GENERALIZED RISK PROCESSES*

V. E. Bening and V. Yu. Korolev†

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*Research supported by the Committee on Knowledge Extension Research of the Society of Actuaries.
†Department of Mathematical Statistics, Faculty of Computational Mathematics and Cybernetics, Moscow State University, Vorobyovy Gory, Moscow, 119899, Russia. E-mail: vkorolev@cmc.msk.su

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Abstract

In this paper we give new general criteria for the weak convergence of one-dimensional distributions of generalized risk processes and describe the class of possible limit laws under an infinite growth of (stochastic) portfolio size and (stochastic) intensity of insurance payments. This makes it possible to construct asymptotic approximations to the distributions of generalized risk processes. We also present the convergence rate estimates which make it possible to evaluate the accuracy of asymptotic approximations, give exponential estimates for the probabilities of negative values of generalized risk processes which can be also interpreted as ruin probabilities in the so-called static model of functioning of an insurance company, formulate some statements concerning the asymptotic expansions for the distributions of generalized risk processes and their quantiles. The latter results can be used to improve the asymptotic approximations mentioned above. Finally, we present the statistical estimators (both point and interval) of the ruin probabilities for a generalized risk process, given the pre-history of such a process.

KEY WORDS AND PHRASES: *Poisson process, Cox process, classical risk process, generalized risk process, weak convergence, mixtures of distributions, heavy tails, convergence rate, asymptotic expansions, exponential bounds, ruin probability, nonparametric estimation, consistent estimators, unbiased estimators, confidence intervals.*

Contents

1 Introduction.

In this paper we give an account of the research we started in (Bening and Korolev, 1998a) and (Bening and Korolev, 1999a) and present new results on the asymptotic behavior of generalized risk processes and the statistical estimation of ruin probabilities for these processes.

The base for our generalizations is the classical risk process

$$R_0(t) = ct - \sum_{j=1}^{N_1(t)} X_j, \quad t \geq 0, \tag{1.1}$$

where $c > 0$ is the premium rate, $\{X_j\}_{j\geq 1}$ are insurance claims (payments) which are assumed to be independent identically distributed random variables (r.v.'s) with mean value $\mathsf{E}X_j = a$, $N_1(t)$ is the number of claims up to time t which is assumed to be the standard Poisson process (a homogeneous Poisson process with unit intensity) independent of $\{X_j\}_{j\geq 1}$. We consider the classical risk process in the form which slightly differs from the traditional one. Usually it is assumed that the counting process which is equal to the number of claims is a homogeneous Poisson process with some intensity $\lambda > 0$. Here we assume that $\lambda = 1$. This assumption by no means restricts the general character of our reasoning. This means only that the time unit is chosen so that, on the average, one payment occurs per unit time interval. As this is so, c has the meaning of the increment of the capital of the insurance company on the unit time interval chosen in this way.

Let $\Lambda(t)$, $t \geq 0$, be a random process independent of $N_1(t)$, with infinitely icreasing almost surely finite continuous trajectories starting from the origin. A Cox process (also called a doubly stochastic Poisson process controlled by the process $\Lambda(t)$ is defined as

$$N(t) = N_1(\Lambda(t)), \quad t \geq 0, \tag{1.2}$$

see, e. g., (Grandell, 1976). Consider the process

$$R(t) = c\Lambda(t) - \sum_{j=1}^{N(t)} X_j, \quad t \geq 0, \tag{1.3}$$

This process is a natural generalization of the classical risk process (1.1) and is a more flexible mathematical model for the surplus of an insurance company since it takes into account both risk and portfolio fluctuations. It can be shown that under risk fluctuations (non-constant intensity of insurance payments), in *reasonable* strategies of the insurer the premium rate or, which is in some sense the same, the current size of the portfolio *must not* be constant (Bülmann, 1989), (Embrechts and Klüppelberg, 1993). On the other hand, the intensity of payments should be proportional to the current number of insurance contracts in the portfolio resulting in that the cumulated intensity $\Lambda(t)$ of payments should be proportional to the total number of contracts in the portfolio or, which is in some sense the same, to the cumulated premiums. Following (Bening and Korolev, 1998a), we will call $R(t)$ a generalized risk process.

