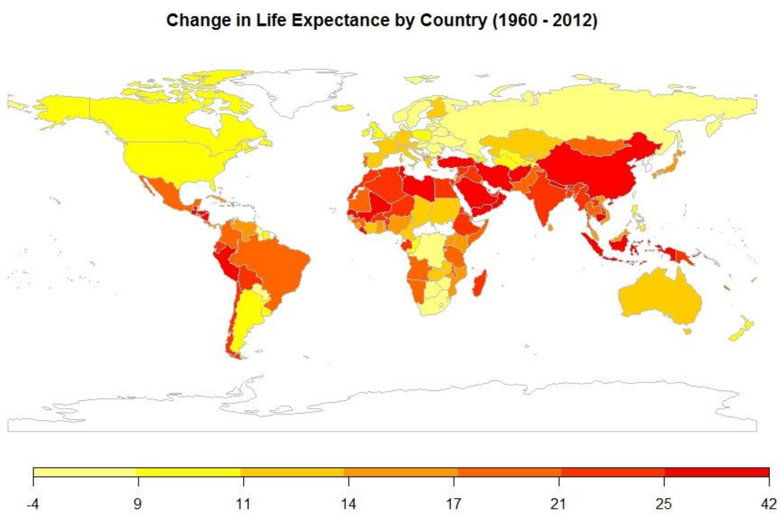
Insurance risk includes mortality risk, longevity risk, morbidity risk, catastrophe risk, terrorism risk, pandemic risk, behavioral risk, and P&C risk. For some risk types, historical experience exists to guide assumption setting. Take longevity risk as an example. Figure 2.3 shows life expectancy in 2012 by country. It is clear that for some low-income countries the opportunity exists for significant improvements in life expectancy, which must be considered by insurers writing life annuity business in those countries. Mortality improvement also varies by country. Figure 2.4 shows the increase in life expectancy from 1960 to 2012; note that changes vary from a decrease of four years to an increase of 42 years!

Figure 2.3 Life Expectancy by Country (2012)



Data source: World Bank.

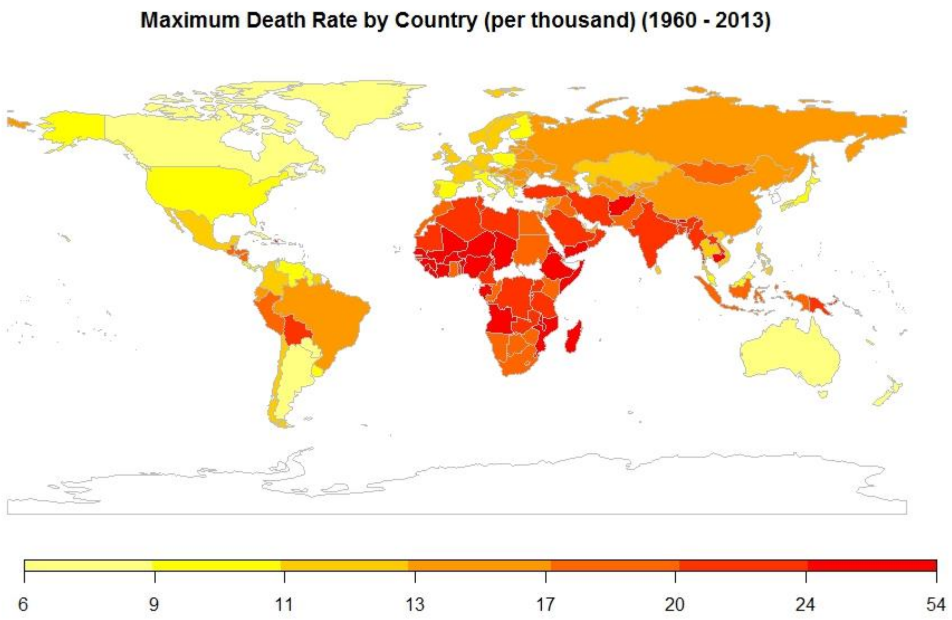
Figure 2.4 Change in Life Expectancy by Country (from 1960 to 2012)



Data source: World Bank.

The maximum death rate also varies significantly by country. The highest death rate in each country can provide insight into the severity of extreme events such as epidemics or pandemics, natural disasters, and war. Figure 2.5 shows the historical maximum death rate by country.

Figure 2.5 Maximum Death Rate by Country (per Thousand) (1960–2013)



Data source: World Bank. Some countries do not have data for all years in the selected data period. Countries with less than 11 years of data have been excluded (appear as transparent) to make sure enough variation is reflected.

Catastrophes such as natural disasters can result in extreme casualty counts and economic loss. Several data sources provide historical data for catastrophes:

1. Wikipedia maintains a list of natural disasters by death toll.[4] Separate lists by type of natural disaster can be found as well.
2. The National Climatic Data Center (NCDC)[5] records all weather disasters that cause an economic loss greater than or equal to $1 million. Table 2.2 lists the top 10 extreme weather events in terms of estimated loss.

[4] http://en.wikipedia.org/wiki/List_of_natural_disasters_by_death_toll.

[5] http://www.ncdc.noaa.gov/billions/events.

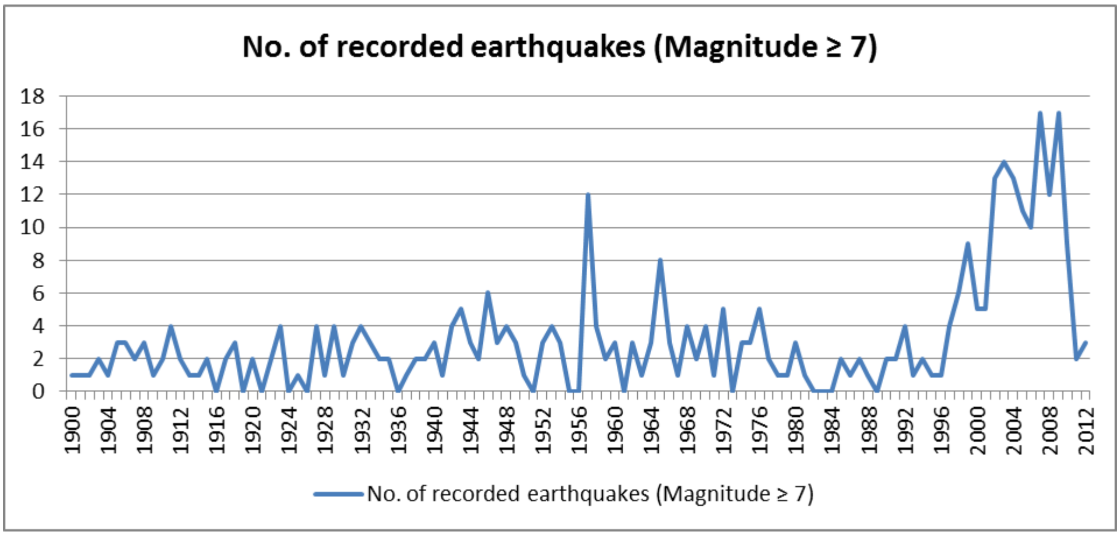
Table 2.2 U.S. Top 10 Weather Disasters (1980–2013)

Event	Time	CPI-Adjusted Estimated Loss ($ Billions; Adjusted to 2013)	Deaths
Hurricane Katrina	Aug. 2005	$149	1,833
Hurricane Sandy	Oct. 2012	$66	159
Hurricane Andrew	Aug. 1992	$45	61
U.S. Drought/Heat Wave	Summer 1988	$39	454
Midwest Flooding	Summer 1993	$34	48
Hurricane Ike	Sept. 2008	$32	112
U.S. Drought/Heat Wave	2012	$30	123
Central/Eastern Drought/Heat Wave	Summer–Fall 1980	$28	1,260
Hurricane Ivan	Sept. 2008	$25	57
Hurricane Wilma	Oct. 2005	$23	35

Data source: National Oceanic and Atmospheric Administration (NOAA) National Climatic Data Center (NCDC).

3. The U.S. Geological Survey (USGS) collects data on earthquakes, volcanos, and landslide hazards. Figures 2.6 and 2.7 illustrate the frequency and associated casualties of historical earthquakes with a magnitude greater than or equal to 7, which is rated as very strong.[6]

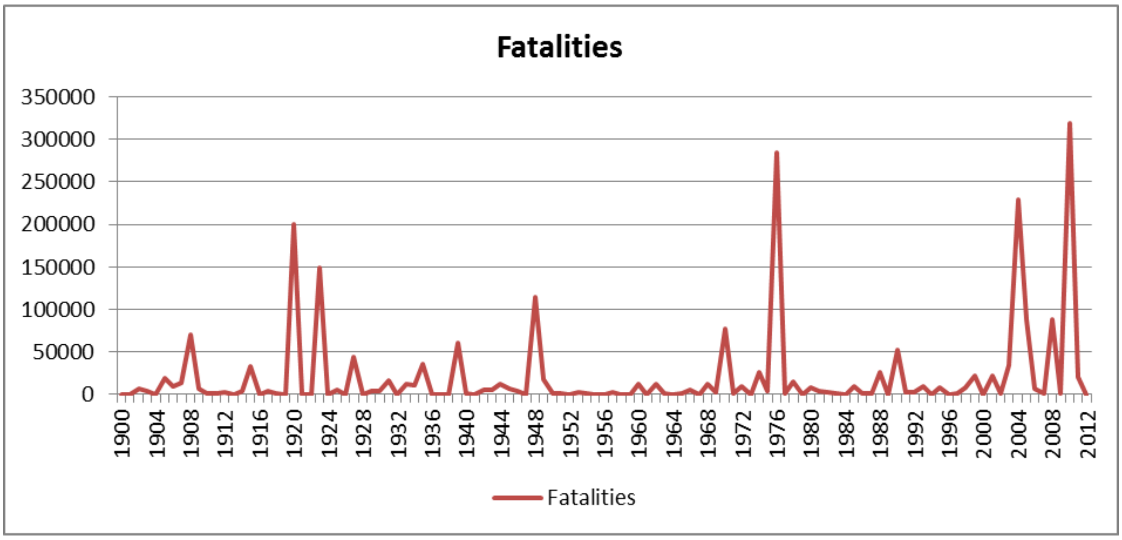
Figure 2.6 Number of Recorded Earthquakes with Magnitude Greater than or Equal to 7 (1900–2012)



Data source: U.S. Geological Survey (http://earthquake.usgs.gov/earthquakes/world/historical.php).

[6]Detailed descriptions of earthquake magnitude and intensity can be found at http://en.wikipedia.org/wiki/Mercalli_intensity_scale.

Figure 2.7 Fatalities Associated with Earthquakes with Magnitude Greater than or Equal to 7 (1900–2012)



Data source: U.S. Geological Survey (http://earthquake.usgs.gov/earthquakes/world/historical.php).

The Global Terrorism Database compiles records of terrorist attacks that took place since 1970. The most extreme terrorist attack in terms of casualties and property damage was the 9/11 attack in the United States (NY, VA, and PA). Table 2.3 lists the top 10 terrorist attacks in terms of death toll.

Table 2.3 Top 10 Extreme Terrorist Attacks (1970–2013)

Time	*Country*	*Attack Type*	*Deaths*	*Property Damage*	*Extent of Property Damage*
Sept. 11, 2001	United States	*Hijacking*	*2,996*	*Yes*	*Catastrophic (likely > $1 billion)*
Apr. 13, 1994	*Rwanda*	*Armed Assault*	*1,180*	*Yes*	*Unknown*
Mar. 21, 2004	*Nepal*	*Armed Assault*	*518*	*Yes*	*Unknown*
Aug. 19, 1978	*Iran*	*Facility/Infrastructure Attack*	*422*	*Yes*	*Minor (likely < $1 million)*
July 18, 1987	*Mozambique*	*Armed Assault*	*388*	*Yes*	*Unknown*
May 23, 1996	*Burundi*	*Unknown*	*375*	*Yes*	*Unknown*
Sept. 1, 2004	*Russia*	*Armed Assault*	*344*	*Yes*	*Minor (likely < $1 million)*
Jun. 23, 1985	*Canada*	*Bombing/Explosion*	*329*	*Yes*	*Unknown*
Feb. 1, 1998	*Sri Lanka*	*Unknown*	*320*	*No*	
Jul. 20, 1996	*Burundi*	*Unknown*	*304*	*Yes*	*Unknown*

Data source: Global Terrorism Database (http://www.start.umd.edu/gtd).

Pandemic flu is another threat to human beings and may cause significant loss to insurers.

Four significant outbreaks of pandemic flu have occurred since 1900. As illustrated in Table 2.4, the most severe one was the 1918–1919 Spanish flu.

Table 2.4 Major Influenza Pandemics in the Twentieth and Twenty-First Centuries

Time	*Type*	*Death Toll Estimate*
1918–1919 Spanish Flu	H1N1	50 million worldwide 67,500 in the United States
1957–1958 Asian Flu	H2N2	1 to 1.5 million worldwide 69,800 in the United States
1968–1969 Hong Kong Flu	H3N2	1 million worldwide 33,800 in the United States
2009–2010 Swine Flu	H1N1	284,500 worldwide Between 8,870 and 18,300 in the United States

Data source: Flu.gov (http://www.flu.gov/pandemic/history).

2.1.3 Other Risks

In addition to extreme events that have industry-wide impacts, other events may cause significant losses to a single company. Table 2.5 includes a short list of extreme events that caused bankruptcies. In general, poor risk management was to blame.

Table 2.5 Company-Specific Extreme Events

Risk Type	*Year of Event*	*Event Description*
Concentration Risk	2001	Taisei Fire and Marine Insurance Co. went bankrupt because of catastrophe insurance claims of $2.5 billion following the 9/11 terrorist attack.
Strategic Risk	2008	Washington Mutual went bankrupt. One key reason was its aggressive growth strategy: low lending criteria and acquisition.
Operational Risk	2002	The collapses of Enron, Tyco, and WorldCom, caused by failed corporate governance.
Operational Risk	1995	The bankruptcy of Baring Brothers & Co. Ltd, caused by bad internal controls.

2.2 Future Extreme Events

Historical extreme events provide useful information for modeling potential future extreme events, but the practitioner should not stop there. History may repeat, but not all the time and not always in the same way. After a structural change or an unforeseen extreme event (black swan), our frameworks need to be revised. For example, consider the following changes:

1. Following the 2008 financial crisis, regulators put more stringent capital requirements in place, and risk models were recalibrated to reflect a higher level of volatility.
2. After several recent severe earthquakes (2010 in Chile, Haiti, and New Zealand, and 2011 in Japan), RMS Inc., a market leader in catastrophe (CAT) modeling, released a new version of models with updated natural catastrophe model parameters.
3. After the 9/11 terrorist attack, reinsurance costs spiked. The assumptions around the severity and frequency of a future terrorist attack changed. The U.S. Terrorism Risk Insurance Act was signed into law to ensure that any future terrorism-related U.S. losses would be shared between the federal government and insurers.

With sufficient knowledge of the causes of historical extreme events (root cause analysis), we have the opportunity to generate new potential future extreme events by experimenting with different combinations and regimes of the underlying drivers. Such events may be derived by way of a thought experiment or the use of scientific interaction models. For example, a forest fire model may be used to test different combinations of weather variables (drivers) and generate new extreme forest fire events. These extreme forest fire events could then be assessed and leveraged in an insurance context.

Emerging risks can also give rise to future extreme events. Many organizations such as the World Economic Forum (2013) and the Joint Risk Management Section of the Casualty Actuarial Society, Canadian Institute of Actuaries and the Society of Actuaries (Rudolph, 2014) have conducted or sponsored studies to compile a list of the most likely emerging risks. Potential extreme events related to emerging risks such as climate change, cyber risk, genetically modified food, nanotechnology, regional instability, international terrorism, and failed regulatory reform were identified. Not only are events related to these risks difficult to predict, but their impacts are equally difficult to quantify. Subject matter experts should be consulted when assessing future extreme events to get the most out of multidisciplinary human knowledge.

3. Individual Tail Risk Modeling

This section provides an overview of various tools and techniques applicable to the modeling of individual risk drivers. In particular, this section provides the practitioner with the following:

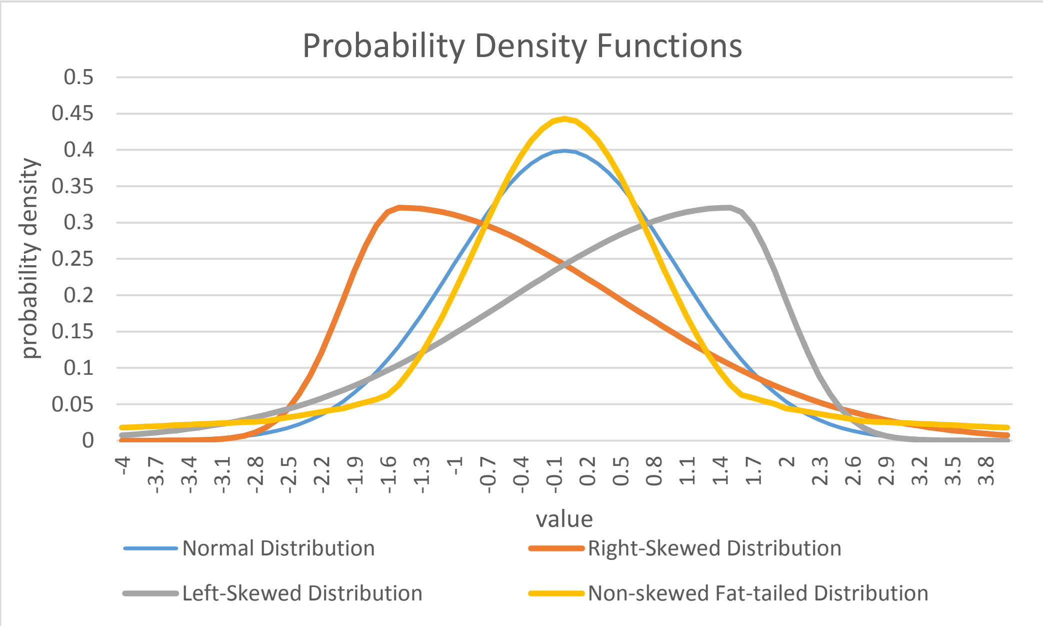
1. The background necessary to identify individual risk drivers exhibiting “extreme behavior” via the detection of fat tails and clustering and
2. A comparison of several univariate modeling approaches for extreme events.

3.1 Detecting Heavy Tails

No general consensus is available on the precise definition of a “heavy-tailed” distribution; furthermore, the terms “heavy-tailed” and “fat-tailed” are often used interchangeably in practice despite differentiation within the academic literature. Some take a more conceptual perspective and define a heavy-tailed distribution to be one whose left or right tail carries more probability density than the corresponding tail from an analogously fitted normal distribution. For example, in Chapter 2 of the book *Handbook of Heavy Tailed Distributions in Finance,* Bradley and Taqqu (2003) state that “[in] a heavy-tailed distribution the likelihood that one encounters significant deviations from the mean is much greater than in the case of the normal distribution.” Others, such as Asmusen (2003) in his book *Applied Probability and Queues,* define a heavy-tailed distribution to be one with tail density heavier than the exponential distribution (i.e., $\lim_{x \to \infty} \mathrm{e}^{\lambda x} \Pr[X > x] = \infty$ for all $\lambda > 0$). Yet others, including Kozubowski et al. (2003) in Chapter 4 of *Handbook of Heavy Tailed Distributions in Finance,* specifically define a *fat-tailed* distribution to be one with tail density that follows the power law probability distribution (i.e., $\Pr[X > x] \sim (x^{-\alpha}) L(x)$ as x goes to infinity, where $L(x)$ is a slowly varying function). Note that because the exponential function goes to zero faster than the power law function, fat-tailed distributions are always heavy-tailed, but not necessarily vice versa.

Although many ways are used to detect heavy tails, perhaps the most intuitive is to graphically compare the distribution in question to a normal distribution with comparable mean and standard deviation. Figure 3.1 shows a reference normal distribution along with three heavy-tailed distributions; note that a heavy-tailed distribution can be left-skewed, right-skewed, or nonskewed.

Figure 3.1 Heavy Tail Visualization

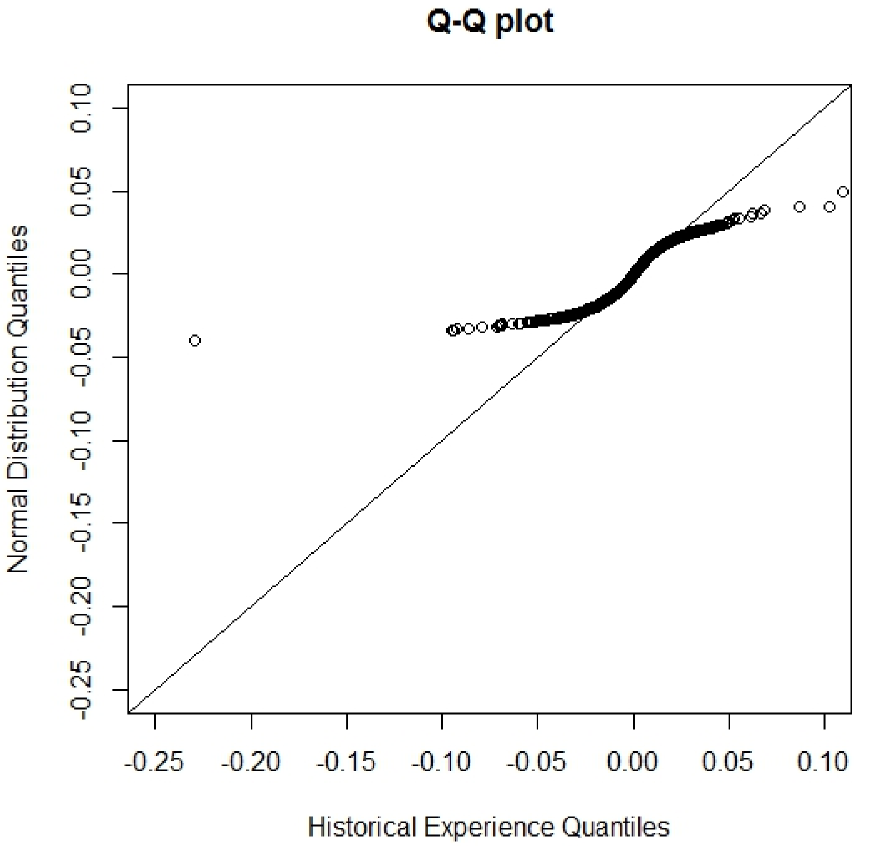


A second way to detect heavy tails is via a quantile-quantile plot of the fitted normal distribution quantiles (y axis) against those from the empirical distribution (x axis). If the quantiles of the theoretical and empirical distributions agree, the plotted points fall on or near the line $y = x$. However, if the left end of the pattern is above the line, we conclude that the empirical distribution has a heavy left tail; similarly, if the right end of the pattern is below the line, we conclude that the empirical distribution has a heavy right tail.

Figure 3.2 shows a quantile-quantile plot (Q-Q plot) for daily S&P 500 index returns from January 3, 1950, to September 24, 2014. Daily returns are calculated as $\ln(P_t/P_{t-1})$, where P_t denotes the adjusted closing level of the S&P 500 index on day t.[7] Points above the line at the left end indicate a heavier left tail for the x-axis distribution compared to the y-axis distribution. Points below the line at the right end indicate a heavier right tail for the x-axis distribution. Note that the tail quantiles for the empirical distribution of historical daily S&P 500 index returns are more dispersed than those from the analogously fitted normal distribution. It is therefore evident from this plot that the historical experience has heavy tails at both ends.