In this paper we will give new general criteria for the weak convergence of one-dimensional distributions of generalized risk processes $R(t)$ and describe the class of possible limit laws as $\Lambda(t)$ infinitely grows which makes it possible to construct asymptotic approximations to the distributions of generalized risk processes (Section 2), present the convergence rate estimates which make it possible to evaluate the accuracy of asymptotic approximations (Section 3), give exponential estimates for the probabilities of negative values of generalized risk processes which can be also interpreted as ruin probabilities in the so-called static model of functioning of an insurance company (Section 4), formulate some statements concerning the asymptotic expansions for the distributions of generalized risk processes (Section 5) and their quantiles (Section 6). The latter results can be used to improve the asymptotic approximations mentioned above. Finally, in Section 7 we will present the statistical estimators (both point and interval) of the ruin probabilities for a generalized risk process, given the pre-history of such a process.

The symbols $\xrightarrow{P}$ and $\stackrel{d}{=}$ will denote convergence in probability and the coincidence of distributions, respectively. The standard normal distribution function will be denoted $\Phi(x)$. Throughout the paper we will assume that there exists $\mathsf{D}X_1 = \sigma^2 > 0$.

2 Convergence criteria for generalized risk processes. Limit laws.

In our paper (Bening and Korolev, 1998a) we considered the case where $\mathsf{E}\Lambda(t)$ is finite and equals t. The aim of the present section is to present new results which are free from this rather restrictive assumption. As we will see below, the finiteness of $\mathsf{E}\Lambda(t)$ makes the situation considerably much simpler as compared to that where no moment-type assumptions are made concerning $\Lambda(t)$.

We will consider the asymptotic behavior of the generalized risk process as $t \to \infty$. As this is so, *t will not necessarily have the meaning of time*. For example, we can assume that t is some location parameter of the controlling process, say, its median. This interpretation will enable us to consider the asymptotic behavior of generalized risk process at a fixed time, but under infinitely increasing (say, in the sense of convergence in probability) size of the portfolio, or, which is the same within the considered model, under infinitely increasing cumulated intensity of insurance payments.

From the reasoning of "expected nonruin" it follows that we should assume that $c > a$. However, for the sake of generality we will assume that $c \neq a$, thus formally admitting the unfavourable case $c < a$. The investigation of the "critical" case $c = a$ requires another technique. This case will be considered in one of our following papers.

2.1 Auxiliary results.

Recall some notions which will be intensively exploited in what follows. Assume that all the r.v.'s, random vectors and random processes are defined on the same probability space $(\Omega, \mathcal{A}, \mathsf{P})$.

A sequence of distribution functions (d.f.'s) $F_1, F_2, \ldots$ is said to converge weakly to a d.f. F (which will be denoted as $F_n \Longrightarrow F$) as $n \to \infty$, if $F_n(x) \to F(x)$ as $n \to \infty$ at every point x, atn which the limit d.f. is continuous. If $X, X_1, X_2, \ldots$ are r.v.'s with the d.f.'s

$F, F_1, F_2, \ldots$, respectively, then we shall say that the sequence $\{X_n\}$ weakly converges to X (denoting this as $X_n \Longrightarrow X$), if $F_n \Longrightarrow F$.

A family of r.v.'s $\{X_n\}$ is called weakly compact if each sequence of its elements contains a weakly convergent subsequence. As is known, a family of r.v.'s $\{X_n\}$ is weakly compact if and only if

$$\lim_{R\to\infty} \sup_n \mathsf{P}(|X_n| > R) = 0$$

(see, e.g., (Gnedenko and Kolmogorov, 1954), Chapter 2, Section 9).