[7] S&P 500 index data from http://finance.yahoo.com/q/hp?s=%5EGSPC+Historical+Prices. The adjusted closing level accounts for dividends and splits.

Figure 3.2 Q-Q Plot of Historical S&P 500 Returns



A third way to detect heavy tails is to determine whether the shape of the tail in question follows a power law distribution of the form

- $\Pr[X > x] \sim (x^{-\alpha})\ as\ x \to \infty, \alpha > 0$, for the right tail; or
- $\Pr[X < x] \sim ((-x)^{-\alpha})\ as\ x \to -\infty, \alpha > 0$, for the left tail,

where α is referred to as the tail index. Recall that a fat-tailed distribution follows the power law and that fat tails imply heavy tails (but not necessarily vice versa). The smaller the tail index, the heavier the tail. As a general rule of thumb, $\alpha \in (0, 2)$ is suggestive of a fat-tailed distribution. Under this latter condition, the random variable X has infinite variance.

Linear regression can be used to estimate α based on the following equations:

- $\ln(\Pr[X > x]) = \ln(m) - \alpha \ln(x) + \epsilon$, for the right tail or
- $\ln(\Pr[X < x]) = \ln(m) - \alpha \ln(-x) + \epsilon$, for the left tail,

where *m* denotes a constant. Note that the practitioner must select a threshold beyond which to fit the linear regression equations since they only apply for large *x* and small *x*, respectively.

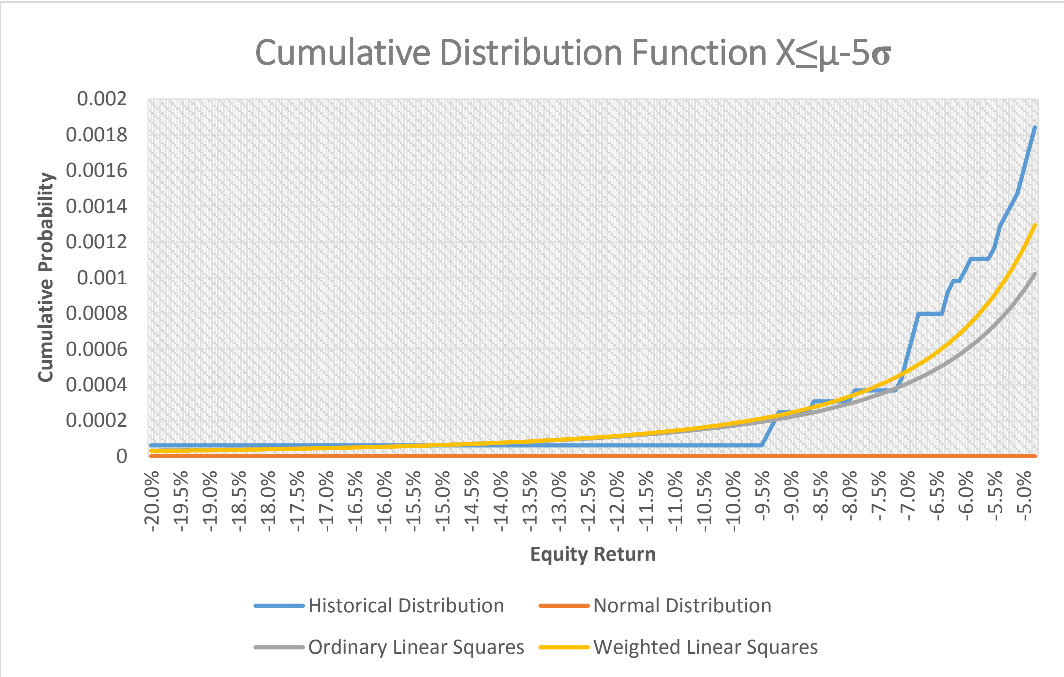
Applying the aforementioned regression approach to the left tail of S&P 500 daily index returns from January 3, 1950, to September 24, 2014, with a threshold of 5-sigma, we derive the tail index estimates shown in Table 3.1. The estimated tail index based on ordinary least squares (OLS) was determined to be 2.44. Using the method of weighted least squares (WLS), which assigns more weight to outliers, α was estimated to be 2.65. The corresponding cumulative distribution functions are shown in Figure 3.3.

Table 3.1 Model Estimation for S&P 500 Daily Returns

Measure	Value
Minimum Daily Return	−22.9%
Maximum Daily Return	11.0%
Average Daily Return (μ)	0.03%
Volatility (σ)	0.97%
Degree of Tail Fatness (α)	
- Ordinary Least Squares	2.44
- Weighted Least Squares* Weight: $1/\text{Ln}(\Pr[X < x])^2$	2.65

* Heavy weights are assigned to outliers in the regression process.

Figure 3.3 CDF of S&P 500 Daily Returns (5-Sigma Events)



Although many tools are used to detect heavy tails, the effectiveness of detection depends on data availability and data quality. An additional consideration is data frequency; generally, the more frequent the data (i.e., the smaller the time step), the better. An inappropriate choice of data frequency may underestimate the severity of extreme events. Less frequent data tend to smooth out the extremes as intra-period anomalies may average out. Given that a company may survive a weekly stock market return of −30 percent but not a daily stock market return of −50 percent, this consideration is important. Table 3.2 compares the analysis of S&P 500 index returns on daily, weekly, and monthly bases. Low frequency leads to less extremeness. For example, the probability of left-side 8-sigma events is 0.04 percent for daily returns, 0.03 percent for weekly returns, and zero for monthly returns. The degree of tail fatness of daily returns is also higher than weekly and monthly returns.

Table 3.2 Tail Heaviness of S&P 500 Index Returns by Data Frequency

	Daily Return	**Weekly Return**	**Monthly Return**
μ Annualized	7.40%	7.30%	7.30%
σ Annualized	15.40%	15%	14.50%
Pr[$X < \mu - 5\sigma$]* **Left-Side 5-Sigma Events**	0.18%	0.12%	0.13%
Pr[$X < \mu - 8\sigma$]* **Left-Side 8-Sigma Events**	0.04%	0.03%	0.00%
(Minimum Return − μ)/σ **(Worst Case in Terms of Volatility Multiple)**	−23.6	−9.7	−6.0
α (OLS)	2.44	2.12	2.10
α (WLS)	2.65	2.27	2.19

* Empirical probability.

Some concise statistical measures can serve as useful secondary indicators of fat tails. Value at Risk (VaR) and conditional tail expectation (CTE) are two such measures.

Given a confidence level p and a time horizon, Value at Risk is the threshold value such that the probability of having a value greater than the VaR within the time horizon is p:

$\Pr[X < \text{VaR}] = p.$

The definition of VaR above is usually used to study the right tail of a distribution. When studying the left tail of a distribution, it can be defined as $\Pr[X > \text{VaR}] = p$. Comparing the VaR with the corresponding percentile of the fitted normal distribution, a large deviation is indicative of a heavy tail.

VaR does not consider the magnitude of experience worse than the threshold. Furthermore, it may not be stable because it depends on the value of only one percentile. CTE can overcome these shortcomings to a certain extent. CTE is the average of values greater (less) than the VaR for the right (left) tail. Comparing the CTE of experience data with that of the fitted normal distribution is also useful for detecting heavy tails.

Table 3.3 uses these statistical measures to assess the tail heaviness of S&P 500 index returns. The increasing gap between the experience data and the fitted normal distribution with increasing confidence level also shows the left tail heaviness. Note that a relatively low confidence level such as 95 percent may conceal the true story of the left tail, as the 95 percent

VaR based on the experience data is larger than that of the normal distribution. Multiple confidence levels can be chosen to mitigate these shortcomings.

Table 3.3 Statistical Measures of S&P 500 Index Daily Returns (1950–2014)

	EXPERIENCE DATA	NORMAL DISTRIBUTION *N*(0.03%, 0.97%)
95% VAR	−1.4%	−1.6%
99% VAR	−2.6%	−2.2%
99.95% VAR	−7.0%	−3.2%
99.99% VAR	−22.9%	−3.6%
95% CTE	−2.3%	−2.0%
99% CTE	−3.9%	−2.6%
99.95% CTE	−10.2%	−3.4%
99.99% CTE	−22.9%	−3.8%

3.2 Extreme Value Theory

Extreme Value Theory (EVT) provides the practitioner with the limiting distributions of the extremes of a random variable. EVT relies on the Fisher-Tippett-Gnedenko theorem,[8] with Emil Julius Gumbel formalizing the theory in his 1958 classic *Statistics of Extremes*. Generally two ways are available to characterize extremes in a data series, block maxima and minima or excesses beyond a threshold, and EVT can provide useful insight into both.

3.2.1 Modeling Block Maxima and Minima

Block maxima and minima are defined to be the maxima and minima within discrete and nonoverlapping data blocks or "periods." For example, the distribution of the yearly minima of daily equity returns can be framed as a block minima problem. In this case, the block or period would be one year, and the event of interest would be the minimum daily equity return within each year.

If we let $M_n = \max\{X_1, X_2, \dots, X_n\}$ be a random variable denoting the block maxima, where the n variables X_1 to X_n are identically and independently distributed, then according to the Fisher-Tippett theorem, the distribution of M_n is asymptotic to the Generalized Extreme Value (GEV) distribution. More formally,

$$\Pr(M_n < z) = \Pr\left(\frac{M_n - b}{a} < x\right) \text{ converges to } F(x; \mu, \sigma, \xi), \text{ where}$$

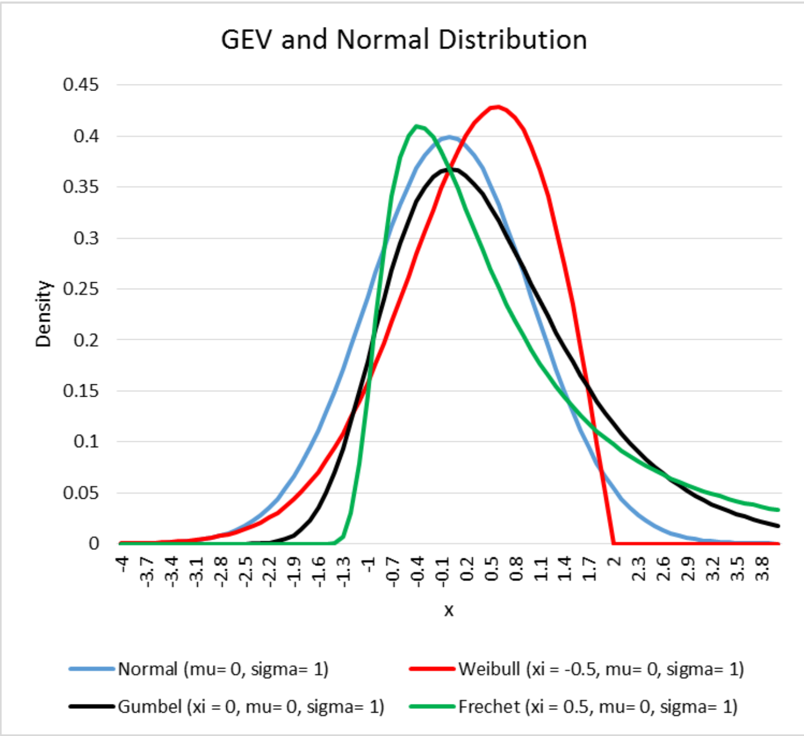
[8]http://en.wikipedia.org/wiki/Fisher%E2%80%93Tippett%E2%80%93Gnedenko_theorem.

$$F(x;\mu,\sigma,\xi) = \begin{cases} \exp\left(-\left(1+\xi\left(\frac{x-\mu}{\sigma}\right)\right)^{-\frac{1}{\xi}}\right) & \xi \neq 0 \\ \exp\left(-\exp\left(-\frac{x-\mu}{\sigma}\right)\right) & \xi = 0 \end{cases}$$

The asymptotic result holds in the limit of the block size going to infinity. In this parameterization, μ is the location parameter, σ is the scale parameter, and ξ is the shape parameter. The scale and location parameters represent the unknown norming constants a and b, respectively.

The advantage of the GEV is that it is effectively three distributions in one. When $\xi = 0$, M_n has exponentially decaying tails and follows the Gumbel distribution ("Type I"). When $\xi > 0$, M_n has a heavy tail and follows the Frechet distribution ("Type II"). When $\xi < 0$, M_n has a finite light tail and follows the Weibull distribution ("Type III"). Figure 3.4 shows the probability density functions of select GEV distributions against the standard normal distribution.

Figure 3.4 Probability Density of GEV and Normal Distribution



Legend: Blue: Normal (μ=0, σ=1); red: Weibull (ξ = −0.5, μ=0, σ=1); black: Gumbel (ξ = 0, μ = 0, σ = 1); green: Frechet (ξ = 0.5, μ = 0, σ = 1).

The methodology above applies to block maxima problems. For block minima problems, since $\max\{X_1, X_2, \ldots, X_n\} = -\min\{-X_1, -X_2, \ldots, -X_n\}$, the practitioner can simply reverse the signs of his or her data series, apply the block maxima equations, and finally reverse the sign of the result.

An applied step-by-step example of the block maxima approach can be found in Section 6.1.

3.2.2 Modeling Exceedances

An alternative way of characterizing extremes is to look at exceedances over a certain threshold using the peak over threshold (POT) method. For example, the distribution of daily equity returns greater than 10 percent (or less than −10 percent) can be framed as a POT problem. If the threshold u is large, the distribution of exceedances $y = x - u$ follows a generalized Pareto distribution (GPD) such that

$$\Pr(X - u < y | X > u) \sim F(y) = \begin{cases} 1 - \left(1 + \xi \dfrac{y}{\sigma_u}\right)^{-\frac{1}{\xi}} & if\ \xi \neq 0 \\ 1 - e^{-\frac{y}{\sigma_u}} & if\ \xi = 0 \end{cases}$$

where ξ denotes the tail index and σ_u is a scaling parameter. Note that in this context, the larger the tail index, the heavier the tail.[9] Exceedances without an upper bound will have $\xi \geq 0$. If the practitioner wishes to model a bounded series of exceedances, then $\xi < 0$ and the upper bound is defined by $\frac{-\sigma_u}{\xi}$.

3.2.3 Parameter Estimation

The parameters in the GEV and GPD distributions can be estimated using a variety of statistical techniques, the most common being Maximum Likelihood Estimation.

Semiparametric estimators, such as the Hill estimator (1975) and Pickands estimator (1975) for the tail index ξ, are also available to the practitioner. The advantage of these semiparametric estimators is that they are often more efficient and rely on fewer assumptions than their parametric counterparts. Take the Hill estimator for example:

$$\xi^{\text{Hill}}(n, k) = \frac{1}{k}\sum_{j=1}^{k}\left(\ln\left(X_{n-j+1}\right) - \ln(X_{n-k})\right),$$

where $X_1 < X_2 < \cdots < X_n$ are the order statistics for the quantity of interest (e.g., maxima, exceedances, etc.), n is the sample size, and the k largest observations are used in the estimation. This is a computationally straightforward way to estimate the tail index parameter assuming that $\xi > 0$. There is no hard rule regarding the selection of k, but it should be selected such that the estimate of the tail index is stable. Paulauskas and Vaiciulis (2011) provide a comparison of tail index estimators, including the Pickands, Hill, and moment estimators.

Continuing with the example of S&P 500 daily returns, the application of block minima and POT approaches are illustrated below. Annual block minima of S&P 500 daily equity index returns from 1950 to 2014 are used to calibrate the GEV distribution. ξ is estimated to be 0.51

[9]The reader should note that ξ defined in the EVT context behaves inversely to the α defined in Section 3.1, even though both are called tail indices. The larger the ξ, the heavier the tail, whereas the smaller the α, the heavier the tail.

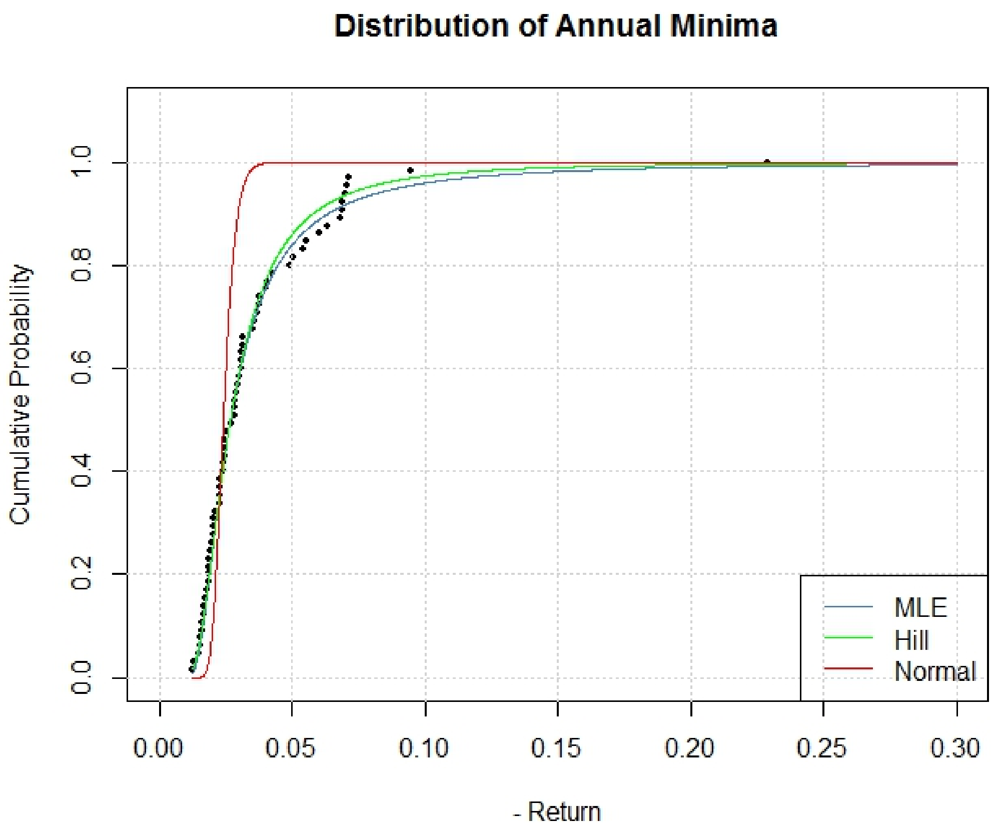
using MLE and 0.40 using the Hill estimator with k determined as the number of daily returns that are greater than the maximum of the annual block minima of the experience data. Table 3.4 compares the ξ estimated from the experience data to those from a Normal distribution calibrated to the experience data. It is easy to tell from the ξ that the experience data have a much heavier tail than the fitted Normal distribution.

Table 3.4 Annual Block Minima Shape Parameters: Daily S&P 500 Index Return

	MLE	Hill Estimator
Experience Data	0.51	0.40
Fitted Normal Distribution	−0.15	0.16

Figure 3.5 compares the cumulative probability functions for the GEV distribution with shape parameter estimated by MLE and Hill, the normal distribution, and the empirical distribution (black dots). Note how the GEV distributions fit the data much better than the normal distribution in this annual block minima example.

Figure 3.5. Annual Block Minima: Daily S&P 500 Index Return



Note: Signs have been reversed here so that minima are represented as maxima

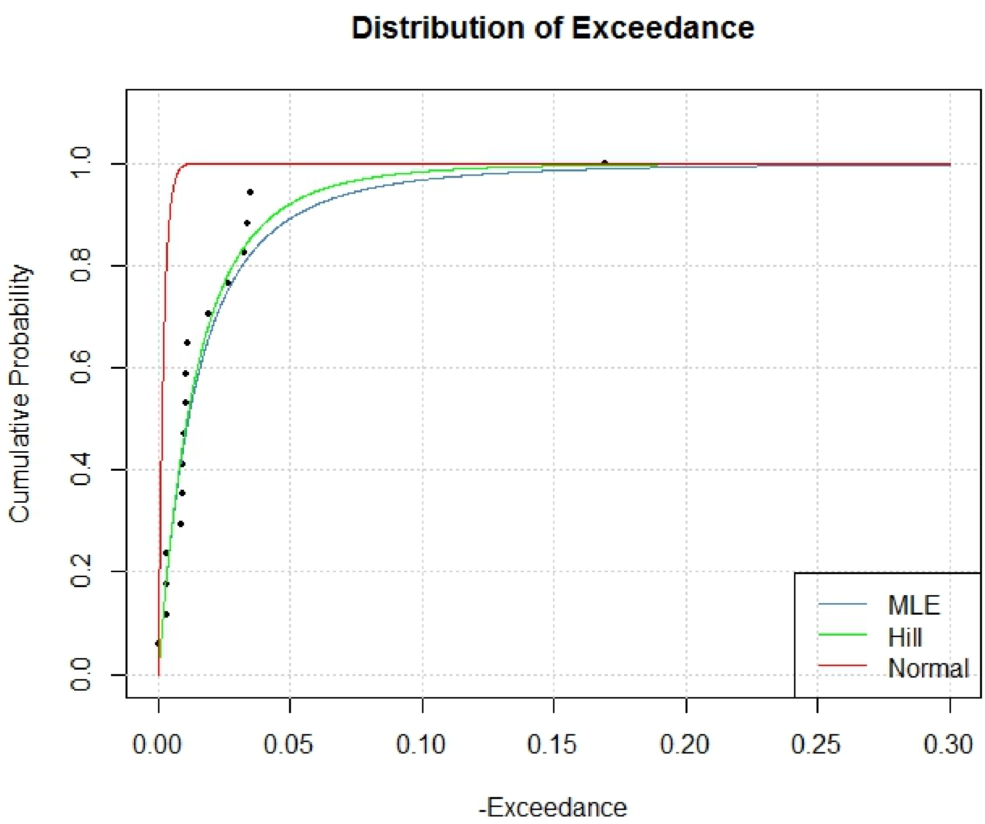
To leverage not only on the annual minima but also other very negative returns, the POT method can be used. Table 3.5 shows the estimated shape parameter ξ under various fitting assumptions. As expected, the experience has a heavier left tail than the fitted normal distribution.

Table 3.5 POT Shape Parameters: Daily S&P 500 Index Return

Block Minima Model	Experience Data (S&P 500 Daily Index Return)		Fitted Normal Distribution	
Threshold	ξ – MLE	ξ – Hill	ξ – MLE	ξ – Hill
Worst 0.1% cases	0.37	0.26	−0.04	0.06
Worst 0.5% cases	0.24	0.38	−0.26	0.13
Worst 1% cases	0.39	0.33	−0.10	0.13
Worst 5% cases	0.27	0.37	−0.11	0.22

Using the worst 0.1 percent of cases, the distribution of exceedance (the distance to the 99.9th percentile) based on the experience and fitted Normal distribution is illustrated in Figure 3.6. Note that the GPD fits the data much better than the Normal distribution.