The Lévy distance (metric) between d.f.'s F_1 and F_2 is defined as

$$L_1(F_1, F_2) = \inf\{h > 0: \ F_1(x-h) - h \le F_2(x) \le F_1(x+h) + h \text{ for all } x \in \mathbb{R}\}.$$

If X_1 and X_2 are r.v.'s with d.f.'s F_1 and F_2, respectively, then we will assume that $L_1(X_1, X_2) = L_1(F_1, F_2)$. The L_1-convergence is equivalent to the weak convergence (see, e.g., (Gnedenko and Kolmogorov, 1954), Chapter 2, Section 9).

Let $(E, \mathcal{E}, \rho)$ be a metric space and $\mathcal{P}(E)$ be the set of probability measures on the measurable space $(E, \mathcal{E})$. Let $A \subset E$. Put $\rho(x, A) = \inf\{\rho(x, y) : \ y \in A\}$. Let $\varepsilon > 0$. Denote $A^\varepsilon = \{x \in E : \ \rho(x, A) < \varepsilon\}, \ \ A \in \mathcal{E}$. Let P_1 and P_2 be arbitrary probability measures from $\mathcal{P}(E)$. Put

$$\sigma(\mathsf{P}_1, \mathsf{P}_2) = \inf\{\varepsilon > 0: \ \mathsf{P}_1(A) \le \mathsf{P}_2(A^\varepsilon) + \varepsilon \text{ for any closed } A \in \mathcal{E}\}.$$

The Lévy-Prokhorov distance between distributions P_1 and P_2 is defined as

$$L_2(\mathsf{P}_1, \mathsf{P}_2) = \max\{\sigma(\mathsf{P}_1, \mathsf{P}_2), \sigma(\mathsf{P}_2, \mathsf{P}_1)\}.$$

We will assume that the Lévy-Prokhorov distance between random vectors X and Y is defined as the Lévy-Prokhorov distance between the induced probability distributions: $L_2(X, Y) = L_2(\mathsf{P}_X, \mathsf{P}_Y)$. As is known, the weak convergence of random vectors is equivalent to their L_2-convergence (see, e.g., (Shiryaev, 1984))

LEMMA 2.1. *$N(t) \xrightarrow{P} \infty$ $(t \to \infty)$ if and only if $\Lambda(t) \xrightarrow{P} \infty$ $(t \to \infty)$.*

For the PROOF see (Gnedenko and Korolev, 1996), Section A2.3.

We shall say that a family of r.v.'s $\{Z(t)\}_{t\ge 0}$ is weakly compact at infinity if each sequence $\{t_k\}_{k\ge 1}$ such that $t_k \to \infty$ $(k \to \infty)$ contains a subsequence $\{t_{k_m}\}_{m\ge 1}$ such that the sequence of r.v.'s $\{Z(t_{k_m})\}$ weakly converges as $m \to \infty$. Let $A(t)$, $B(t)$ and $D(t)$ be functions such that $A(t)$ and $B(t)$ are measurable, $B(t) > 0$ and $D(t) > 0$. Let $X(t)$ and $M(t)$, $t \ge 0$, be independent stochastic processes such that $X(t)$ is measurable (by measurability of a random process we mean its measurability with respect to the Cartesian product of the σ-algebra $\mathcal{A}$ of the initial probability space and the Borel σ-algebra of subsets of the real line) and the trajectories of $M(t)$ start from the origin and do not decrease.

The following statement describing necessary and sufficient conditions for the weak convergence of superpositions of independent stocastic processes will play the main role in the proof of our main theorem. It can be regarded as a generalization and sharpening of the famous Dobrushin's lemma (Dobrushin, 1955).