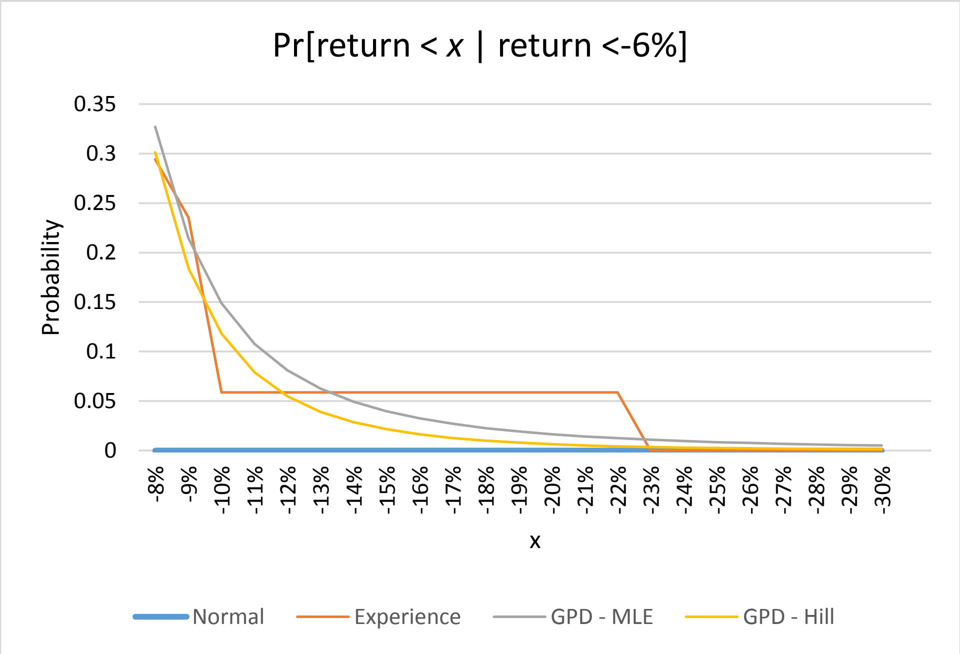
Figure 3.6. Point over Threshold: Daily S&P 500 Index Return



Note: Signs have been reversed here so that shortfalls are represented as exceedances.

The POT distribution also allows us to answer questions such as “What is the probability that the return is less than −10 percent, given that it is less than −6 percent?” Figure 3.7 shows the conditional probability of the return less than a value *x* given that it is less than −6 percent. It is clear that the fitted Normal distribution significantly underestimates the probability while the GPDs smooth the experience. Even for returns like −30 percent that did not happen in history, the GPDs can help estimate the probability of occurrence, which is 0.1 percent based on the Hill estimator and 0.5 percent based on the MLE.

Figure 3.7. Conditional Probability at the Left Tail: Daily S&P Index Return



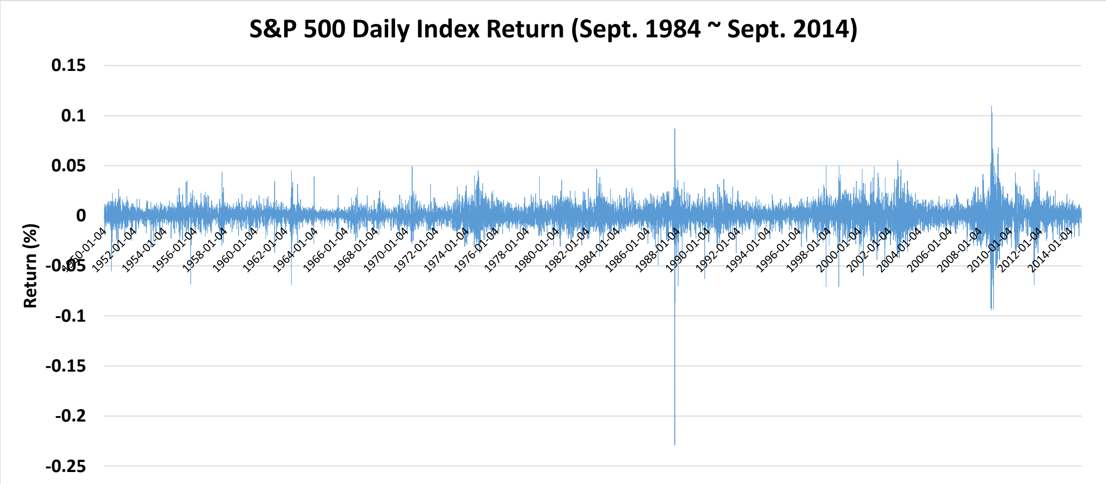
3.3 Detection of Clustering

In addition to the presence of extreme outcomes ("heavy tails"), clustering of adverse experience is an important factor when determining the impact of extreme events. For example, a return of −10 percent for three consecutive months may be worse than a monthly return of −20 percent followed by a market recovery. Moreover, the presence of clustering itself may be indicative of a more significant systemic risk event. This subsection presents the practitioner with various approaches to identify volatility and temporal clustering.

Visualization

Plotting the time series can help identify the existence of volatility clustering. Figure 3.8 shows the daily returns of the S&P 500 index over a 30-year horizon. Notice how high volatility tends to be followed by high volatility, in particular during the period associated with the 2008 financial crisis.

Figure 3.8 S&P 500 Daily Index Return: 30 Years

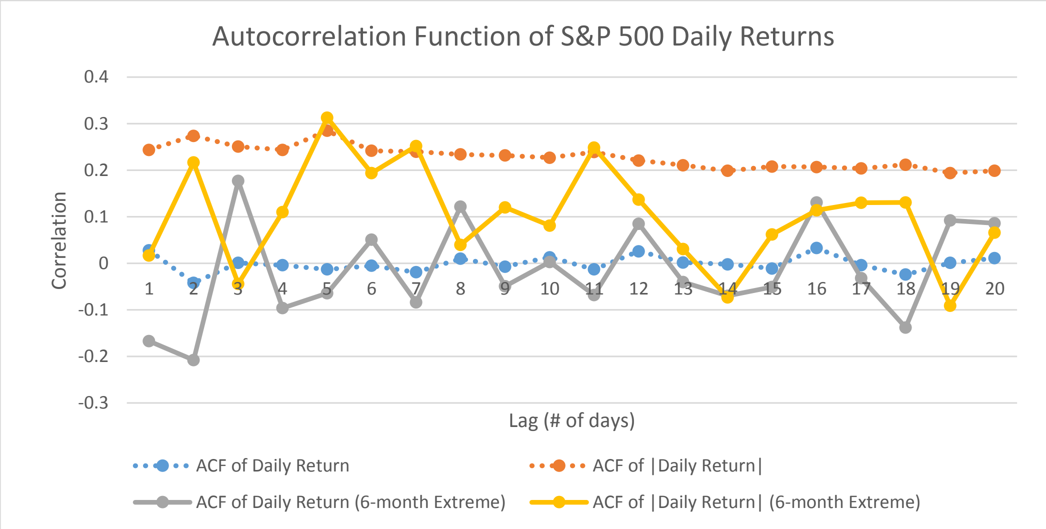


Autocorrelation Function

The autocorrelation function (ACF) describes the correlation as a function of time lag. A positive autocorrelation means that the returns with a certain time lag tend to move in the same direction together. A zero autocorrelation means that the returns are uncorrelated. A negative autocorrelation means that the returns tend to move in the opposite direction. A high ACF indicates a high chance of volatility clustering. Figure 3.9 compares the ACFs between the full data period (Jan. 3, 1950 to Sept. 24, 2014) and the most extreme six months (Sept. 2, 2008 to Feb. 27, 2009). The absolute value of the ACF is much higher during the extreme period, which means they are more correlated.

When studying volatility clustering, in addition to the ACF of the return, people are often interested in the ACF of the absolute value of return as well. When the mean of the return is zero, it does not matter whether a return is negative or positive in terms of its contribution to the volatility. Figure 3.9 also shows the ACF of absolute return. Unlike the ACF of return, the ACF of absolute return is more stable and higher on average than in the extreme period. This means that the market is volatile with either positive movements or negative movements. It can serve as an indicator of possible volatility clustering, but it should be used with other measures. In an extreme event analysis the direction of movement matters. A down-and-up scenario is better than a down-and-down scenario while the ACF of absolute values could be the same.

Figure 3.9 ACF of S&P 500 Daily Return



GARCH

Some models can be used to measure volatility clustering, such as the generalized autoregressive conditional heteroskedasticity (GARCH) model developed by Bollerslev (1986) and the hidden Markov model (HMM) developed by Baum et al.[10] If the subsets of a random variable have different levels of volatility, the variable is heteroskedastic. The subsets could be determined by, for example, time period or location. GARCH can be used to model the evolution of conditional variance. Using the S&P 500 index as an example, we model the return according to a moving average process with a one-day lag (i.e., MA(1)) and the volatility as a GARCH(1,1) process. Other model specifications can be used as well and may fit the data better; however, the chosen model is good enough to illustrate the presence of volatility clustering:

$$r_t = \mu + u_t + \theta u_{t-1}$$ MA(1): moving average with a lag of one day

$$\sigma_t^2 = \omega + \alpha u_{t-1}^2 + \beta \sigma_{t-1}^2$$ GARCH(1,1)

$$u_t \sim N(0, \sigma_t^2)$$

where

r_t: Equity return for time period t

μ: Mean of the equity return

u_t: White noise error term that follows a normal distribution $N(0, \sigma_t^2)$

σ_t: Volatility of the equity return.

Calibrated to the daily returns from January 1950 to September 2014, the estimates of model parameters are given below:

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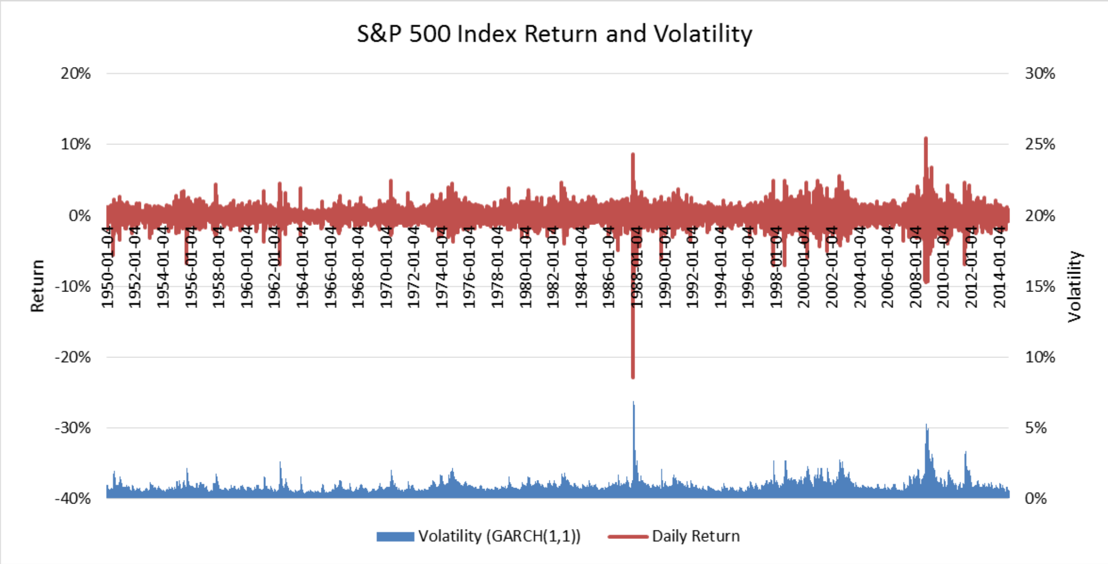
[10] http://en.wikipedia.org/wiki/Hidden_Markov_model.

$$r_t = 0.00 + u_t + 0.10u_{t-1}$$

$$\sigma_t^2 = 0.00 + 0.08u_{t-1}^2 + 0.91\sigma_{t-1}^2.$$

Here β can be used as a measure for volatility clustering. A β of 0.91 means a high degree of volatility clustering, as shown by the fitted time series in Figure 3.10. Note how the estimated volatility using GARCH(1,1) varies over time.

Figure 3.10 S&P 500 Index Return and Estimated Volatility



Note: Daily return uses the left *y* axis. Volatility uses the right *y* axis.

HMM

A Hidden Markov model studies the Markov Process of hidden states with observations that highly depend on the hidden states. The next hidden state depends on the current hidden state, but not the history of the hidden states. In most cases, a transition matrix is used to define the probability of the next state given the current one. The distribution of observations changes with the hidden state. Based on actual observation, an inference system built on Bayes' Rule can be used to predict future hidden states. The states can be considered different phases of a cycle. Like the GARCH model, it can be used to identify high volatility and low volatility. Unlike the GARCH model, it uses randomness to describe the transition between high volatility and low volatility.

The basic setup of HMM is the same as the Markov Chain Monte Carlo (MCMC) model and the regime-switching model. However, in a HMM, states are not observable and can be inferred based only on observation of other related variables whose distribution depends on the state. The state inferred from the observations is important in projecting the future paths of states and their outcomes. Similar to a regime-switching model, an HMM model captures additional variability introduced by state changes that could not otherwise be reflected by a single distribution.

Section 6.2 contains an example of the HMM applied to U.S. earthquake data.

3.4 Modeling Approaches

This section discusses several univariate modeling approaches for extreme events:

(1) *Historical replication.* The past is assumed to be a good guide for the future. Past extreme events are used as future extreme scenarios. The probability assigned to an extreme scenario is based on historical experience.

(2) *Historical simulation, a.k.a. bootstrapping.* Historical data are used to simulate the future distribution by random sampling with replacement. The variable of interest is assumed to be independently and identically distributed. The extreme scenarios are chosen based on the simulated data, as well as the assigned probability.

(3) *Weighted historical simulation.* Unlike historical simulation, weighted historical simulation assigns decreasing probability to historical data the further out they are. More recent experience is assumed to be more relevant.

(4) *State-dependent historical simulation.* Based on the current state, only a subset of the historical data are used for simulation.

(5) *Extreme value theory (EVT)–block maxima.* The block maxima method analyzes the distribution of the maxima. For example, given a time series of stock index daily returns, the distribution of annual maxima (or minima) may be of interest. If so, the Generalized Extreme Value (GEV) distribution may provide an appropriate fit.

(6) *Extreme value theory–peak over threshold (POT).* The POT method analyzes the distribution of exceedances over a large threshold. The distribution exceedances is commonly assumed to follow the GPD asymptotically.

(7) *State-dependent model simulation.* A combination of EVT distributions (or more general distributions) and state-based models like HMM. The inclusion of multiple states allows for the modeling of greater variability.

(8) *Scientific modeling.* Scientific models are designed to replicate the key features of real-world systems. By incorporating all of the driving factors and causality in the system, scientific models can be used to simulate a much broader range of extreme events. For example, there have been many pandemic flu simulations based on the location of outbreak, the type of virus, the method of transmission, the social contact, the strategy of intervention, the development of medical treatment, etc. Scientific models are especially useful in running "what-if" scenarios and designing future extreme scenarios.

(9) *Futurism.* The science of prediction and systems thinking. Futurists usually have interdisciplinary knowledge and carry out thought experiments for a variety of matters including global trends and possible disaster scenarios. These scenarios can be used as future extreme scenarios if the time horizons match.

Table 3.6 compares these nine modeling approaches.

Table 3.6 Comparison of Extreme Event Modeling Approaches

Modeling Approach	*Pros*	*Cons*
Historical Replication	• Easy to implement • Maintains the volatility clustering	• History may not repeat and can be useless following a structural change • Data constrained
Historical Simulation	• Easy to implement • Can help construct extreme events for different time horizons; for example, based on daily historical data, daily, weekly, monthly, and annual extreme events can be generated	• The variable is assumed to be i.i.d. so there is no preservation of volatility clustering • Historical data are given equal weight even though older data may be less relevant, particularly after a structural change • Data constrained
Weighted Historical Simulation	• Can help construct extreme events for different time horizons	• The variable is assumed to be i.i.d. so that volatility clustering is not preserved • Data constrained
State-Dependent Historical Simulation	• Maintains the volatility clustering • Can help construct extreme events for different time horizons	• Data constrained • Difficult to implement
EVT–Block Maxima	• Focuses on the distribution of the most extreme value • Flexible enough to accommodate all levels of tail heaviness	• Data constrained • Difficult to implement • The series of maxima is assumed to be i.i.d so that volatility clustering is not preserved
EVT-POT	• Focuses on the tail of the distribution	• Difficult to implement • Data constrained

Modeling Approach	*Pros*	*Cons*
	• Can help predict the extreme value at a certain confidence level where there are no historical data	
State-Dependent Model Simulation	• Volatility clustering is maintained	• Difficult to implement • Data constrained
Scientific Modeling	• The causes of extreme events are known • Provides more comprehensive coverage of extreme events in addition to the historical ones	• Difficult to implement • Knowledge constrained • Difficult to determine the probability of an extreme event
Futurism	• A long-term vision • Interdisciplinary coverage	• Difficult to determine the probability of an extreme event

3.5 Model Selection

Depending on data availability, distribution, and clustering, some modeling approaches are more appropriate than others for a certain risk type. When data are limited, historical simulation, EVT, and model-based simulation may not have the desired statistical credibility. Scientific modeling requires an in-depth understanding of the system and the underlying driving factors. Futurism is boundless but may be difficult to use in a business context.

Table 3.7 provides the practitioner with an overview of which extreme event models may be appropriate for a particular risk type.

Table 3.7 Comparison of Extreme Event Modeling Approaches

Risk Type	*Models*	*Explanation*
Economic Risk	Historical Replication Weighted Historical Simulation State-Dependent Historical Simulation EVT-POT State-Dependent Model Simulation Scientific Modeling such as macroeconomic models Futurism	Economic risk usually follows a heavy-tailed distribution, has a high level of volatility clustering, and exhibits a cyclical pattern.
Insurance Risk	Historical Replication State-Dependent Historical Simulation EVT–Block Maxima EVT–POT State-Dependent Model Simulation Scientific Modeling such as agent-based models and geological models Futurism	For risks with sufficient experience data, historical replication, EVT, and scientific modeling can be used. For risks with insufficient experience data, scientific modeling or futurism may be appropriate. For insurance risks that exhibit a cyclical pattern, state-dependent models such as the HMM can be used to better model extreme events.
Emerging Risk	Futurism/Expert Opinion	Multidisciplinary expertise needs to be relied on.
Company-Specific Extreme Events	Historical Replication Futurism/Expert Opinion	Relying on the historical extreme events if available; limited experience likely; important to understand impact of potential management failures and operational risk events.

4. Correlation and Contagion

Extreme events often involve highly correlated underlying risk drivers across space and time. For example, given the increasing integration of economic activities, an extreme financial event may well involve a bear equity market, a low interest rate environment, and a poor credit environment. This scenario played out during the 2008 financial crisis, and to avoid a domino or contagion effect, many systemically important financial institutions were saved by government bail-out programs. Even insurance risk may be affected by an extreme economic environment since policyholder behavior is often tied to personal income level.

Correlation is considered the most important assumption in aggregate risk assessment and risk budgeting. A small change in the correlation structure often leads to a significant change in the total required capital. This section discusses modeling approaches for correlation and cause-and-effect relationships between risk drivers.

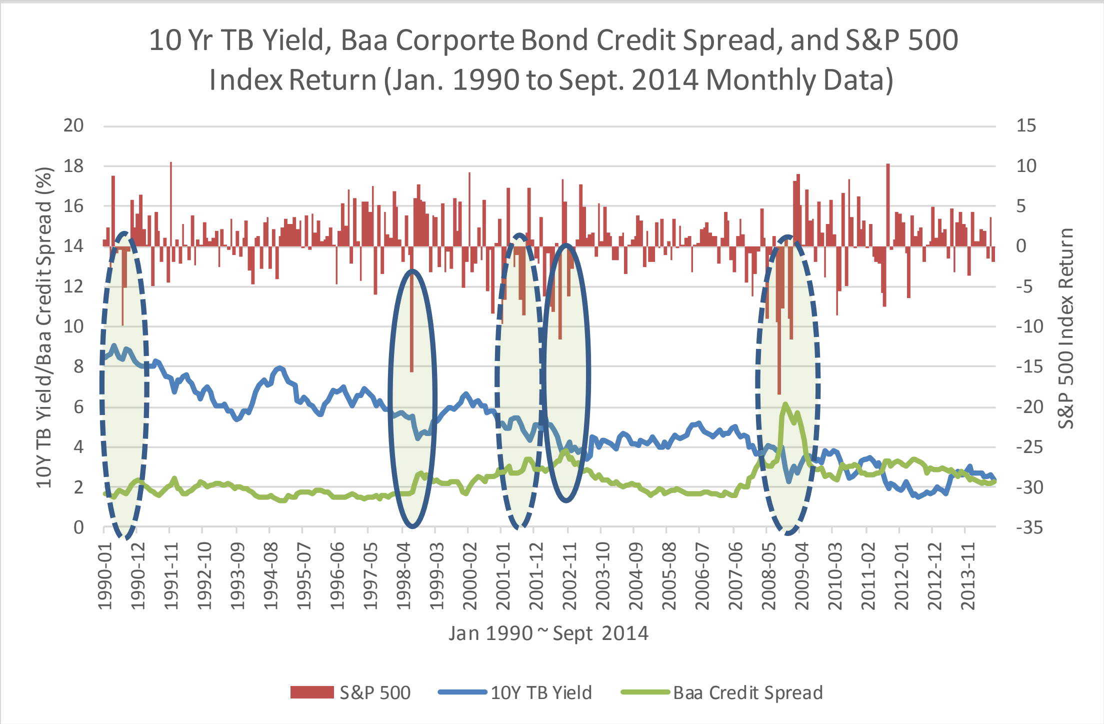
4.1 Historical Correlation of Extreme Event Risk Drivers

The preferred way to model the relationships between extreme events is to model the relationships between their underlying risk drivers. For the purposes of this paper, we define a risk driver to be a random variable that lends itself to univariate statistical modeling and simulation. Some examples of risk drivers would be the S&P 500 daily return, the 10-year U.S. Treasury yield, and the mortality rate for a particular risk class.

To quantify the correlation between extreme event risk drivers, it is important that the practitioner first filter out nonextreme data (i.e., nontail experience) by leveraging techniques presented in early sections. Correlations associated with an extreme event can differ significantly from those observed during nonextreme periods.

In Figure 4.1 and Table 4.1, we illustrate just how different correlations can be during extreme events. Figure 4.1 overlays time series for Treasury bond (TB) yield, corporate bond credit spread (CS), and equity market return from January 1990 to September 2014. Note that in general, TB yield and credit spread exhibit a negative correlation, whereas the relationship between stock return and bond yield/CS is less clear. Now focus on the five ovals, which highlight periods with very negative equity market returns. The three ovals with a dashed outline correspond to U.S. recessionary periods. Note that a much more positive relationship exists between equity return and TB yield within the ovals. Conversely, a much more negative relationship exists between equity return and credit spread within the ovals. The correlation between TB yield and credit spread is also higher during these poor equity market periods.

Figure 4.1 Correlations in Economically Volatile Times



Data sources: 1. U.S. Treasury Bond Yields: U.S. Department of Treasury (http://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yieldAll)

2. Moody's Baa Corporate Bond Credit Spread: Moody's Investor Services.

3. S&P 500 index return: Yahoo Finance

Table 4.1 Correlations in Economically Volatile Times

Time Period	*Equity* Return and *TB Yield**	*Equity Return* and *Credit Spread**	*TB Yield* and *Credit Spread*
Jan. 1990 to Sept. 2014	0.7%	−9.3%	−61.7%
Jul. 2008 to Mar. 2009	83.8%	−62.9%	−73.1%

* With three-month time lag to reflect market reaction time.

In some cases, extreme events may not happen concurrently but one after another. In the analysis above, a three-month time lag is used when calculating the correlation between equity return and TB yield or credit spread for the period July 2008 to March 2009. If concurrent time series are used, the correlation calculation would fail to recognize the impact of slower cause-and-effect relationships (e.g., it takes time for policymakers to collect and absorb market information before reacting, such as reducing interest rates). The correlation between equity return and TB yield drops from 83.8 to 27.7 percent without the time lag. The correlation between equity return and credit spread increases from −62.9 to −29.4 percent without the time lag. Therefore, it is important to understand the order of changes in the system even when historical data are used.

It is worth noting that a negative correlation between risk drivers does not necessarily lead to a high risk diversification benefit. For example, the negative correlation between interest rate and credit spread means that a low interest rate environment is likely to coexist with high credit risk. For a life insurer, low interest rates mean a higher reserve and a low investment return on new money, all else being equal. A high credit spread means a high expected credit loss and an increased cost of new capital. The value of existing bond portfolios may increase depending on the combined change of interest rate and credit spread. As such, a life insurer is unlikely to experience a material diversification benefit in this example.

In addition to economic risk, some risks that are categorized under noninsurance risk are driven by economic factors as well. For example, new business sales are affected by income level, which in turn depends on the economic environment. Policyholders may pay fewer premiums, terminate the insurance policy, or exercise embedded options during an economic crisis. Unemployment insurance or products that offer extra benefits for the unemployed may find their claim rates much higher than expected and the duration of unemployment longer than expected during an economic recession. More bankruptcies, higher liquidity risk, and more operational risk events can be anticipated during an economic crisis.

On the other hand, noneconomic events such as World War II and the Black Death may cause economic upheavals as well. Although we may think insurance risk and economic risk are independent, in the extreme cases such an assumption is rarely true.

If historical data are available, they can be used to help evaluate the correlations as in the example above. However, credible and relevant experience data may be hard to get. The available data may include sparse or no extreme events. Structural changes in products and economy make the data irrelevant. Because of these difficulties, we explore alternative approaches in the next section.

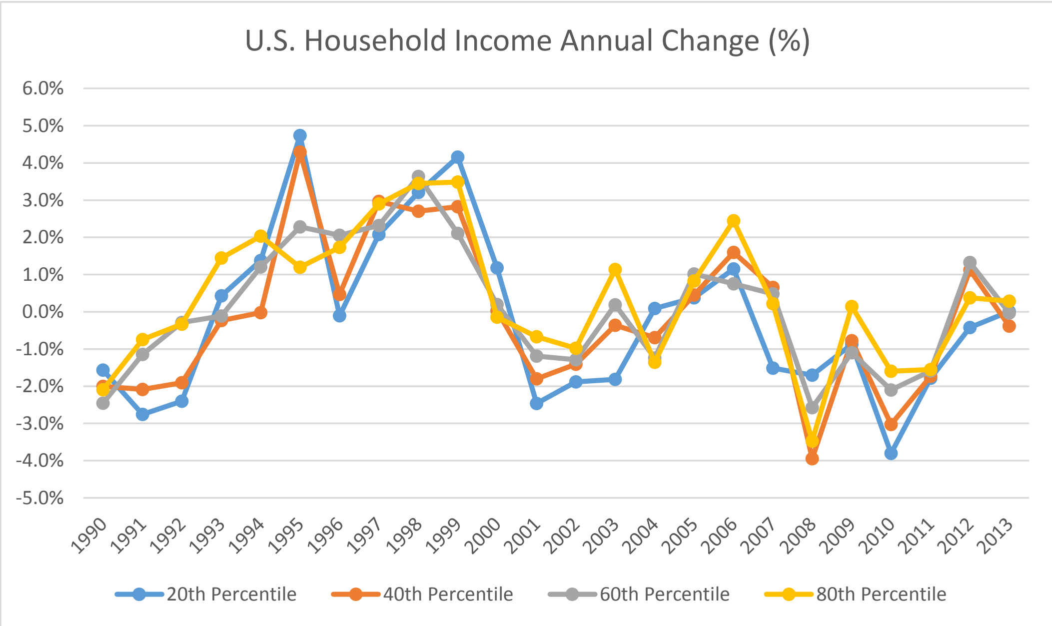
4.2 Contagion and Cause-and-Effect Relationships

With insufficient historical data, studying the cause-and-effect relationships and the chain of contagion becomes important. It requires multidisciplinary knowledge, sound reasoning, and improvisation. Several examples are given below for illustration. Many other cases and applications are found in the real world, as we keep exploring the universe:

1. As mentioned above, new business sales, policyholder premium payments, lapses, and option exercises are affected by the economic environment. Many policyholder behaviors are based on household financial planning and household financial income. A household financial analysis is a key to understanding dynamic policyholder behaviors. In a financial crisis, we would expect lower household income and therefore, for example, lower insurance sales, more lapses, and fewer premium payments. Figure 4.2 lists the annual U.S. household income movement from 1990 to 2013 at four percentiles: 20th, 40th, 60th, and 80th percentiles. Depending on the

income level of the target market, the impact of economic crisis on policyholder behaviors can be roughly estimated. For low- to middle-income families, a drop in income may affect their ability to pay required insurance premiums or buy new policies. They may need to surrender their policies to get the cash value or take the policy loan to meet their liquidity needs. For high-income families, a drop in income may have minimal impact as they have alternative sources of wealth to pay insurance premiums and buy new insurance products. During the last financial crisis, we can see a cumulative 4 to 7 percent income decrease from 2008 to 2010. This needs to be considered when constructing reasonable policyholder behavior extreme scenarios.

Figure 4.2 U.S. Household Income Year-to-Year Change (1990–2013)



Source: U.S. Census Bureau.

In addition to use household income level as a guideline, a couple of caveats are also useful for refining our expectation of extreme policyholder behaviors.

a. *Irrational decision.* People tend to view their financial needs separately because they have different levels of risk tolerance for different needs. For example, when investing for funding their children's education, they tend to be conservative. When funding for entertainment, they could be very aggressive. From a pure economic perspective, this may lead to an inefficient portfolio in aggregate. This helps explain some of the irrational policyholder behaviors such as giving up a deep-in-the-money guarantee for the cash surrender value, as evidenced in the financial crisis.

b. *Sentiment.* People's risk averse changes from time to time. When an economic recession lasts for some time, people's sentiment and level of risk averse may change. They may prefer cash to other asset types.

Considering the target market, the product type, the income level change, the actual approach of household financial planning, and the sentiment change, a lower level model can estimate a likely policyholder behavior extreme event that may happen with a financial crisis.

2. Some insurance product lines are more directly affected by financial crisis in terms of claim experience. Examples include unemployment insurance, mortgage insurance, and credit insurance. A clear understanding of the labor market, household income, financial leverage ratio, and personal loan market in an economic crisis can help estimate the extreme claim experience for these specific product lines.
3. Contagion can happen from people to people, company to company, industry to industry, country to country, region to region, and so on. It acts as an important role in the development of extreme events, in terms of scale, scope, and speed.
    a. A well-known example is the systematically important financial institutions. Due to the high integration of the financial world, financial institutions are the investors, debtors, and creditors of each other. A collapse of a big financial institutions may cause a chain effect that leads to the collapse of the financial industry and then nonfinancial industries.
    b. Sovereign risk has a similar pattern of contagion. Countries hold debt of each other. A default of one country may cause the default of other countries as well.
    c. Pandemic flu is another example of contagion. The origin of the outbreak and population movement have material impact on the speed of spread.

    Understanding the degree and spread of contagion is useful for estimating the scale, scope, and speed of system-wide extreme events. One way to study contagion is using network models. Network models analyses the dependency among individuals. Elliott et al. (2014) used network models to study the consequences of integration and diversification by studying asset cross-holdings. They gave a practical example of interdependencies of six European economies (France, Germany, Greece, Italy, Portugal, and Spain). The debt cross-holdings are used to describe the dependencies via a financial network. With an initial failure caused by a credit crisis, the financial network can be used to propagate the order of failures of the six countries. For example, the order of default could be Greece, Portugal, Spain, France, Germany, and Italy. The financial network can also be used to determine the extent of market turbulence to trigger the first failure. In addition, government may play an active role to mitigate the impact of contagion via bailout programs. Its impact needs to be factored into the model as well.

Cause-and-effect relationships and contagion can be used together to predict dependencies of extreme events when historical data are not enough or do not exist.

4.3 Modeling of Risk Driver Dependencies

Risk driver dependencies are important in risk budgeting and capital allocation. Relying on historical data, cause-and-effect relationship analysis, and contagion analysis, risk driver dependencies can be put into risk models to estimate risk exposure, required capital, and the like. Before the discussion of modeling approaches, it is helpful to differentiate the dependencies of risk drivers and the dependencies of risk exposures and required capital. Two risk drivers can be independent while at the same time their required capital are not. For example, whole life insurance products are subject to economic risk and mortality risk. In the long run, insurance risk and economic risk are not independent because the demographic and population has an impact on economic development. Economic development has an impact on medical improvement and health care that will eventually affect mortality experience as well. However, in a short time horizon such as one year, economic risk and mortality risk can be considered as independent. Using the U.S. GDP growth rate and changes in death rate from 1961 to 2012, the correlation of the annual changes is 11 percent and the correlation of 10-year changes is 76 percent.

Figure 4.3 U.S. Annual GDP Growth Rate and Death Rate Change (1961–2012)

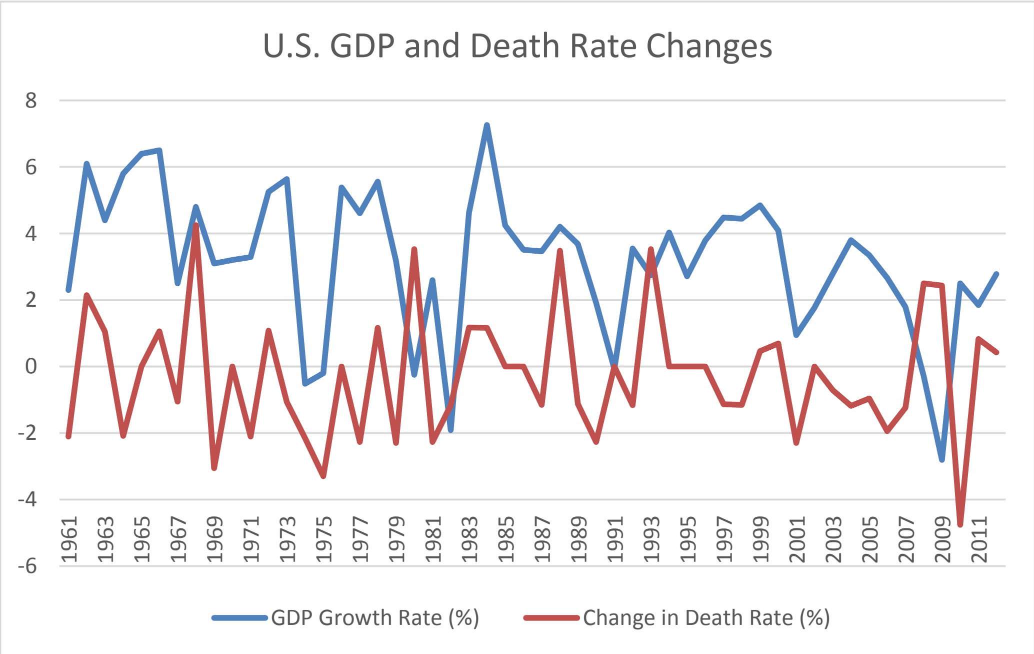
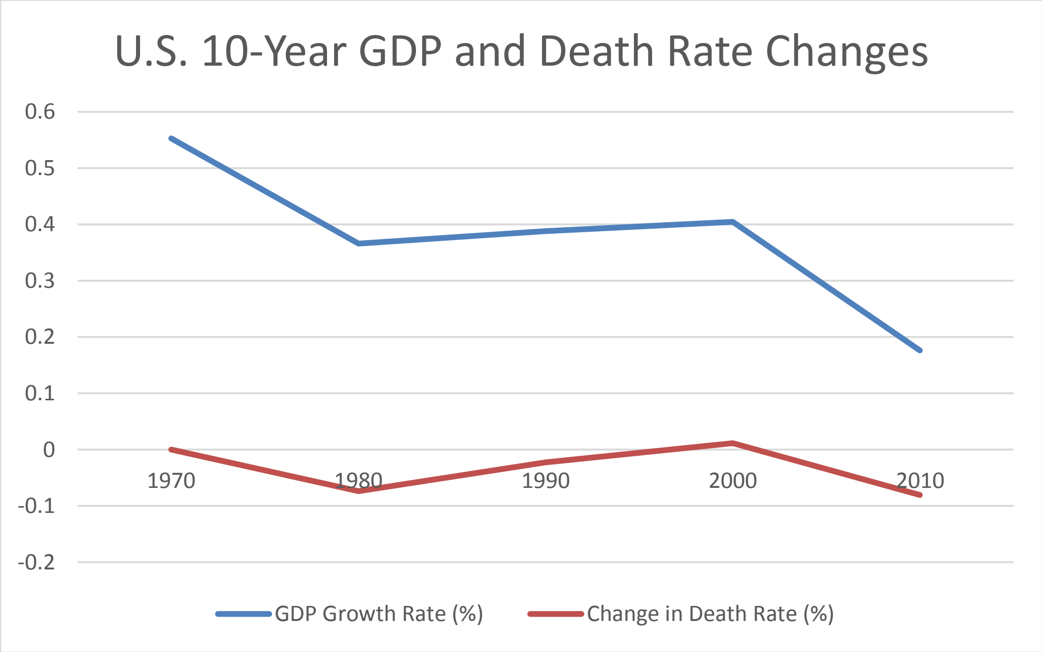


Figure 4.4 U.S. 10-Year GDP Growth Rates and Death Rate Changes (1961–2010)



For the purpose of stress testing, contingent planning, and risk budgeting, a one-year time horizon is normally used in the industry as the time generally needed for management to solve an extreme issue. So it is generally safe to say that economic risk and insurance risk are independent or only slightly correlated in the short term. However, in the context of insurance business, it does not mean that the required capitals for economic risk and insurance risk are independent. If the insurance risk is high, the volatility of benefit outgoes is high. As a consequence, the big mismatch between asset and liability leads to a high exposure to economic risk.

Depending on the time horizon and the purpose, correlation among extreme events can be built in with several approaches: correlation, copula, and structured models:

1. *Stress testing and contingent planning.* A stress scenario based on either historical data or cause-and-effect relationships can encompass either a single extreme event or several extreme events. If events are not highly correlated, a single extreme event scenario can be used. If they are highly correlated, several events can be built into the same stress scenario. For example, a bear stock market, low interest rate, and high credit spread or default rate with adverse dynamic policyholder behavior can be used as a stress scenario. A severe earthquake in California could be another stress scenario. An advantage of using stress testing is that it can incorporate the order and timing of extreme events in a reasonable way instead of assuming all of them happen at the same time.
2. *Risk budgeting and capital allocation.* A company's risk appetite and risk tolerance normally includes a target probability of solvency with a one-year time horizon. To be consistent with this benchmark, it is necessary to associate extreme events with probability of occurrence. We can reflect it in several ways:

a. *Historical correlation.* If historical data are sufficient and available, they can be used for modeling correlated extreme events that happened before.

b. *Correlation matrix.* Individual risk exposures at a chosen confidence level are aggregated using their correlation coefficients. Using the example in Section 4.1, Table 4.2 and Table 4.3 list two sets of correlations using general data and extreme data. With the assumed individual risk exposures, the aggregated risk exposures using two correlation matrices are calculated:

$$\mathrm{RE}_{\mathrm{Total}} = \sqrt{\begin{pmatrix} RE_1 & RE_2 & RE_3 \end{pmatrix} \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix} \begin{pmatrix} RE_1 \\ RE_2 \\ RE_3 \end{pmatrix}}$$

where

RE_{Total} is the aggregated risk exposure

RE_i is the risk exposure for risk factor i and

ρ_{ij} is the correlation coefficent of risk factors i and j.

The aggregated exposure using correlation matrix under extreme cases is 22 percent higher than that using normal correlation matrix.

Table 4.2 Risk Aggregation: Normal Correlation Matrix

Risk Type	Equity Risk	Interest Risk	Credit Risk
Equity Risk	1	0.007	0.093
Interest Risk	0.007	1	0.617
Credit Risk	0.093	0.617	1
Individual Exposure	10	30	5
Aggregated Exposure	35		

Table 4.3 Risk Aggregation: Extreme Correlation Matrix

Risk Type	Equity Risk	Interest Risk	Credit Risk
Equity Risk	1	0.838	0.629
Interest Risk	0.838	1	0.731
Credit Risk	0.629	0.731	1
Individual Exposure	10	30	5
Aggregated Exposure	43		

Ideally the correlation matrix at each confidence level is unique to reflect the nonlinear relationship in reality. However, because of the lack of data, it is difficult to construct them credibly. In addition, the correlation between risk drivers is not necessarily the same as the correlation between risk exposures, as discussed above. Therefore, it may need to be adjusted to reflect product features that can strengthen or weaken the relationship. Many studies have been done by the insurance industry and regulators. The CRO Forum issued an article “Calibration Recommendation for the Correlations in the Solvency II Standard Formula” in 2009 that contains a suggested range of correlation among major risk types. CEIOPS also provided information on the appropriate correlation matrix used in solvency requirement calculation at a confidence level of 99.5 percent. These can be used as a reference so that our customized correlation matrix does not deviate much from the industry standard.

c. *Copula*. Copula is a statistical method to derive the joint distribution based on marginal distributions and a copula function. Its theoretical foundation is Sklar’s theorem (1959), which says that every multivariate cumulative distribution function can be written as a function of the marginal distributions functions. For a bivariate cumulative distribution function, $\mathrm{P}(X \leq x, Y \leq y) = C\big(\mathrm{P}(X \leq x), \mathrm{P}(Y \leq y)\big)$. The copula function *C* is a parameterized model that describes the relationship of multiple variables. Table 4.4 illustrates several copula functions for bivariate analysis, all of which can be extended to multivariate analysis to accommodate three or more variables. With the same marginal distributions, different copulas exhibit different joint distributions. The correlation at the tail implied by the Gumbel copula is the highest in the example. The Clayton copula shows a negative correlation in the example. Although the example is for two variables, copulas can be easily applied to multiple variables as well.

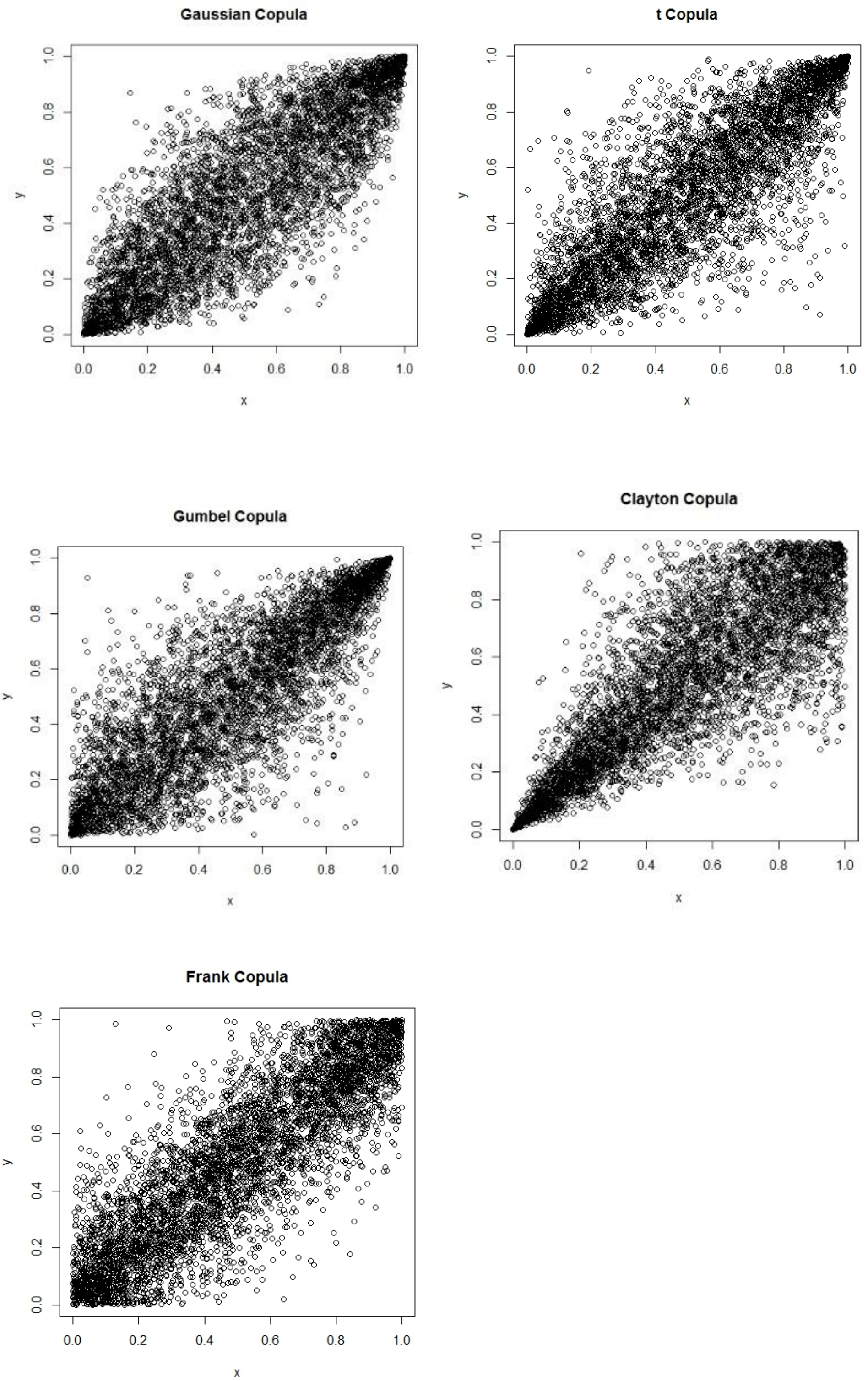
Table 4.4. Copula Example

Marginal Distribution		
u	$\mathrm{P}(X \leq x)$	0.95
v	$\mathrm{P}(Y \leq y)$	0.95
Joint Distribution		
Gaussian Copula	Bivariate Normal distribution Φ with correlation coefficient ρ $C(u, v) = \Phi_\rho(x, y)$	$\mathrm{P}(X \leq x, Y \leq y) = 0.928$ when $\rho = 0.85$

t Copula	Bivariate *t* distribution with correlation coefficient ρ and the number of the degrees of freedom *v*	$\mathrm{P}(X \leq x, Y \leq y) = 0.932$ when $\rho = 0.85$ and $v = 5$
Gumbel Copula	$C(u,v) = \exp\left(-\left((-\log(u))^{\theta} + (-\log(v))^{\theta}\right)^{1/\theta}\right)$	$\mathrm{P}(X \leq x, Y \leq y) = 0.937$ when $\theta=3$
Clayton Copula	$C(u,v) = (u^{-\theta} + v^{-\theta})^{-1/\theta}$	$\mathrm{P}(X \leq x, Y \leq y) = 0.799$ when $\theta=4$
Frank Copula	$C(u,v) = -\frac{1}{\theta}\log\left(1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right)$	$\mathrm{P}(X \leq x, Y \leq y) = 0.916$ when $\theta=9.5$
Independent	$C(u,v) = u \times v$	$\mathrm{P}(X \leq x, Y \leq y) = 0.9025$

Unlike the correlation matrix which a nonlinear relationship needs to use multiple matrices, a copula can describe a nonlinear relationship. Copulas allow us to parsimoniously reflect a nonlinear dependence in stochastic scenario generations. Figure 4.5 illustrates a few simulated copulas as used in Table 4.4. We show five sets of simulated data, each set with two variables that have a correlation coefficient of about 0.85. Gaussian copula models the linear relationship, which is exactly the same as the correlation matrix approach. The *t* copula has a higher correlation at both ends; the Gumbel copula has a higher correlation at the right end; the Clayton copula has a higher correlation at the left end; and the Frank copula has a lower correlation at both ends. However, like the correlation matrix approach, it is difficult to consider the order and timing of extreme events. Copulas are a complicated statistical concept with many more types and possible applications than discussed above. Bouye (2000) has a more comprehensive explanation of the concept and applications of copulas. Nelson Roger's well-known book An *Introduction to Copulas* provides more theoretical background.