LEMMA 2.2. *Assume that $B(t) \to \infty$ and $D(t) \to \infty$ as $t \to \infty$ and the families of r.v.'s*

$$\left\{\frac{X(t)-A(t)}{B(t)}\right\}_{t>0} \qquad \text{and} \qquad \left\{\frac{B(M(t))}{D(t)}\right\}_{t>0}$$

are weakly compact at infinity. Then one-dimensional distributions of appropriately centered and normalized superposition of the processes $X(t)$ and $M(t)$ weakly converge to the distribution of some r.v. Z as $t \to \infty$:

$$\frac{X(M(t))-C(t)}{D(t)} \Longrightarrow Z, \quad t \to \infty,$$

with some real function $C(t)$ if and only if there exists a weakly compact at infinity family of triples of r.v.'s $\{(W(t), U(t), V(t))\}_{t\geq 0}$ such that

1) for each $t > 0$, $Z \stackrel{d}{=} W(t)U(t) + V(t)$ with the r.v. $W(t)$ and pair $(U(t), V(t))$ independent;

2) $L_1\left(\frac{X(t)-A(t)}{B(t)}, W(t)\right) \to 0$ as $t \to \infty$;

3) $L_2\left(\left(\frac{B(M(t))}{D(t)}, \frac{A(M(t))-C(t)}{D(t)}\right), (U(t), V(t))\right) \to 0$ as $t \to \infty$.

This statement was proved in (Korolev, 1996) and then presented in (Gnedenko and Korolev, 1996), Section A2.2.

REMARK 2.1. In general, the requirement that the family of triples $\{(W(t), U(t), V(t))\}$ should be weakly compact at infinity is unnecessary, that is, it is automatically fulfilled under the conditions of the lemma. To prove this we need not introduce any changes into the proof of the "only if" part as compared with (Korolev, 1996) or (Gnedenko and Korolev, 1996). Some slight changes should be made in the "if" part. Namely, the weak compactness at infinity of the families $\{Y(t)\}_{t\geq 0}$ and $\{U(t)\}_{t\geq 0}$ directly follows from that of the families $\{(X(t) - A(t))/B(t)\}_{t\geq 0}$ and $\{B(M(t))/D(t)\}_{t\geq 0}$ by virtue of conditions 2 and 3. So, all we have to do is to make sure that the family $\{V(t)\}_{t\geq 0}$ is weakly compact at infinity. But by virtue of the inequality

$$\mathsf{P}(|Y_1 + Y_2| > R) \leq \mathsf{P}\left(|Y_1| > \frac{R}{2}\right) + \mathsf{P}\left(|Y_2| > \frac{R}{2}\right) \tag{2.1}$$

which is valid for any r.v.'s Y_1 and Y_2 and any $R > 0$, due to condition 1, for an arbitrary $R > 0$ we have

$$\mathsf{P}(|V(t)| > R) = \mathsf{P}(|W(t)U(t) + V(t) - W(t)U(t)| > R) \leq$$

$$\leq \mathsf{P}\left(|W(t)U(t) + V(t)| > \frac{R}{2}\right) + \mathsf{P}\left(|W(t)U(t)| > \frac{R}{2}\right) =$$

$$= \mathsf{P}\left(|Z| > \frac{R}{2}\right) + \mathsf{P}\left(|W(t)U(t)| > \frac{R}{2}\right).$$

The first summand on the right-hand side does not depend on t. The family of products $\{W(t)U(t)\}_{t\geq 0}$ of independent factors is weakly compact at infinity, since as we have established above, each factor family is weakly compact at infinity. Therefore R can be chosen large enough to provide arbitrary smallness of the right-hand side of (2.1) irrespective of t. Now the reference to (Korolev, 1996) or (Gnedenko and Korolev, 1996) completes the proof.

2.2 Main results.

The main result of this paper is the following Theorem 2.1. It can be regarded as a generalization and modification of one well-known result due to Rootzén (Rootzén, 1975), (Rootzén, 1976) (also see (Grandell, 1976)) who considered simple Cox processes.