Figure 4.5 Copula Illustrations



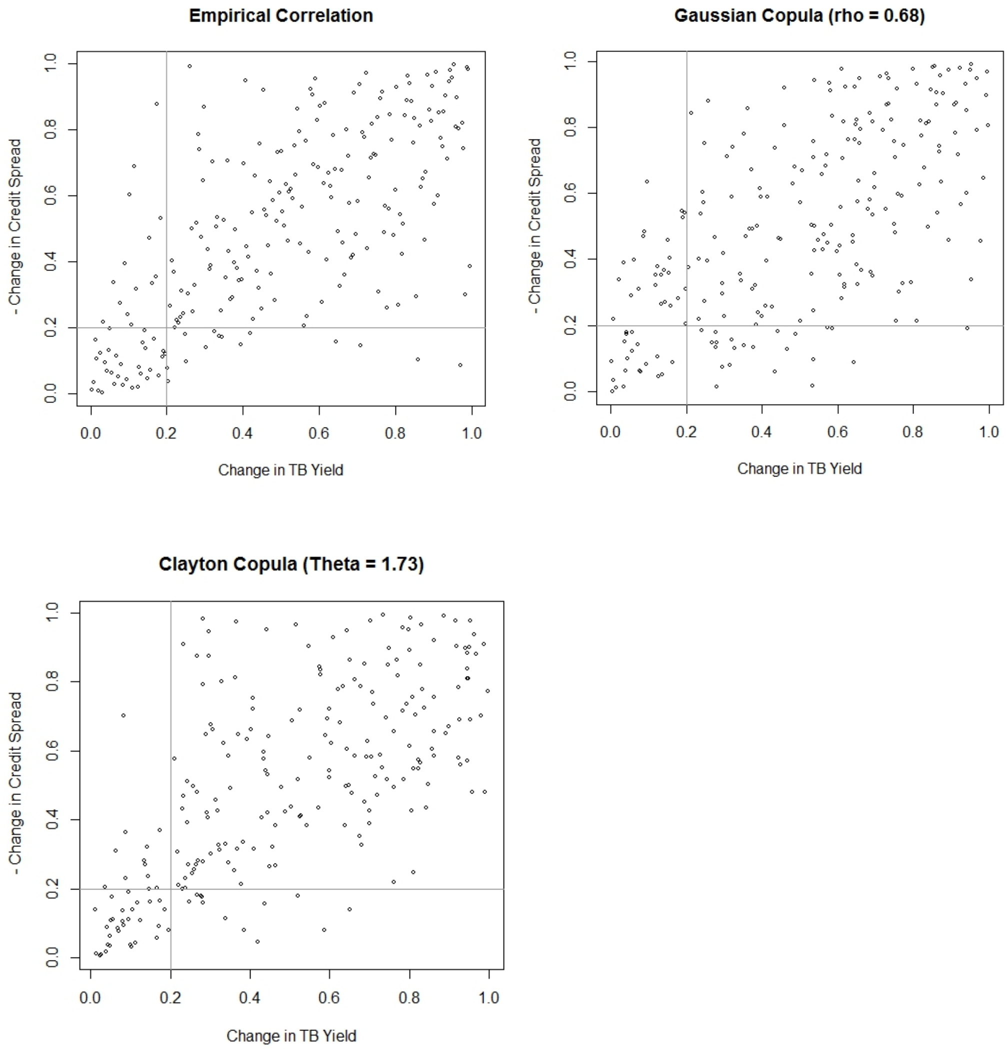
Using the quarterly changes in 10-year Treasury bond yields and Baa-rated bond credit spread from 1955 to 2014 as an example, the correlation coefficient is estimated to be −0.65. It has a higher correlation when the Treasury bond yield has a material decrease and the credit spread has a material increase. To incorporate this pattern, the Clayton copula, which allows a high correlation at the left end, is used.

The changes of credit spread are negated before fitting the data to a copula. Figure 4.6 plots the percentile of the change in the negated credit spread against the percentile of the change in the Treasury bond yield. In the bottom left area of each graph, the Treasury bond yield has a big decrease, and the credit spread has a big increase. It can be seen that the calibrated Gaussian copula has fewer data points in the bottom left box, and the calibrated Clayton copula has more data points, similar to the experience data. It means that the Clayton copula better captures the high correlation at the left end. To compare it numerically,

$$\Pr(X \leq \text{the 20th percentile of } X, Y \leq \text{the 20th percentile of } Y) = \begin{array}{ll} 13.7\% & \text{Experience} \\ 11.0\% & \text{Gaussian copula} \\ 13.5\% & \text{Clayton copula} \end{array}$$

Using the Clayton copula combined with the marginal distributions of the Treasury bond yield and credit spread, the high correlation in stressed scenarios can be preserved.

Figure 4.6 Copula Examples



Although the copula approach provides more flexibility and preserves the parsimony, it is not an easy task to find the most appropriate copula. The data used for copula calibration may be sparse, and the goodness of fit can be low. The goodness of fit can be measured by comparing the empirical multivariate distribution and the fitted distribution using statistical tests such as the Cramer–von Mises test and Kolmogorov-Smirnov test. Genest et al. (2007) reviewed and compared a variety of goodness-of-fit tests for copulas.

d. Structured models. Correlated simulation models are used to reflect nonlinear correlation and timing of extreme events. Take equity return and interest rate for example, they can be simulated using the following structured models:

Equity Return: Lognormal Model

$$\ln\left(\frac{S_{t+1}}{S_t}\right) = \mu - \frac{\sigma_s^2}{2} + \sigma_s \varepsilon_{1t}$$

where

S_t: stock price at time t

μ: expected stock return

σ_s: equity volatility

ε_{1t} ~ N (0, 1): A random sample of standard normal distribution.

$Interest\ Rate: One - factor\ Hull\ White\ Model$

$$r_{t+1} = r_t + (\theta_t - \alpha_t r_t) + \sigma_r(\rho\varepsilon_{1t} + \sqrt{1-\rho^2}\varepsilon_{2t})$$

where

r_t: short rate at time t

μ: expected stock return

σ_r: interest rate volatility

ε_{2t} ~ N (0, 1): A random sample of standard normal distribution. $\left(\rho\varepsilon_{1t} + \sqrt{1-\rho^2}\varepsilon_{2t}\right)$ and ε_{1t} are both samples of standard normal distribution and have a correlation of ρ.

ρ represents the correlation between equity return and interest rate. It can be defined as a function of equity return and change in interest rate: $\rho\left(\ln\left(\frac{S_{t+1}}{S_t}\right), \Delta r_t\right)$.

To incorporate the timing of extreme events, the correlation function can be used for equity return and interest rate in a different period. In the example below, the correlation between equity return and interest rate has a lag of three periods:

$$\ln\left(\frac{S_{t+1}}{S_t}\right) = \mu - \frac{\sigma_s^2}{2} + \sigma_s \varepsilon_{1t}$$

$$r_{t+1} = r_t + (\theta_t - \alpha_t r_t) + \sigma_r(\rho\varepsilon_{1,\boldsymbol{t-3}} + \sqrt{1-\rho^2}\varepsilon_{2t})$$

5. Managing Tail Risk

A good understanding of fat tails and their dependency is the basis of tail risk management. Tail risk management is an important component of the risk management system. Given their significant impact, tail events are those that may cause the bankruptcy of a company. With an effective management of tail risk, the company is protected against tail events, and therefore the chance of default can be reduced. Tail risk management does not focus on the optimal use of financial resources to achieve a high risk-adjusted return, which is also an important goal of risk management. Tail risk management usually looks at events that are rarer than the confidence level used in capital allocation and risk budgeting. Therefore, the focus is on the risk side instead of a balance between risk and return.

5.1 Determining Risk Tolerance for Extreme Events

A company's risk appetite describes the level of risk that the company is willing to and able to take to achieve its strategic objectives. Tail risk management should be consistent with a company's risk appetite. A risk appetite framework usually contains three levels with increasing details: enterprise risk tolerance, risk appetite for each risk category, and risk limit. The highest level of risk appetite framework is the most relevant to tail risk management because extreme events are likely to devour more capital than allocated to a single risk category. Financial resources supporting other risk categories are likely to be used to survive extreme events. On the other hand, business decisions are not made under the assumption of extreme events. Capital allocation is likely to be made at a confidence level that extreme events do not happen. Some capital allocation methods such as allocation by percentile layer proposed by Bodoff (2009) uses all loss layers to the chosen confidence level. Even extreme events are considered in capital allocation and risk budgeting, they are not the only considerations in the decision-making process.

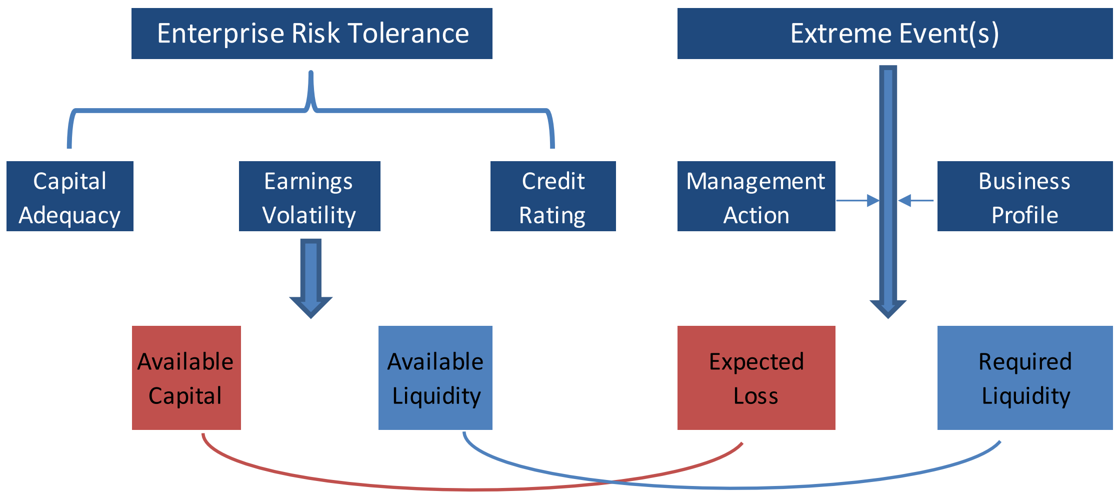
When determining the risk tolerance for extreme events, the exposure to extreme events can be compared to the target aggregate risk level as defined in the enterprise risk tolerance. Aggregate risk tolerance is usually expressed in terms of capital adequacy, earnings volatility, and credit rating. Enterprise risk tolerance determines a company's target capital level, available liquidity, and business profile. An example of a quantitative description of enterprise risk tolerance is given below. In reality, companies may use different ways to describe their enterprise risk tolerance:

1. Capital at risk (CaR): The tolerance for capital loss:
   a. The probability that a loss of 40 percent available statutory capital is less than 0.5 percent.
   b. The probability that a capital deficiency under economic framework is less than 0.05 percent.
2. Earnings at risk (EaR): The tolerance for earnings volatility. The probability of negative annual earnings is less than 5 percent.

3. Credit rating: The company needs to maintain an A financial strength rating and a buffer of more than 100 percent annual net income above the minimum capital requirement for the A rating.

When an extreme event happens, the available capital can be used to absorb the loss. The business profile will determine the amount of loss caused by the extreme event. Material additional liquidity requirement may occur during an extreme event and in some cases is the direct cause of a bankruptcy. In addition, competent management could take active actions to reduce the extreme event's adverse impact. When assessing whether an extreme event is above the company's risk tolerance, all these factors should be considered. Figure 5.1 summarizes such a process. By comparing the available capital and available liquidity with the predicted losses and required liquidity caused by an extreme event, the chance of surviving the extreme event can be estimated.

Figure 5.1. Risk Tolerance for Extreme Events



The available capital and available liquidity used in the comparison depends on the enterprise risk tolerance. For example, the available capital could be 40 percent of available statutory capital, or 100 percent available economic capital, as stated in the CaR. In the risk tolerance statement, EaR and target credit rating may serve as additional constraints, but they usually focus on severe situations, not extreme ones. To test if an extreme event exceeds the risk tolerance, the predicted loss is compared to 40 percent available statutory capital, or 100 percent available economic capital. If the event causes a large amount of cash payment during a short period, the market impact of large-sale asset selling should be considered as well. In the example below, available capital is less than the final adjusted loss after considering liquidity cost and the impact of management actions. The extreme event is therefore above the company's enterprise risk tolerance. The company should take actions to reduce its exposure to the event. These derisking actions may include stopping writing business for some product lines, reinsurance, hedging, selling some existing business, and capital raising.

Available Capital	$4 Billion
Predicted Loss	$3.8 Billion
+ Liquidity Cost	$1 Billion
− Management Actions	$0.7 Billion
= Adjusted Loss	$4.1 Billion

Some extreme events happen together. For example, when the equity market crashes, interest rates and credit spreads may change significantly at the same time. Therefore, the loss estimation should be based on all related events instead of an individual event.

Section 6.2 contains an example of liquidity stress testing under extreme scenarios. Section 6.3 explains the necessary considerations for assessing the impact of a pandemic extreme event.

The analysis of extreme events can be fed into the risk appetite framework. For extreme events that exceed the risk tolerance, it can be clearly stated in the risk appetite that the corresponding risk will not be taken. For example, the company will not write business in catastrophe-prone regions.

5.2 Monitoring Tail Risk

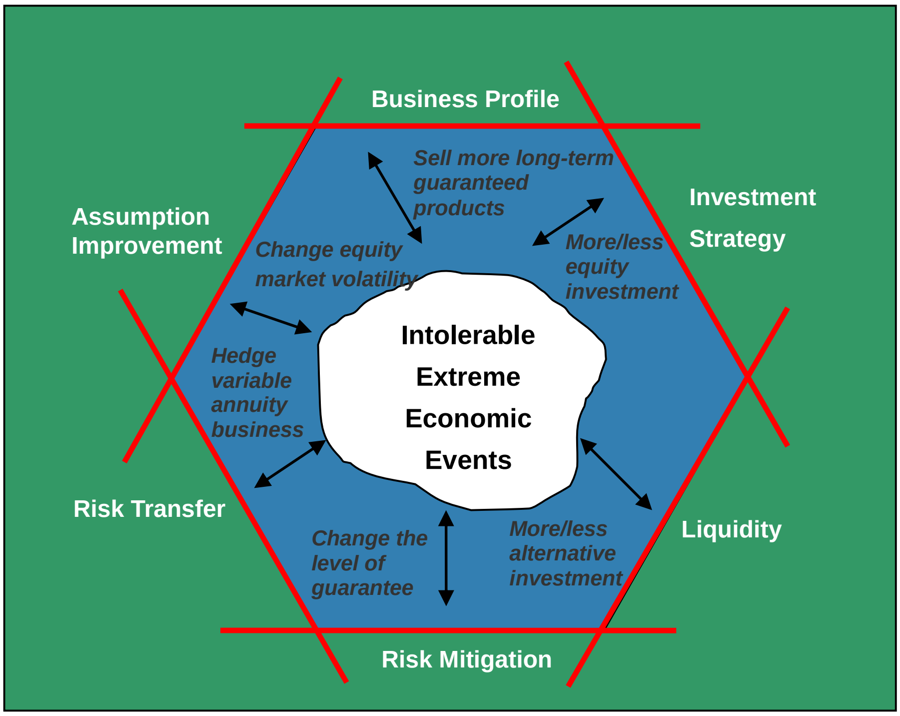
Once the risk tolerance for extreme events is determined, tail risk can be actively monitored. Based on the company's business profile and risk profile, a list of relevant extreme events needs to be compiled. The status needs to be monitored against the maximum level of extremity allowed within the risk tolerance. Changes in the status can be caused by many things. Some common ones are listed below:

1. A change of the business volume and business mix. For example, a significant increase in catastrophe insurance sales may cause an exceedance of risk tolerance.
2. A change of the strategies regarding investment, target market, underwriting, etc. For example, a more aggressive investment strategy may increase the exposure to equity market crash. For life insurance products and retirement products, different target markets may also have different levels of mortality and longevity risk. A change of target market could cause a change of risk exposure as well.
3. A change of the expected level of extremity can change the expected loss and required liquidity given such an extreme event happens. The expectation of the worst case usually changes after a record extreme event happens. Such events could be, for example, an unprecedented market crash or a natural disaster.
4. A change of the risk transfer or risk mitigation arrangement. For example, buying more reinsurance, setting up a risk sharing plan with customers, or buying a catastrophe equity put will reduce the exposure to tail risk.

Basically, anything that has a material impact on the exposure to tail risk needs to be monitored. Figure 5.2 shows an example of tail risk monitoring for economic risk. In the center, intolerable extreme events are those that exceed the company's enterprise risk tolerance. It could be a financial crisis similar to the credit crisis started in late 2007 with equity market crash, low interest rates, and high credit losses. The magnitude of the market turbulence assumed in the intolerable events could be higher or lower than the credit crisis started in 2007, as determined by the company's risk tolerance. For example, it may be described as something like a 30 percent drop in the S&P 500 index, a 10-year Treasury bond yield of less than 2.3 percent, and a total credit loss of more than 10 percent of the high-yield bond portfolio within six months. The hexagon represents the dimensions of the tail risk monitoring system:

1. *Business profile.* The change of the business profile, either in the volume or in the mix, can change the risk exposure. For example, selling more long-term guaranteed products is likely to raise the risk level and get the company closer to the center.
2. *Investment strategy.* An aggressive investment strategy will certainly increase the risk exposure and the chance that it will exceed the risk tolerance. For example, investing more in high-volatility asset types will raise the risk exposure.
3. *Liquidity.* Less liquid asset investment will reduce the available liquidity. A higher chance of credit rating downgrade will increase the required liquidity. For example, investing more in less liquid alternative asset types will reduce the available liquidity and reduce the risk tolerance of the company.
4. *Risk mitigation.* More risk mitigation actions will reduce the level of risk being taken. For example, reducing the level of guarantee provided to the clients can reduce the risk exposure.
5. *Risk transfer.* A risk transfer arrangement is likely to reduce the aggregate risk exposure. At the same time, the counterparty risk could be higher. For example, a hedging program for the variable annuity business can reduce the risk exposure.
6. *Assumption improvement.* An assumption change caused by either new knowledge or new experience will change the assessment of the risk tolerance. For example, a change in the equity market volatility assumption can cause a revaluation of the company's risk exposure and risk tolerance.

Figure 5.2 Monitoring Extreme Economic Events



The blue area can be considered as a dangerous region where the risk tolerance could be easily exceeded with an insignificant change of the situation. If the company lies in the blue area, more frequent monitoring and assessment are required to prevent the situation getting worse. In addition, actions need to be taken to transfer or mitigate the risk to move to the green area. The border of the blue area can be determined so that there is a prespecified margin between the risk exposure and the risk tolerance at the border. Based on a chosen capital measure, the margin could be 10 percent of available capital.

Key risk indicators (KRIs) should be constructed for all important dimensions of monitoring and checked on a regular basis. The risk indicators may not be different from what is used in the business management process. Table 5.1 shows examples of KRIs for monitoring extreme economic events. It shows the current status, the border for the dangerous region, and the border of the intolerable. For example, the current guaranteed credit interest rate is tolerable but is in the blue area. The company may consider lowering the level of guarantee. In practice, more KRIs are likely to be used to monitor the exposure to extreme economic events.

Table 5.1 Monitoring Extreme Economic Events

	Current	**Warning**	**Intolerable**
Equity market volatility	25%	35%	40%
Equity allocation	15%	20%	25%
Real estate allocation	8%	12%	15%
Guaranteed credit interest rate	2.5%	2%	3%
Growth rate of long-term guarantee products	2%	4%	5%
Effective hedging ratio	55%	40%	30%

5.3 Mitigating Tail Risk

Given the uncertainty of both the frequency and the severity of extreme events, tail risk is difficult to mitigate. Many ways are available to reduce tail risk, but they are not always effective.

1. *Risk diversification.* By taking risks that are not highly and positively correlated, the aggregate level of risk is less than the sum of individual risks. However, risk diversification can reduce only nonsystemic risk. Some tail risks are highly correlated where the diversification benefit is small, and in some cases, only one extreme event is needed for the bankruptcy of a company where risk diversification will not help.
2. *Hedging.* The gain and loss are fully or partially offset by taking a position in a contract. The risk is transferred to a third-party after hedging. However, for tail risk, it is difficult to find counterparties that are willing to take the risk because they may also find the tail risk difficult to predict and mitigate. Even if they are counterparties who want to take the risk, the associated credit risk is difficult to assess. How much loss will be paid by the counterparty is difficult to predict. The extreme event may cause extreme losses to the counterparty, and the final payment could be less than specified in the contract.
3. *Reinsurance.* The risk is transferred to reinsurers. For example, catastrophe reinsurance helps protect an insurer against catastrophe events. However, not all tail risk can be transferred to reinsurers, and the cost of transfer can be very high.
4. *Risk sharing.* The risk is shared with the clients. For example, an upper limit on claim payment can help cap the maximum loss. A floating credit interest rate also helps limit the exposure to extreme economic events. Risk sharing is helpful, but it also decreases the value of insurance, which is supposed to provide protection.
5. *Risk avoidance.* For tail risk that exceeds a company's risk tolerance, the company can change its business strategy to avoid taking such risk.
6. *Contingent planning.* A well-established contingent planning system can reduce the losses caused by extreme events by taking timely actions.

The effectiveness of risk diversification depends on the type of tail risk. If the tail risk is systemic risk, risk diversification is ineffective. If the tail risk is nonsystemic risk, risk diversification can be useful. Hedging and reinsurance are better choices for reducing systemic risk, but their benefit can be dampened by the increased counterparty risk. Reinsurance may be more effective because it has a smaller correlation to the capital market than hedging.

The severity of the tail risk affects the choice of mitigation method as well. A more severe tail risk needs more capital to absorb the risk. The reinsurance industry may not have enough capital because it relies on a limited capital base.

Risk sharing and risk avoidance can always reduce the exposure to tail risk. However, they may conflict with the business strategy and may not be used. A company may need to take the risk to stay competitive in the market. Contingent planning can help reduce the impact of extreme events, but its financial impact may be small compared to the total loss amount.

For example, catastrophe (CAT) risk, the tail risk caused by a catastrophe event such as a natural disaster, can be mitigated using several methods:

1. *Catastrophe reinsurance.* It pays the buyer of reinsurance the losses in excess of a specified amount caused by a catastrophic event or a series of them.
2. *Hedging.*
   a. *Catastrophe bond.* By issuing a CAT bond, the catastrophe risk can be transferred from the issuer to investors.
   b. *Catastrophe equity put.* The insurer can sell its stock to investors at a predetermined price if the catastrophic losses exceed a prespecified threshold. The insurer will have more financial resources to absorb the catastrophic loss.
3. *Risk avoidance.* When writing business, measures can be taken to limit the exposure to catastrophe risks such as not writing business in catastrophe-prone regions.
4. *Risk diversification.* Geographic diversification of businesses can also help reduce the catastrophe risk at the aggregate level.

5.4 Hedging Strategies

Many hedging strategies can reduce tail risk. It is useful to know the available options even though their effectiveness need to be analyzed before using them. A brief introduction of these strategies is given below.

Economic Risk Hedging

1. *Equity put option.* The buyer of an equity put option can sell the equity at the strike price if the actual equity price is lower than the strike price. Therefore, the buyer is protected from the price dropping below the strike price. Sometimes, because of the high cost of equity put options, the buyer may sell call options at a higher strike price to reduce the cost. The buyer gives up some upside potential for downside protection.
2. *Volatility swap/variance swap.* Volatility swap is a forward contract that allows investors to trade the volatility of an asset. The underlying asset could be a foreign exchange rate, interest rate, or equity index. Variance swap is similar to volatility swap except that variance, the square of volatility, is traded. Variance swap is more popular than volatility swap because variance swap is easier to replicate. The payoff of the variance swap is the notional amount × (realized variance – variance strike). The realized variance is the annualized variance calculated based on the actual prices at some prespecified time points. For example, daily close prices during a six-month period can be used to determine the realized variance of the asset return. Volatility swap and variance swap focus on the volatility of the asset prices but not the direction of movement. But during a financial crisis, the volatility is usually much higher. Therefore, these two types of swaps can be used to hedge tail risk as well.

3. *VIX option/futures.* Like volatility swap and variance swap, VIX options and futures are another two ways to trade on the volatility of the equity market. VIX is a volatility index developed by the Chicago Board Options Exchange that tracks the implied volatility based on the prices of options on the S&P 500 index. The options and futures on VIX can be used to protect the buyer from the increase in the VIX and therefore the volatility of the stock market.
4. *Credit derivative.* A credit default swap (CDS) or total return swap can help hedge the credit risk of an obligor. The loss caused by a credit default can be fixed by buying a CDS. But the market is thin, and it is not appropriate for hedging at a macrolevel. Credit default swap indices are constructed from a group of obligors, and the market is more liquid. They are more suitable for macrohedging.
5. *Sovereign risk hedging.* Sovereign risk can cause significant loss as evidenced in the financial crisis started in 2007. Yet it is difficult to hedge. When a sovereign crisis happens, money will fly to high-quality assets such as Treasury bonds or gold. The currency of the stressed economy is likely to depreciate. Therefore, a long position in the high-quality assets and a hedge on the foreign exchange rate help offset the loss caused by a sovereign crisis.
6. *Asset allocation based on tail risk.* To hedge the extreme economic risk, a traditional asset allocation method based on expected mean and volatility can be changed so that the risk measure focuses on the left tail of the return distributions. Tail risk–based asset allocation can reduce the expected return but also protect the company from the tail losses. However, whether this is the best choice for business management depends on the risk appetite of the organization. Alankar et al. in their report "An Introduction to Tail Risk Parity" (2012) introduce an asset allocation method based on expected tail losses. It can serve as an alternative to financial derivatives for hedging economic tail risk.
7. *Tail risk index.* Some investment banks also provide sophisticated tail risk protection products that dynamically adjust the investment according to the market volatility structure. A good example is the tail hedging algorithm explained by Tom et al. (2014) in their article "Tails, You Lose: Making Sense of Tail-Hedging Indexes." The equity dynamic tail hedge index purchases S&P 500 index put options dynamically. It reduces the downside hedge in normal markets and increases the hedge if a tail event is anticipated. The anticipation of tail event depends on the skew and CDS spreads. The skew is the difference between the implied volatility of options with lower strike price and the implied volatility of options with higher strike price. The skew and CDS spreads normally go high in a bear market. This type of tail risk index can be used for hedging economic tail risk.

Insurance-Linked Securities

8. *CAT bond.* Catastrophe bonds help transfer catastrophe risk from an issuer to investors. A CAT bond usually has a high coupon rate. CAT bonds are rated below investment grade in most cases. If no catastrophic event happens, investors will get their principal back. If a catastrophic event happens and the loss trigger the principal impairment, investors will lose the principal. The issuer can then use the principal to pay for the claims caused by the catastrophic event.
9. *Extreme mortality securitization.* It helps transfer catastrophic mortality risk that may be caused by, for example, pandemics, natural disasters, or terrorism. The buyer will pay the principal to the seller if the mortality experience is worse than the predetermined level in the contract.
10. *Longevity swap.* By paying fixed premiums and receiving floating payments, longevity risk can be hedged. The floating payments are higher if people live longer.
11. *Longevity bond.* It can be used to transfer longevity risk as well. Like a CAT bond, the principal will be used by the issuer to pay for the cost of longevity if the experience is worse than the specified level in the contract.
12. *Industry loss warranty.* It pays the buyer a fixed amount if the total loss caused by an event, normally a hurricane, exceeds a predetermined amount. It can help insurer reduce its risk exposure to weather disasters.

Others

13. *Catastrophe equity put.* Like a CAT bond, a catastrophe equity put can hedge the loss caused by catastrophe events. It gives insurers the rights to sell its shares to investors at a predetermined price if the catastrophe loss exceeds the specified amount.
14. *Contingent capital.* As a type of hybrid security, contingent capital is an innovative way of recapitalization given the occurrence of a specified event, such as the capital adequacy ratio falling below the threshold. The debt is either written down or converted to equity at a predetermined price. It provides additional financial resources at a fixed price to absorb the loss caused by extreme events. In addition, it does not differentiate the type of risk that causes the worsening capital position and therefore can be used to hedge the tail risk at the aggregated level. It is similar to the catastrophe equity put except that the triggers are different. Section 6.4 contains an example of using contingent capital to reduce systemic risk.
15. *Contingent liquidity swap.* It is a swap agreement that the buyer of the swap pays cash in exchange for less liquid assets in the event of a liquidity event. Those less liquid assets may include stocks or high-yield bonds. The seller of the CLS makes a series of payments for the contingent liquidity. It helps provide additional liquidity and reduce liquidity cost when a liquidity crisis occurs.

With further development of the financial market, more and more extreme risks can be transferred to the capital market through securitization. Therefore many new hedging strategies may emerge in the future.

When choosing a hedging strategy, the cost of hedging is an important factor. It is affected by the liquidity of the market and the timing of implementation. Time horizon, size of the hedging, basis risk, and counterparty risk also have an impact on the effectiveness of a hedging strategy and need to be considered.

5.5 Emerging Risks

Emerging risks are risks that are new, quickly evolving, or unexpected, but are not well understood. Because of the lack of knowledge and lack of experience data, it is hard to estimate the impact of emerging risks. However, a few things still can be done to improve the assessment of emerging risks. Many organizations such as the OECD, the CRO Forum, and the SOA publish research reports on emerging risks. They can be used to identify potential extreme risk events and assess their impact on the business. These potential extreme scenarios covers economic, geopolitical, societal, environmental, and technological categories. Instead of relying on the historical experience, these scenarios are forward-looking incorporating new information.

For some emerging risks, it is difficult to quantify their potential impact. An alternative is to rely on the impact of historically unexpected events. For example, the impact of the 2008 financial crisis that were not expected in the risk models can be used as an estimate of a future unexpected extreme economic event. The same thing can be done for other risk categories.

6. Examples

This section contains examples illustrating how to apply several of the tail risk modeling and mitigation techniques presented throughout this paper. These examples are provided for illustrative purposes only and may reflect simplified views of reality.

6.1 Step-by-Step Block Maxima Example: U.S. Tornado Deaths

In this example we model the distribution of deaths arising from the most extreme historical tornado events in the United States. Since it is often the “big one” that is of most concern to an insurer and/or reinsurer, especially if coverage is spatially concentrated, we focus here on the modeling of singular extreme events, rather than on the modeling of cumulative impacts. Our dataset is based on publicly available information from the NOAA’s National Weather Service Storm Prediction Centre[11] and was constructed by determining the number of fatalities associated with the most deadly U.S. tornado annually from 1991 to 2013 inclusive. Table 6.1 summarizes the data compiled by the authors. Note that this dataset has 23 blocks (each represented by a year) with the event count (number of deadly tornados) per block ranging from 10 in 2009 to 59 in 2011. Yearly periods were chosen to avoid seasonal effects. The practitioner should keep in mind that the aforementioned results for the GEV distribution assume a large

[11]http://www.spc.noaa.gov/climo/torn/fatalmap.php?yr=1991.

dataset (in terms of both size and number of blocks), so to the extent one is dealing with a smaller dataset, as we are here, caution is warranted.

Table 6.1 US Tornado Fatalities (1991 to 2013)

Year	No. of Deadly Tornados	Total No. of Deaths	No. of Deaths for Most Severe Event (Max)	Year	No. of Deadly Tornados	Total No. of Deaths	No. of Deaths for Most Severe Event (Max)
1991	15	39	17	2003	16	54	11
1992	16	39	12	2004	19	35	8
1993	16	33	7	2005	12	38	24
1994	22	69	22	2006	25	67	16
1995	15	30	6	2007	26	81	13
1996	13	26	7	2008	37	126	22
1997	23	68	27	2009	10	22	8
1998	32	130	32	2010	21	45	10
1999	30	94	36	2011	59	553	158
2000	13	41	11	2012	22	69	11
2001	23	40	6	2013	14	55	24
2002	25	55	7				

The R package *ismev* provides the basic tools to fit the GEV/GPD to one's data, produce the associated diagnostic plots, and calculate quantiles. Below is a six-step procedure for analyzing this dataset in the software package R. Lines that begin with ">" indicate code that can be entered directly into the R command line to execute the specified action. This example leverages R version 3.1.2 and version 1.39 of the *ismev* package.

Step 1: Identify Event of Interest, Variable of Interest, and Block

- Event of Interest: Deadly Tornados in the United States
- Variable of Interest: Number of Deaths per Deadly Tornado
- Block: Calendar Year

Step 2: Construct and Import Extreme Value Dataset

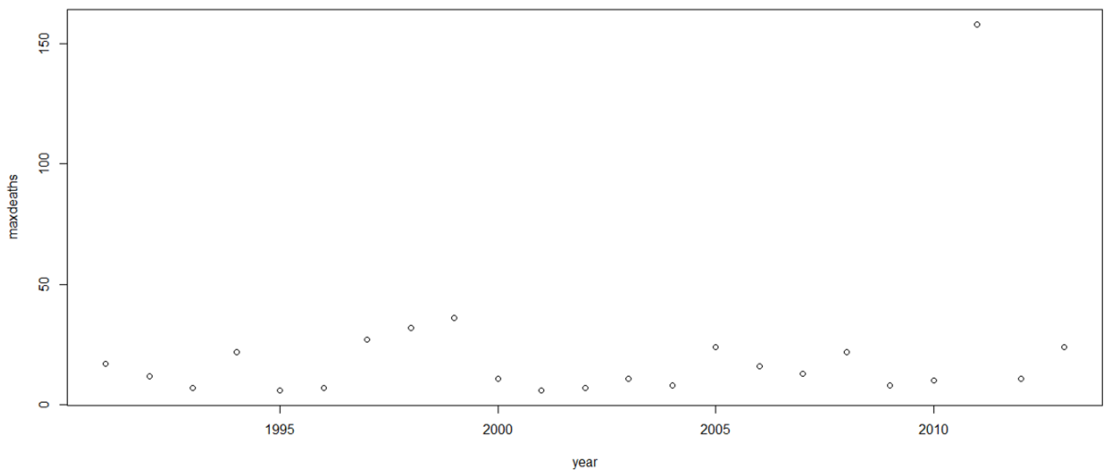
- Construct a comma-separated value (CSV) file called *US_Tornado_Deaths.csv* with two columns, the block (*year)* and the maximum of the variable of interest within each block (*maxdeaths).* For example, of the 59 tornados in 2011, the most severe one killed 158 people, so for 2011 *maxdeaths* is recorded as 158.

```
year,maxdeaths
1991,17
1992,12
1993,7
1994,22
1995,6
1996,7
1997,27
1998,32
1999,36
2000,11
2001,6
2002,7
2003,11
2004,8
2005,24
2006,16
2007,13
2008,22
2009,8
2010,10
2011,158
2012,11
2013,24
```

- Import the file into R, replacing {Path} with the location of the csv file:
  - ➢ mydata <- read.table("{Path}/US_Tornado_Deaths.csv", header=TRUE, sep=",")

Step 3: Visualize Data to Confirm Absence of Trend

- ➢ plot(mydata)



- Through this plot we are able to confirm that our maxima have no obvious trend. Consequently, it is reasonable to assume that the maxima are independent of one another, in which case we can proceed with fitting the GEV distribution to the data. Had we observed a trend, more complex covariate modeling would have been required.

Step 4: Fit GEV Distribution to Data

- Install the R package *ismev* by navigating through the following menus in R: Packages → Install package(s)… → {Select closest site} → ismev
- Load the ismev package and then fit the GEV distribution to the data
  - ➢ library(ismev)
  - ➢ fit <- gev.fit(mydata$maxdeaths)

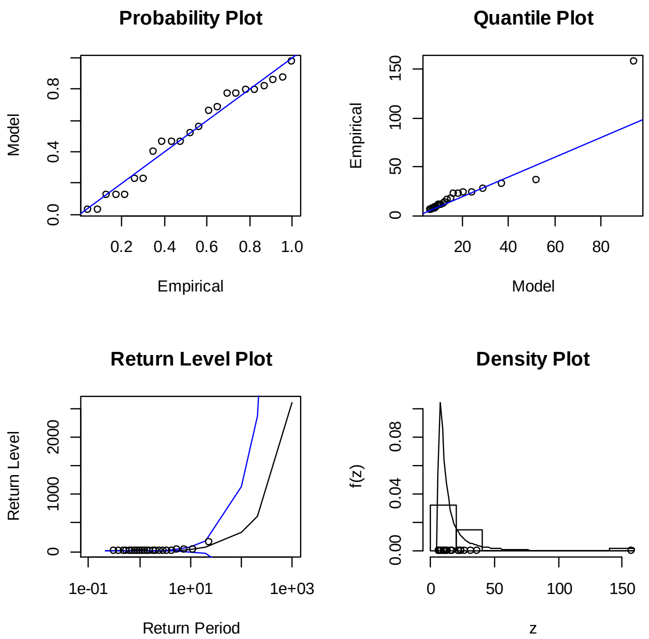
- The following parameter estimates based on Maximum Likelihood Estimation along with their standard errors are automatically output by the *gev.fit* function:

	μ	σ	ξ
MLE	9.61	4.82	0.89
Standard Error	1.26	1.50	0.34

- *Approximate* 95 percent confidence intervals can be obtained by adding or subtracting 1.96 times the standard error from each of the point estimates, returning [7.14, 12.08] for μ, [1.88, 7.76] for σ, and [0.22, 1.56] for ξ. More accurate estimates of the confidence intervals can be obtained through the use of profile likelihood.
- Note that since the estimate of ξ is greater than zero and its approximate 95 percent confidence interval lies entirely above zero, we conclude that the distribution of *maxdeaths* is heavy-tailed and most consistent with the Frechet distribution.

Step 5: Review Model Fit Diagnostics

- The function *gev.diag* automatically returns four key diagnostic plots: the Probability Plot, Quantile Plot, Return Level Plot, and Density Plot.
    - ➢ gev.diag(fit)

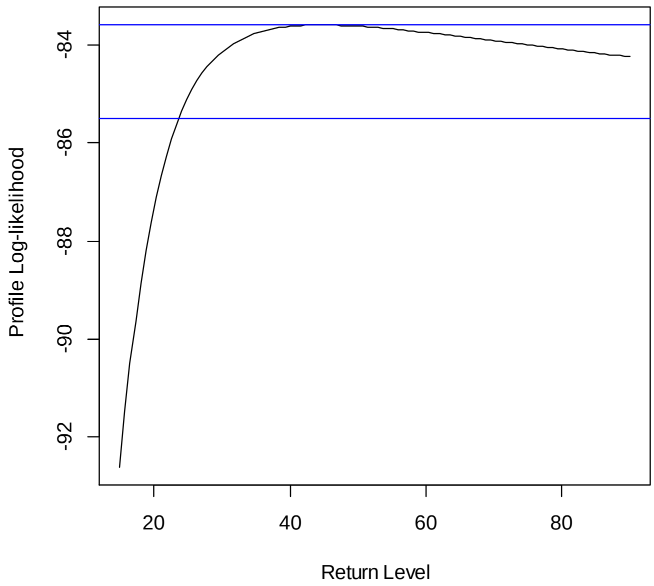


- Although the Probability Plot supports our model fit in that the points are roughly linear, the Quantile Plot and Density Plot suggest that the model distribution is unable to replicate the behavior shown by the two largest *maxdeaths*. The Density Plot overlays the model distribution (in this case Frechet) on the histogram of data.
- The Return Level Plot provides the practitioner with information on the quantiles of the fitted distribution. If we let R^k denote the return level for a return period of *k*-

periods (where one period equals one block size), then R^k can also be thought of as the $(1 - 1/k)$ quantile of the fitted distribution. For example, if $R^{10} = 20$, it means that the 90th percentile of the distribution is 20, or equivalently that we expect to see only an extreme beyond 20 once every 20 years.

Step 6: Determine Quantiles of Interest

- The function *gev.prof* enables the practitioner to produce a profile log-likelihood for return level given a specified return period. The return level that corresponds to the peak in the profile log-likelihood is the best estimate return level.
- For example, if we wanted to calculate the 90th percentile of the fitted *maxdeaths* distribution, we would execute the command below, where the second argument represents the return period (recall that a return period of 10 years is equal to the 90th percentile), and the third and fourth arguments represent the smallest and largest values at which to evaluate the profile log-likelihood.
  - ➢ gev.prof(fit, 10, 15, 90)

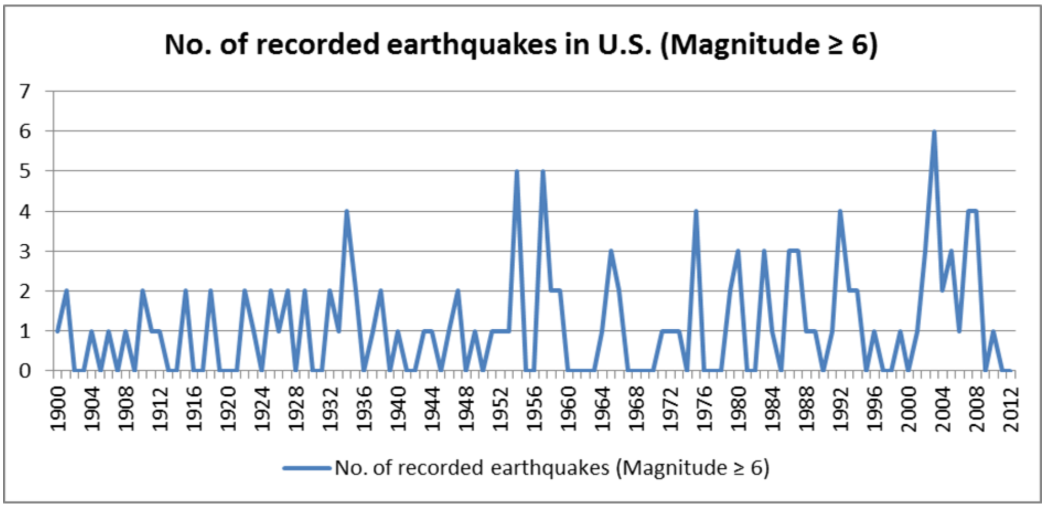


- Based on this plot, we conclude that the 90th percentile of our *maxdeaths* distribution is approximately 44.5. In other words, once every 10 years, we expect a major U.S. tornado to result in more than 44 deaths.
- Note that since our tornado dataset contained only 23 annual blocks, it is not advisable to extrapolate beyond the 95th percentile (1 – 1/23 = 0.957).

6.2 Hidden Markov Model Example: U.S. Earthquake Frequency

Figure 6.1 shows the historical annual frequencies of earthquakes in the United States with a magnitude of no less than six. A magnitude of six means that the earthquake is very strong with an estimated frequency of 1 in 100 years[12]. Volatile cycles can be seen from the historical magnitudes of U.S. earthquakes.

Figure 6.1 Number of Recorded Earthquakes in the United States with Magnitude Greater than or Equal to 6 (1900–2012)



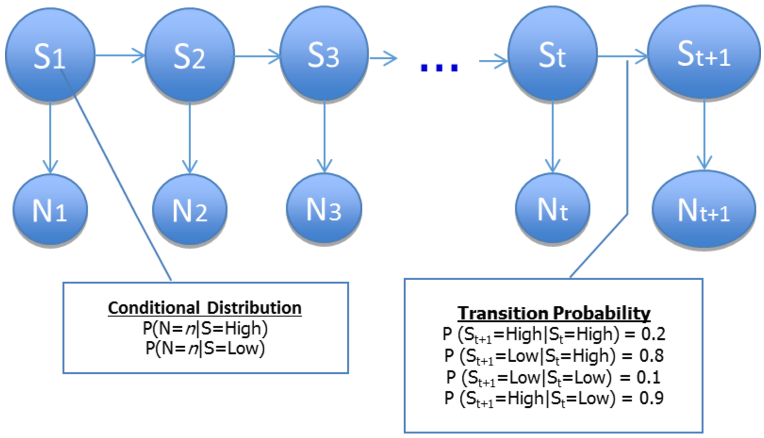
Data source: U.S. Geological Survey: Historical World Earthquakes

In this example, the earthquake frequency in the United States with a magnitude no less than six is modeled using an HMM for illustration. In practice, the historical insurance loss caused by earthquakes may be more appropriate for modeling the cyclical effect of earthquakes using HMM. The purpose of this example is to show how the HMM can be applied but does not indicate that the HMM is appropriate for the same issues in practice. We further caution that in practice the analysis would need to be done at a more granular level than presented here because the location of the earthquake has a material impact on the insurance loss.

In different phases of crustal activity, the distributions of earthquake frequency are different. There are two states assumed: "High" frequency and "Low" frequency. The annual frequency of earthquakes with a magnitude of no less than six can be any integer from 0 to 6. Figure 6.2 illustrates the model structure.

[12] Information on earthquake magnitude scale and the frequency and effects of each magnitude can be found at http://en.wikipedia.org/wiki/Richter_magnitude_scale.

Figure 6.2 HMM Structure Example



Based on historical U.S. earthquake frequency data from 1900 to 2012, the Baum-Welch algorithm[13] is applied, and the calibrated HMM is given in Table 6.2. Note that the calibrated conditional probability of having three earthquakes is 4.9 percent in a "Low" state and 0 percent in a "High" state. This result is counterintuitive as we would expect the "High" state to have a higher conditional probability than the "Low" state. The reason is that the chance of having three earthquakes is small based on the historical data, and when it did happen it was close to other small-frequency years. A longer period of data or a different grouping of data is likely to solve this problem.

Table 6.2 HMM Calibration

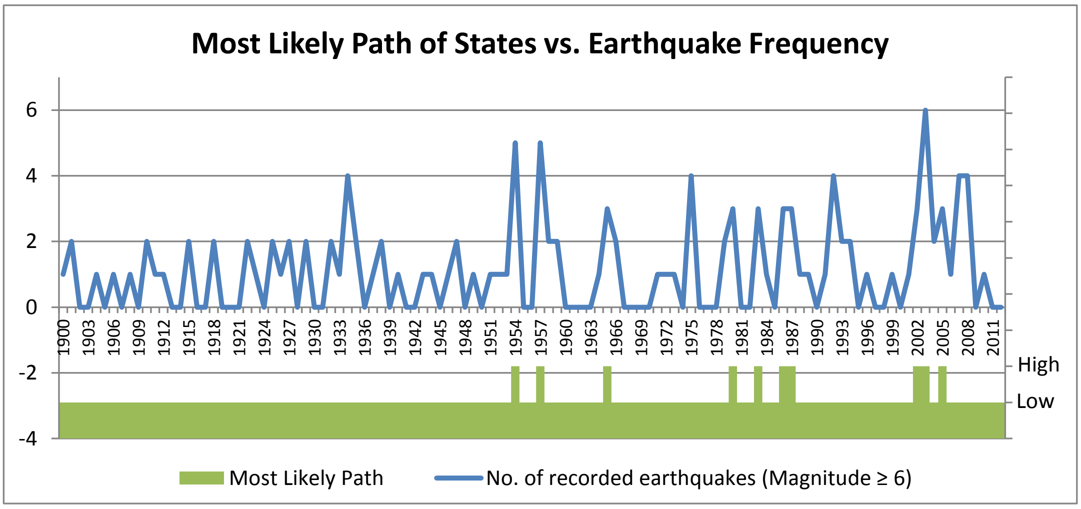
Two States	"H": High, "L": Low		
Initial State Probability	$\pi_H = 20\%$, $\pi_L = 80\%$		
Transitional Probability	Probability	S_{t+1} = "H"	S_{t+1} = "L"
	S_t = H	**20.0%**	**80.0%**
	S_t = L	**7.8%**	**92.2%**
Earthquake Frequency	0, 1, 2, 3, 4, 5, and 6, denoted as a, b, c, d, e, f, and g, respectively.		
Conditional Distribution of Earthquake Frequency	Probability	S_i = "H"	S_i = "L"
	$P(a \mid S_i)$	0.0%	46.6%
	$P(b \mid S_i)$	0.0%	30.1%
	$P(c \mid S_i)$	0.0%	18.4%
	$P(d \mid S_i)$	69.9%	0.0%

[13]Details can be found at http://en.wikipedia.org/wiki/Baum%E2%80%93Welch_algorithm.

	$P(e \mid S_i)$	0.0%	4.9%
	$P(f \mid S_i)$	20.1%	0.0%
	$P(g \mid S_i)$	10.0%	0.0%

Using the Viterbi algorithm[14] widely used in dynamic programming, the most likely path of states can be found. Figure 6.3 shows the most likely path against the historical earthquake frequency. The path fits the historical earthquake frequency but with some uncertainty. For some years with a high number of earthquakes, the state is "Low."

Figure 6.3 Most Likely Path for a U.S. Earthquake with Magnitude Greater than or Equal to 6 (1900–2012)



With the calibrated HMM model, the future distribution of earthquake frequency can be simulated. We will test two scenarios: (a) where the current state is "High" and (b) where the current state is "Low." Given the updated HMM, the state and frequency for the next period are simulated. For example, if the current stat is "High," the simulator will first simulate the state of the next period based on the transition matrix. After that, the model will simulate the number of earthquakes based on the conditional distribution given the state. Simulation results are summarized below.

[14] Details can be found at http://en.wikipedia.org/wiki/Viterbi_algorithm#History.

Table 6.3 Summary of the HMM Simulations versus Historical Data

Statistic	No. of Earthquakes		
	Historical Experience	Simulation: Initial State Low	Simulation: Initial State High
Average	1.12	1.043	1.369
CTE(50)	2.07	2.008	2.558
CTE(70)	2.79	2.68	3.34
CTE(80)	3.17	3.065	3.87
CTE(90)	4.09	3.96	4.74
CTE(95)	4.67	4.56	5.36
CTE(99)	6.00	5.7	6

*CTE(X): conditional tail expectation at the *X*th percentile.

In general, we can see that the CTE values of the simulated paths starting in the high state are higher than those of the historical experience. For extreme risk modeling and management, the simulated paths starting in the high state are the most useful of the three datasets because they more fully capture the potential adverse outcomes.

6.3 Extreme Liquidity Event Testing

Liquidity stress testing is a difficult task because of the uncertain evolvement and outcome of a liquidity event. As mentioned in a BIS report titled "Liquidity Stress Testing: A Survey of Theory, Empirics and Current Industry and Supervisory Practices" (2013), "liquidity risk modelling still remains in its infancy, especially in macro stress tests." For insurers, the fundamental method of assessing liquidity adequacy is using cash flow testing. The cash flows from the asset portfolio and the liability portfolio are compared to assess the liquidity adequacy under different scenarios, expected or stress. However, in a liquidity crisis, the liquidity requirement may increase sharply in a short period, which is not covered in a standard cash flow testing scenario. In addition, the liquidity stress testing is usually done on an individual basis without considering the impact of other players in the market. In a systemic event, the contagion among counterparties makes the liquidity issue more complicated. With these missing pieces, liquidity stress testing result could be distorted. The example below shows a more advanced yet simplified liquidity stress testing for an insurance company.

ABC Life Insurance Company sells two types of products: (a) long-term guaranteed insurance products that provide a death benefit, guaranteed surrender benefit, and maturity benefit; and (b) variable annuities that offer a guaranteed death benefit and guaranteed minimal withdrawal benefit. The risk committee is reviewing its company's liquidity risk profile. One of the goals is to understand bad liquidity scenarios that the company may face and prepare contingent plans for these scenarios.

The balance sheet and expected cash flows of the company are given in Tables 6.4 and 6.5.

Table 6.4 Simplified Balance Sheet of Company ABC

Assets ('000)		Liabilities ('000)	
Cash	1,000	Reserve - Guaranteed Products	90,000
Treasury Bond	50,000	Reserve - Variable Annuities	40,000
Sovereign Bond	15,000	Long-term Debt	40,000
Aa-rated Corporate Bond	45,000	**Total Liabilities**	**170,000**
Baa-rated Corporate Bond	55,000	**Sharholder's Equity ('000)**	
Stock	50,000	**Total Equity**	**46,000**
Total Asset	**216,000**	**Total Liabilities & Equity**	**216,000**

Table 6.5 Expected Cash Flows of Company ABC (Six Months)

Month	Inv. Inc. & Redemptions	Premium Income	Debt Repayment	Benefit Payment	Expense	Net Cash Flow
1	1,100	650	400	1,000	300	50
2	1,100	650	400	1,000	300	50
3	2,600	650	400	1,000	300	1,550
4	1,100	650	400	1,000	300	50
5	1,100	650	400	1,000	300	50
6	1,100	650	400	1,000	300	50

For Company ABC, an insurance risk–driven liquidity crisis could be caused by, for example, a catastrophic event, terrorism, or pandemic that leads to deaths and significant economic losses. The company may need to pay a huge amount of death benefits and surrender benefits. A systemic economic crisis could be even worse because of its additional impact on the asset side. The risk committee focuses on these two types of liquidity crisis and wants to test their impact. Two stress scenarios were constructed, and their impact on the company's liquidity is studied.

Scenario 1: 1918–1919 Spanish flu

The "Spanish flu" in 1918–1919 caused about 50 million deaths worldwide. The death rate was about 28.6 per thousand compared to 7.8 per thousand for the world population in 2014. About 500 million people were infected, which was about 29 percent of the world population. Because of the lack of economic data for those years and World War I, it is difficult to estimate the impact of the Spanish flu on economies. According to the study "Economic Effects of the 1918 Influenza Pandemic" by Garrett (2007), the economic impact of the Spanish flu was short term. The shortage of labor resulted in higher wages, and some industries such as services and entertainment experienced contractions, whereas other industries such as health care experienced a growth.

If a similar pandemic event were to happen today, a direct impact on Company ABC is the increased death claims. The general impact of the 1918 Spanish flu is a 270 percent increase from normal death rate (28.6/7.8−1). It can be applied to the expected death rate. In practice, some refinements can be made. The death rate from the Spanish flu can be compared to the death rates in the 1910s instead of 2014. The adjustment can be done by age because the death rate increase was much higher for younger people aged between 15 and 34. This better suits the company where the age distribution of the clients is different from the age distribution of the population. However, other types of pandemic flu may have a different pattern regarding the relationship between age and death rate and could be tested. The location of the clients can also help improve the accuracy of the estimation. People who live in big cities are more likely to be infected. Unlike in 1918, new developments may change the death rate to a different extent. Some pandemic flus spread much faster than previously. For example, the 2009 swine flu spread about three times faster than past pandemics. Air transportation is certainly a big contributor to this and may lead to more death claims in a shorter time. Medical advances may make such an extreme pandemic less likely today. For this analysis, a flat 270 percent increase in the death rate is assumed for six months. In addition, with the expectation of a widespread pandemic and the increase in medical expenses, clients may surrender their long-term guaranteed product for cash surrender value or apply for policy loans. A 20 percent increase in the surrender benefit payments is assumed. Given the changed expectation of the mortality rate and lapse rate during the pandemic, the reserve is also changed. A 3 percent increase of the reserve is assumed in this stress scenario.

No concrete conclusion can be made on the impact of pandemic flu on the economy for pandemics in recent history. A short-term turbulence is likely to happen but be quickly followed by a recovery. Therefore, it is reasonable to assume a small impact on the asset side. For a more severe event such as the Black Death in fourteenth-century Europe, which killed more than 30 percent of the population, the impact on economies was significant. A 5 percent value reduction of the bond and equity portfolio is assumed. A 20 percent reduction of the stock dividend payment is assumed as well. In practice, a more detailed analysis of the exposure of the bond issuers and the stock companies to the pandemic can be conducted for a more realistic estimation.

People are also willing to purchase life insurance products that provide high death benefits and health insurance. At that same time, they may reduce the premium for investment-type products such as the variable annuities that ABC sells. The new premium income is assumed to be unchanged in this example.

With a 5 percent reduction in the noncash assets and 3 percent increase in the reserve, the shareholders' equity value drops from $46 million to $31 million. The projected cash flows in the next six months under the stress scenario are shown in Table 6.6. Benefit payments including both death benefit and surrender benefit increased by 170 percent. The third month investment income decreased because of reduction of the stock dividend. The total net cash flows for the next six months are estimated to be $8.7 million. This further reduce the shareholders' equity from $31 million to $23 million. The company has $1 million cash. The company needs to sell noncash assets for the remaining $7.7 million cash payments. It may want to retain some cash and sell more noncash assets to maintain the liquidity of the asset portfolio. On average, the company needs to sell less than $1.5 million in assets each month. As long as the company holds a diversified asset portfolio, it will not have a material market impact.

Table 6.6 Expected Cash Flows of Company ABC (Six Months): Spanish Flu

Month	Inv. Inc. & Redemptions	Premium Income	Debt Repayment	Benefit Payment	Expense	Net Cash Flow
1	1,100	650	400	2,700	300	-1,650
2	1,100	650	400	2,700	300	-1,650
3	2,300	650	400	2,700	300	-450
4	1,100	650	400	2,700	300	-1,650
5	1,100	650	400	2,700	300	-1,650
6	1,100	650	400	2,700	300	-1,650

The projected balance sheet after six months considering the net cash flows is shown in Table 6.7.

Table 6.7 Simplified Balance Sheet of Company ABC: Spanish Flu

Assets ('000)		Liabilities ('000)	
Cash	1,000	Reserve - Guaranteed Products	92,700
Treasury Bond	45,477	Reserve - Variable Annuities	41,200
Sovereign Bond	13,643	Long-term Debt	40,000
Aa-rated Corporate Bond	40,929	**Total Liabilities**	**173,900**
Baa-rated Corporate Bond	50,024	**Sharholder's Equity ('000)**	
Stock	45,477	**Total Equity**	**22,650**
Total Asset	**196,550**	**Total Liabilities & Equity**	**196,550**

The equity is estimated to be wiped out by 50 percent. However, the company can still survive, and there is no severe liquidity issue. Given the analysis, some actions can be taken to mitigate the risk in this stress scenario:

1. The company can invest more in the health care industry, which is likely to boom after a pandemic occurs.
2. The company needs to have a plan to raise capital once such an extreme pandemic happens.
3. The company may use extreme mortality securitization or reinsurance to transfer the risk caused by a high death rate.

Scenario 2: Economic Crisis

An economic crisis such as the credit crisis that began in late 2007 can cause more severe problems than a pandemic flu. In such an environment, almost all market participants strive to get liquidity and capital at the same time, which drives up the liquidity cost significantly. The worst economic crisis in the twentieth century was the Great Depression in the 1930s and 1940s. During the Great Depression, the highest annual default rate was 8.4 percent in 1933. In the financial crisis that began in 2007, the highest annual default rate was 5.4 percent in 2009. The government intervention during the Great Depression was very different from and much less mature than what was used nowadays. For stress testing purposes, it is more appropriate to rely on the latest financial crisis, which was the second worst economic crisis in the past 100 years. Using the 2007–2009 financial crisis as the basic reference, Company ABC will be affected by an economic crisis in the following aspects:

1. *The value reduction of the stock holdings.* The S&P 500 Index value dropped by 57 percent from October 9, 2007, to March 9, 2009. The decrease does not happen evenly across time. It dropped by 47 percent in six months from May 20, 2008, to November 20, 2008, and 27 percent in one month from September 10, 2008, to October 10, 2008. It can significantly reduce the asset value and therefore available liquidity. To consider a six-month period liquidity condition, it is assumed that the value of Company ABC's stock holdings will decrease by 47 percent in six months. It drops 4 percent each month except in one month when a 27 percent drop is expected. The dividend income in the third month is also removed in the cash flow analysis. In practice, many refinements can be made. The beta of the stock portfolio that represents the level of systemic risk can be computed to adjust the magnitude of the value reduction. The analysis can also be performed at a more detailed level that analyzes the performance of each stock or each industry.
2. *The value reduction of the bond portfolio.* The decrease of the bond portfolio's value can be caused by two factors: (1) the increase in the bond yield and (2) the default of payments.
   a. *Corporate bond portfolio*

      Using Moody's Baa-rated bond yield as a benchmark, the total yield increased by 2.48 percent from May 2008 to November 2008. After November 2008, the bond yield dropped because of government intervention, which pushed down the Treasury bond yield significantly. Assuming the corporate bond portfolio has a duration of 10 years, an increase of 2.48 percent in the bond yield will cause an approximate 24.8 percent decrease in the value of the corporate bond portfolio. For a more detailed analysis, the change of the bond yield by credit ratings and industry can be used. In addition, some bond payments may default. The default rate is higher and the recovery rate is lower during the crisis. In 2008 the Aa-rated bond default rate is 0.556 percent, compared to 0 percent in 2003 to 2007. The Baa-rated bond default rate is 0.472 percent, compared to an average of 0.036 percent in 2003 to 2007. Using the lower bond recovery rate experienced in 2008, the loss due to default is 0.341 percent for an Aa-rated bond portfolio and 0.301 percent for a Baa-rated bond portfolio (Moody's 2011). The same level of reduction is also applicable to the coupon payments and redemptions.
   b. *Sovereign bond portfolio*

      The European debt crisis, as part of the financial crisis, caused a significant increase in the sovereign bond yield. The Greek government experienced the worst case where its 10-year government bond yield jumped from 6.46 percent in February 2010 to 29.24 percent in February 2012. In addition, the sovereign-debt restructuring in March 2012 caused the bondholders to lose about 50 to 75 percent of the bond value. For Company ABC, a 50 percent reduction in the value of sovereign bond portfolio is assumed in the example. In practice, other factors such as the issuer of the sovereign bonds and their default probability and loss given default can be considered for a more accurate estimation.

3. *The decrease of government bond yield.* After November 2008, the 10-year government bond yield decreased by 1.75 percent in two months. Its impact on the company's equity is twofold. Existing government bond portfolio's value will increase. However, the new investment return will decrease, and the liability reserve will increase.
4. *Credit rating downgrade or credit default.* The net impact of a bear equity market, a higher default rate and credit spread, and a lower interest rate can significantly devour shareholders' equity and trigger a credit rating downgrade or default. Policyholders are more likely to surrender their policies for fear of the credit issue the insurer faces. Company ABC's long-term guaranteed products provide guaranteed cash surrender value. Policyholders may line up to surrender their policies after a rating downgrade announcement. It is assumed that there a 400 percent increase will occur in the surrender benefit payments. In addition, new premium income may become smaller as well.

Based on the financial crisis that began in 2007, a stress scenario is constructed to test the liquidity risk, as shown in Table 6.8. It starts with a stock market crash and high default rates followed by a low interest rate environment and sovereign bond defaults.

Table 6.8 Stress Economic Scenario

Month	Stock Return	Δ 10Y TB Yield	Δ Credit Spread	Credit Default Loss (Aa-rated)	Credit Default Loss (Baa-rated)	Sovereign Bond Loss Ratio
1	-4%	0%	0.83%	0.06%	0.05%	5%
2	-27%	0%	0.83%	0.06%	0.05%	5%
3	-4%	0%	0.83%	0.06%	0.05%	5%
4	-4%	-0.875%	0.00%	0.06%	0.05%	5%
5	-4%	-0.875%	0.00%	0.06%	0.05%	20%
6	-4%	0.000%	0.00%	0.06%	0.05%	20%

Δ: monthly change

The cash flow projection in the next six months under the stress scenario is shown in Table 6.9. A small haircut is made for investment income and redemptions due to the payment defaults. Premium income, debt repayment, and expense are assumed to be unchanged. The benefit payment increases materially because of the increased surrender benefit payment.

Table 6.9 Cash Flow Projection: Economic Crisis

Month	Inv. Inc. & Redemptions	Premium Income	Debt Repayment	Benefit Payment	Expense	Net Cash Flow
1	1,099	650	400	2,600	300	-1,551
2	1,099	650	400	2,600	300	-1,551
3	1,099	650	400	2,600	300	-1,551
4	1,099	650	400	2,600	300	-1,551
5	1,099	650	400	2,600	300	-1,551
6	1,099	650	400	2,600	300	-1,551

This projection assumes no need to raise additional capital, and the company is still solvent. However, by taking a look at the projected balance sheet, shareholders' equity becomes zero in the third month, and additional capital needs to be raised. However, at that time it may be very difficult to raise capital in the market because in such a systemic event, the cost of capital raising can be very high. In addition, at the end of the second month, the shareholders' equity has dropped by 77 percent. It is likely that the company may have insufficient capital to meet regulators' requirements and rating agencies' expectations. Regulators' intervention and a credit rating downgrade is expected in this situation. Given the size of the company, it is not a systemic important financial institution and is unlikely to get the government's financial support. Therefore, Company ABC will fail to fulfill its obligations if such an extreme economic crisis happens.

Table 6.10 Balance Sheet Projection: Economic Crisis

Month	0	1	2	3	4	5	6
Cash	1,000	500	500	500	500	500	500
Treasury Bond	50,000	48,949	47,399	45,848	48,309	50,986	49,435
Sovereign Bond	15,000	14,250	13,538	12,861	12,218	9,774	7,819
Aa-rated Corporate Bond	45,000	41,254	37,821	34,673	34,653	34,633	34,614
Baa-rated Corporate Bond	55,000	50,426	46,232	42,387	42,366	42,344	42,323
Stock	50,000	48,000	35,040	33,638	32,293	31,001	29,761
Total Assets	**216,000**	**203,380**	**180,529**	**169,907**	**170,338**	**169,239**	**164,452**
Reserve - Guaranteed Products	90,000	90,000	90,000	90,000	97,875	106,439	106,439
Reserve - Variable Annuities	40,000	40,000	40,000	40,000	43,500	47,306	47,306
Long-term Debt	40,000	40,000	40,000	40,000	40,000	40,000	40,000
Total Liabilities	**170,000**	**170,000**	**170,000**	**170,000**	**181,375**	**193,745**	**193,745**
Equity	**46,000**	**33,380**	**10,529**	**-93**	**-11,037**	**-24,507**	**-29,293**

Company ABC needs to take actions to mitigate its exposure to an economic crisis that can cause an extreme liquidity issue:

1. Company ABC needs to increase its capital materially so that it can survive such an extreme event. In a normal economic environment, it can issue more bonds, sell more shares, or set up prearranged capital such as contingent capital. An example is given in Section 6.5.
2. Company ABC may want to reduce its business on long-term guaranteed products. It may transfer some of the risk through reinsurance or securitization. It may provide lower guaranteed cash surrender value or sell some short-term products that are less vulnerable to economic risk.

Sections 6.4 and 6.5 discuss risk management considerations for the two stress scenarios.

6.4 Managing Extreme Pandemic Events

An extreme pandemic can cause a high infection rate, high death rate, and interruption of regular economic activities. The most extreme pandemic event in the recorded history was the 1918–1919 Spanish flu, with an estimated death rate of more than 3 percent, and one-third of the world population was infected. For insurers, it meant high death claims, health insurance claims, fewer annuity payments, short-term economic turbulence, and potential interruption of regular business operations. Like the pandemic example in Section 6.1, the impact of an extreme pandemic can be assessed considering the following aspects:

1. *Death claims.* The increased death rate will increase the death claims. A 3 percent or higher death rate can be assumed to estimate its impact on the company's business. The speed of spread is also important. The 1918 Spanish flu killed about 25 million people in the first 25 weeks, compared to 50 to 100 million total deaths that occurred in this event. More refined analysis considering the age and location of the insured is also helpful.
2. *Health claims.* Insurance that covers medical costs such as prescription drugs and hospital stays will experience an increase in claims. Assuming that 29 percent of the population get infected, the claim amount can be estimated. Other factors such as the capacity of the medical and hospital system, the speed of infection, the time from infection to death, and the potential subsidy from the government should be considered.
3. *Annuity.* An increase in the death rate will decrease future annuity payouts. For people who survived the 1918 Spanish flu, no evidence suggests that they lived shorter lives than expected. Life expectancy increased steadily after 1918. Research conducted by Yu et al. (2008) showed that the survivors of the 1918 Spanish flu contained neutralizing antibodies in their cells that made them immune to the virus. Therefore, the positive impact on the annuity business comes only from the high death rate due to the pandemic.
4. *Long-term care.* The high death rate could cause higher than expected deaths of people to receive long-term care services. On the other hand, the pandemic infection is unlikely to trigger long-term care services, and no definite conclusion can be made that any severe circumstances were seen after recovery from the pandemic.
5. *Short-term economic turbulence.* Such an extreme event will have a negative impact on economies. Regular economic activities may be disturbed to reduce the chance of infection. However, the aggregate impact on the economy is unlikely to be very big. Except for the fourteenth-century Black Death, none of the pandemics caused an economic crisis.

6. *Business interruption.* An outbreak of pandemic may interrupt normal business operations. Many companies have set up contingent plans for business interruption to minimize the impact on business.

After assessing the impact of such an extreme pandemic event on the company's business, the company can understand its risk tolerance to extreme pandemic events. Given the changing business profile, the company needs to actively monitor its exposure to extreme pandemic events against its risk tolerance. To calculate the current exposure, a database that records policyholders' information such as the benefit type, benefit amount, location, age, and gender needs to be built. The expected loss for each policy given an extreme pandemic event can be estimated using these information considering, for example, the origin of the outbreak, the speed of spread, the death rate, and the medical cost. By aggregating the individual expected losses, the total exposure is calculated. If it is higher than the risk tolerance, then derisking is needed.

6.5 Managing Extreme Systemic Risk

Continuing with the extreme economic scenario discussed in Section 6.2, the company needs to raise its capital level to be able to withstand an economic crisis similar to the financial crisis started in 2007. As illustrated in Table 6.10, Company ABC's equity value changes from \$46 million to −\$29 million. It is assumed that the company wants to maintain at least an equity value of \$20 million in the extreme scenario. The \$20 million could be determined such that the corresponding capital adequacy ratio and liquidity position meet the minimum requirements of regulators, rating agencies, and investors.

The company has many ways to improve the situation. It can either decrease its risk exposure or get additional financial resources to support the risk. The company can transfer the risk by securitization, reinsurance, or hedging. Reinsurance and hedging normally involve new counterparty risk, which could overwhelm the benefit of risk transfer, especially in a systemic event. Without reducing the risk exposure, Company ABC needs to raise about \$49 million capital. The company can get the additional capital through bank loans, issuing bonds, selling extra shares, or some prearranged source of capital that will be converted to capital in a stress situation. It is less costly to raise the capital in a normal economic environment. Bank loans and bonds tends to increase the liquidity risk because of the requirement of repayment, although they cost less. Selling additional shares will increase the cost of capital. Contingent capital, as a prearranged source of capital, has a lower cost of capital.

Many types of contingent capital are available. In this example, it is assumed the contingent capital is the contingent convertible (CoCo) bond. CoCo bonds have no difference from plain vanilla bonds in normal situations. The buyers receive coupon payments and redemption values. When the company is in a distressed situation, CoCo bonds are automatically converted to equity. The conversion depends on some predetermined conditions such as the company's capital adequacy ratio, equity price, and credit default swap. The buyers will get a higher yield than from an ordinary bond to compensate the risk of conversion. The company gets to use the capital when it is needed with a lower cost than selling additional shares, and with the conversion, it does not create additional liquidity requirement. In the extreme economic crisis scenario example in Section 6.2, the company's equity drops quickly and becomes negative in the third month. Now, assume that the company has issued a $49 million CoCo bond before the crisis. The bond has a coupon rate of 9 percent with quarterly payments. The CoCo bond is converted to equity at the face amount during the third month. Table 6.11 shows the projected balance sheet with the CoCo bond. With this prearranged capital, the company can survive the economic crisis.

Table 6.11 Balance Sheet Projection: With CoCo Bond

Month	0	1	2	3	4	5	6
Cash	1,000	500	500	500	500	500	500
Treasury Bond	99,000	97,949	96,399	93,746	100,398	107,632	106,082
Sovereign Bond	15,000	14,250	13,538	12,861	12,218	9,774	7,819
Aa-rated Corporate Bond	45,000	41,254	37,821	34,673	34,653	34,633	34,614
Baa-rated Corporate Bond	55,000	50,426	46,232	42,387	42,366	42,344	42,323
Stock	50,000	48,000	35,040	33,638	32,293	31,001	29,761
Total Assets	**265,000**	**252,380**	**229,529**	**217,804**	**222,427**	**225,885**	**221,099**
Reserve - Guaranteed Products	90,000	90,000	90,000	90,000	97,875	106,439	106,439
Reserve - Variable Annuities	40,000	40,000	40,000	40,000	43,500	47,306	47,306
Long-term Debt	89,000	89,000	89,000	40,000	40,000	40,000	40,000
Total Liabilities	**219,000**	**219,000**	**219,000**	**170,000**	**181,375**	**193,745**	**193,745**
Equity	**46,000**	**33,380**	**10,529**	**47,804**	**41,052**	**32,140**	**27,353**

7. Conclusion

Given the severe impact of extreme events, managing exposures to them is crucial for the financial soundness of insurance companies. One extreme event or several extreme events occurring together may cause the downfall of an insurance company. A close monitoring of extreme risk exposure and an actionable contingent plan help a company prepare for these extreme events. However, the effectiveness depends on the robustness of extreme risk analysis.

Unlike typical risk analysis, extreme risk analysis usually requires improvements over traditional models to better examine the patterns in a small set of data. These improvements are useful for identifying extreme situations, measuring the level of extremity and dependency, and designing risk management strategies. Lack of experience and knowledge with extreme events will not disappear with these models. However, wise application can help reveal the patterns of extreme events that may be neglected using traditional models.

With a better understanding of how extreme events may affect a company, the exposure to extreme events can be monitored. Decisions can be made regarding the mitigation of tail risk given the company's risk appetite. Tail risk can be avoided, transferred, or hedged in many ways. Contingent plans for extreme events can also be set up to reduce the impact of extreme events.

With further advances in human knowledge, some extreme events may become more predictable. At the same time, new types of extreme events or events with an unprecedented level of extremity will occur from time to time. Therefore, it is unlikely that we can ever fully understand and predict extreme events. The more actionable solution is to maintain a sufficient capital level to absorb the negative impact of extreme events or reduce the risk exposure through hedging, securitization, and the like.

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Appendix A: Glossary

Autocorrelation function (*ACF*): Describes the correlation as a function of time lag. A positive autocorrelation means that the values of a variable with a certain time lag tend to move in the same direction together. A zero autocorrelation means that they are uncorrelated. A negative autocorrelation means that they tend to move in the opposite direction.

Copula: A statistical method to derive the joint distribution based on marginal distributions and a copula function. $\mathrm{P}(X \leq x, Y \leq y) = C\big(\mathrm{P}(X \leq x), \mathrm{P}(Y \leq y)\big)$. The copula function C is a parameterized model that describes the relationship of multiple variables.

Conditional tail expectation (*CTE*): The average of values greater (less) than the VaR for the right (left) tail.

Extreme value theory (*EVT*): It studies the tail thickness of a distribution. Generally there are two types of problems studied in the EVT: *block maxima* and *point over threshold* (POT). The block maxima (minima) focuses on maxima or minima of a discrete series. Its distribution follows the *generalized extreme value* (*GEV*) *distribution* asymptotically. The POT method analyzes the distribution of the size of exceedance over a large threshold. The distribution is commonly assumed to follow the *generalized Pareto distribution* (GPD) asymptotically.

Fat-tailed distribution (a.k.a. *heavy-tailed distribution*): A distribution whose left or right tail carries more probability density than the corresponding tail from an analogously fitted normal distribution

Generalized autoregressive conditional heteroskedasticity (*GARCH*) model: It models the evolution of conditional variance. For example, GARCH(1,1) is specified as:

$$\sigma_t^2 = \omega + \alpha u_{t-1}^2 + \beta \sigma_{t-1}^2 \quad u_t \sim N(0, \sigma_t^2)$$

u_t: White noise error term that follows a normal distribution $N(0, \sigma_t^2)$

σ_t: Volatility of the variable

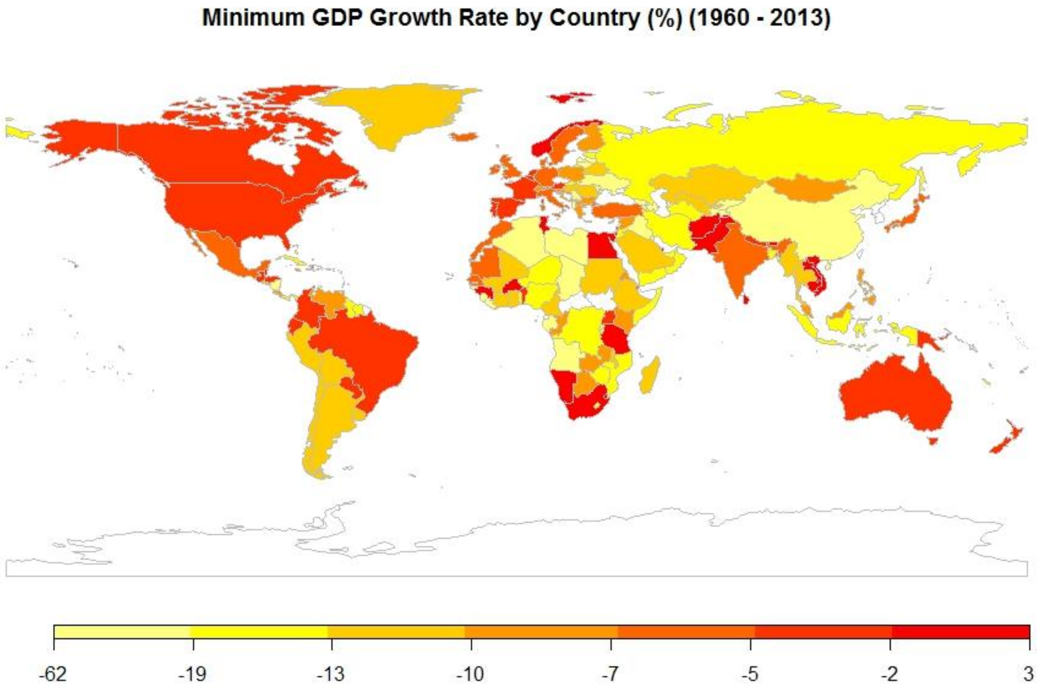
Hidden Markov model (*HMM*): It studies the Markov Process of a hidden state with observations that highly depend on the hidden state. The next hidden state depends on the current hidden state, but not the history of the hidden state. A transition matrix is used to define the probability of the next state given the current one. The distribution of observations changes with the hidden state. Based on the actual observation, an inference system built on Bayes' Rule can be used to predict future hidden states. The states can be considered as different phases of a cycle.

Q-Q plot: It compares two distributions by plotting their quantiles against each other. If the two distributions are identical, the plots should lie on the line $y = x$ exactly.

Value at Risk (*VaR*): Given a confidence level p and a time horizon, Value at Risk is the threshold value such that the probability of having a value greater than the VaR within the time horizon is p. Pr[X < VaR] = p. It is usually used to study the right tail of a distribution. When studying the left tail, it can be defined as Pr[X > VaR] = p.

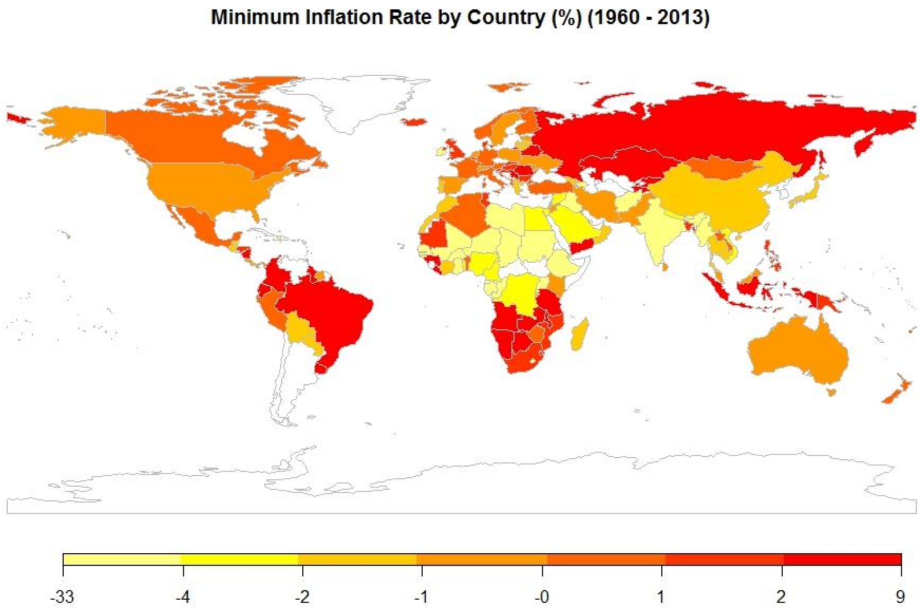
Appendix B: Economic Indicator Heat Maps

Figure B.1. Minimum GDP Growth Rate by Country (%) (1960–2013)



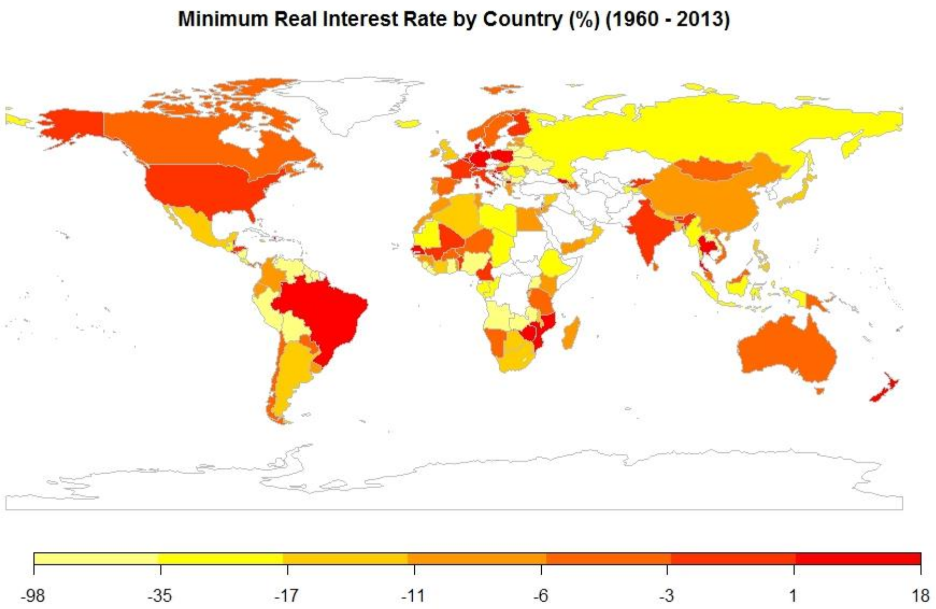
Data source: World Bank. Some countries do not have data for all years in the selected data period. Countries with less than 11 years of data have been excluded (appear as transparent) to make sure enough variation is reflected.

Figure B.2. Minimum Inflation Rate by Country (%) (1960–2013)



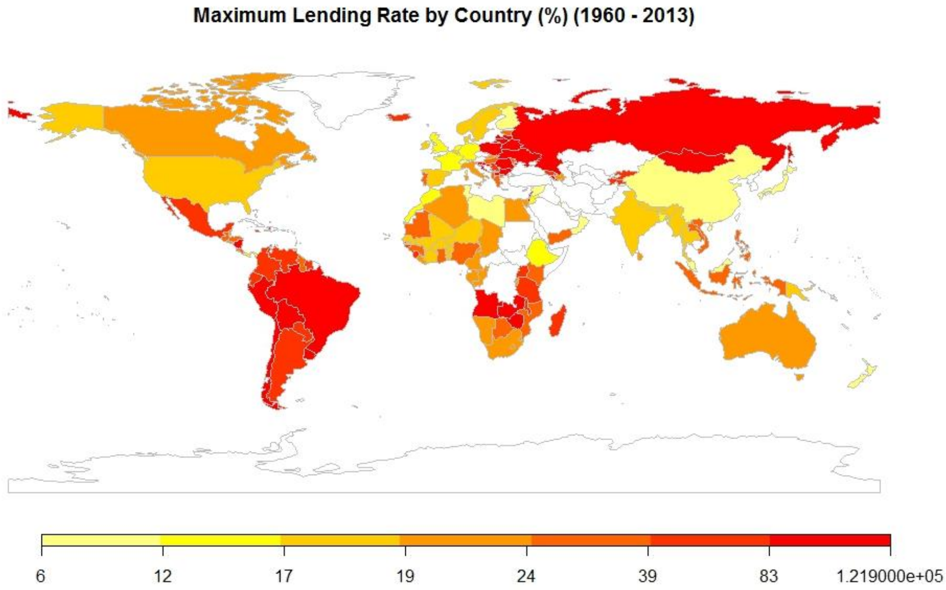
Data source: World Bank. Some countries do not have data for all years in the selected data period. Countries with less than 11 years of data have been excluded (appear as transparent) to make sure enough variation is reflected.

Figure B.3. Minimum Real Interest Rate by Country (%) (1960–2013)



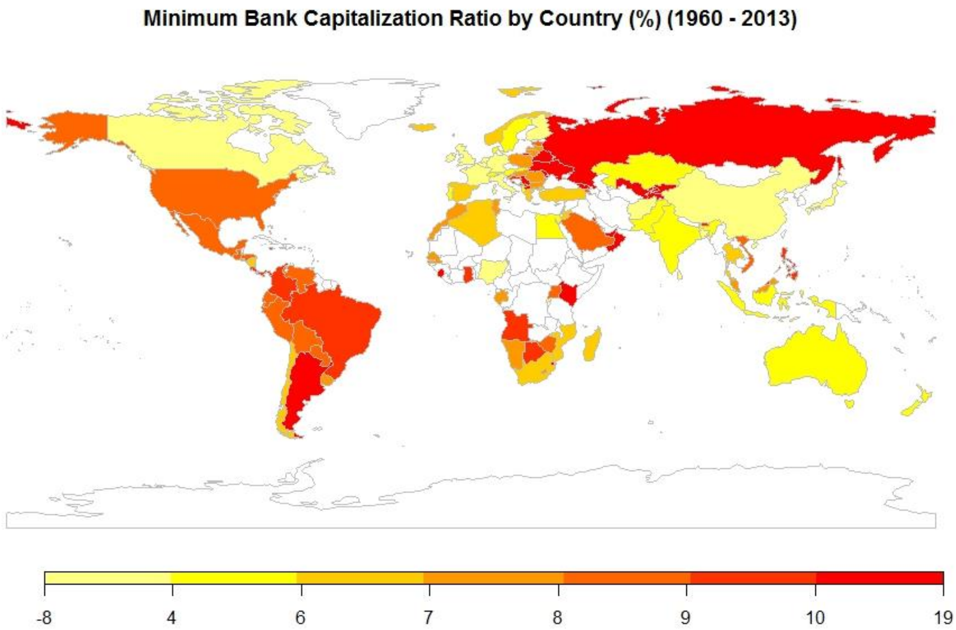
Data source: World Bank. Some countries do not have data for all years in the selected data period. Countries with less than 11 years of data have been excluded (appear as transparent) to make sure enough variation is reflected.

Figure B.4. Maximum Lending Rate by Country (%) (1960–2013)



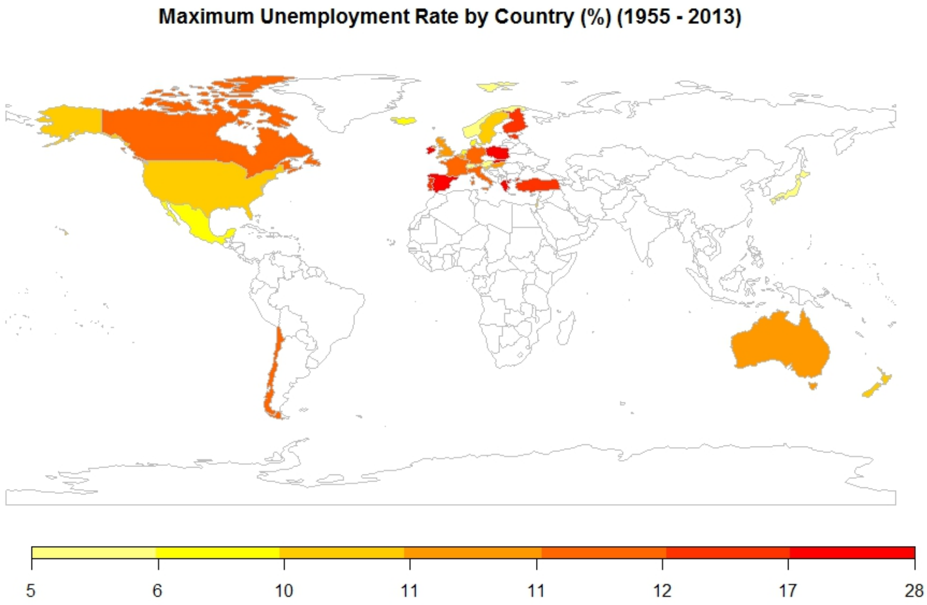
Data source: World Bank. Some countries do not have data for all years in the selected data period. Countries with less than 11 years of data have been excluded (appear as transparent) to make sure enough variation is reflected.

Figure B.5. Minimum Bank Capitalization Ratio by Country (%) (1960–2013)



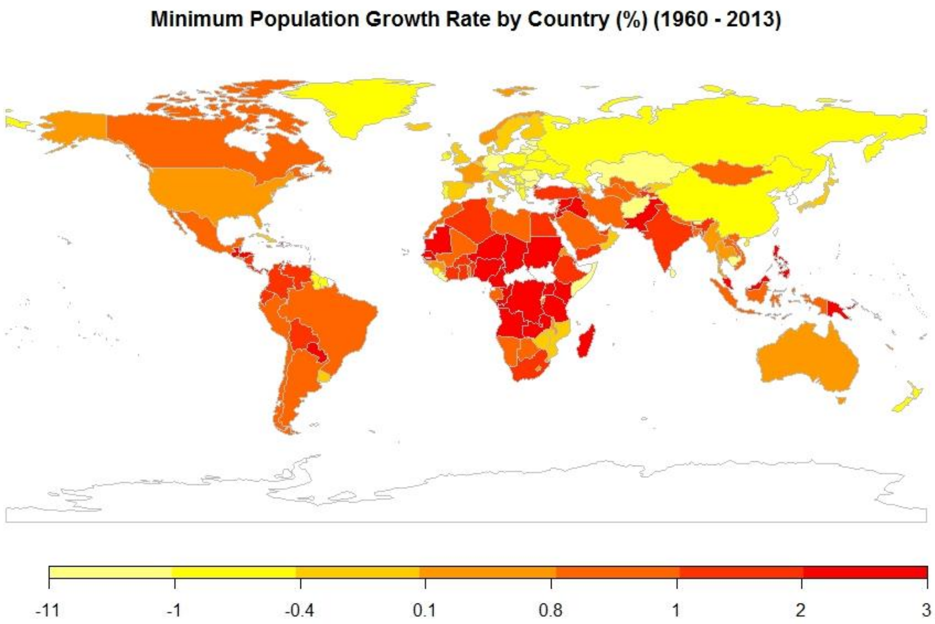
Data source: World Bank. Some countries do not have data for all years in the selected data period. Countries with less than 11 years of data have been excluded (appear as transparent) to make sure enough variation is reflected.

Figure B.6. Maximum Unemployment Rate by Country (%) (1955–2013)



Data source: OECD Statistics. Some countries do not have data for all years in the selected data period. OECD adjusts the unemployment data to make them more comparable across countries.

Figure B.7. Minimum Population Growth Rate by Country (%) (1960–2013)



Data source: World Bank. Some countries do not have data for all years in the selected data period. Countries with less than 11 years of data have been excluded (appear as transparent) to make sure enough variation is reflected.