THEOREM 2.1. *Assume that* $\mathsf{E}X_1 = a \neq c$ *and* $\Lambda(t) \xrightarrow{P} \infty$ *as* $t \to \infty$. *Let* $D(t) > 0$ *be a function such that* $D(t) \to \infty$ *as* $t \to \infty$. *Then one-dimensional distributions of an appropriately centered and normalized generalized risk process* $R(t)$ *weakly converge to the distribution of some r.v.* Z *as* $t \to \infty$, *that is,*

$$\frac{-R(t) - C(t)}{D(t)} \Longrightarrow Z \quad (t \to \infty) \tag{2.2}$$

with some real function $C(t)$ *if and only if*

$$\limsup_{t\to\infty} \frac{|C(t)|}{D^2(t)} \equiv k^2 < \infty \tag{2.3}$$

and there exists a r.v. V *such that*

$$Z \stackrel{d}{=} k \cdot \sqrt{\frac{a^2 + \sigma^2}{|a - c|}} \cdot W + V \tag{2.4}$$

where W *is the r.v. with the standard normal distribution, independent of* V *and*

$$L_1\left(\frac{(a - c)\Lambda(t) - C(t)}{D(t)}, V(t)\right) \to 0 \quad (t \to \infty), \tag{2.5}$$

with the distribution of the r.v. $V(t)$ *defined by the characteristic function*

$$\mathsf{E}\exp\{isV(t)\} = \exp\left\{-\frac{s^2(a^2+\sigma^2)}{2|a-c|}\left[k^2 - \frac{|C(t)|}{D^2(t)}\right]\right\}\mathsf{E}\exp\{isV\}, \qquad s \in \mathbb{R}. \tag{2.6}$$

PROOF. We will derive this theorem as a consequence of Lemma 2.2.

The "only if" part. We oviously have $R(t) = R_0(\Lambda(t))$. It is well known that the classical risk process $R_0(t)$ is asymptotically normal, that is,

$$\frac{-R_0(t) - (a - c)t}{\sqrt{t(a^2 + \sigma^2)}} \Longrightarrow W \quad (t \to \infty), \tag{2.7}$$

where the r.v. W has the standard normal distribution (see, e.g., Theorem 3.4.1 in (Gnedenko and Korolev, 1996)). Therefore in Lemma 2.2 it is reasonable to set $X(t) = -R_0(t)$, $M(t) = \Lambda(t)$, $A(t) = (a-c)t$, $B(t) = \sqrt{t(a^2+\sigma^2)}$. Then the weak compactness at infinity of the sequence $\{(X(t) - A(t))/D(t)\}$ required in Lemma 2.2 is a direct consequence of (2.7). To provide the possibility of the direct use of Lemma 2 we should prove that the family of r.v.'s

$$\left\{ \frac{B(M(t))}{D(t)} = \frac{\sqrt{(a^2+\sigma^2)\Lambda(t)}}{D(t)} \right\}_{t \geq 0}$$

is weakly compact at infinity. For this purpose we first prove that the family $\{N(t)/D^2(t)\}$ is weakly compact at infinity. Let $X_1', X_2', \ldots$ be identically distributed r.v.'s such that $X_1 \stackrel{d}{=} X_1'$ and the r.v.'s $X_1, X_2, \ldots, X_1', X_2', \ldots$ are independent. Denote $X_j^{(s)} = X_j - X_j'$, $j \geq 1$,

$$T(t) = \frac{1}{D(t)} \sum_{j=1}^{N(t)} X_j^{(s)}, \quad t \geq 0.$$

We obviously have

$$\mathsf{P}(|T(t)| \geq x) = \int_0^\infty \sum_{n=1}^\infty \mathsf{P}(N_1(\lambda) = n) \mathsf{P}\left(\left| \frac{1}{D(t)} \sum_{j=1}^n X_j^{(s)} \right| \geq x \right) d\mathsf{P}(\Lambda(t) < \lambda) =$$

$$= \int_0^\infty \sum_{n=1}^\infty \mathsf{P}(N_1(\lambda) = n) \mathsf{P}\left(\left| \frac{1}{D(t)} \left[\left(\sum_{j=1}^n X_j - c\lambda - C(t) \right) - \right.\right.\right.$$

$$\left.\left.\left. - \left(\sum_{j=1}^n X_j' - c\lambda - C(t) \right) \right] \right| \geq x \right) d\mathsf{P}(\Lambda(t) < \lambda) \leq$$

$$\leq 2 \int_0^\infty \sum_{n=1}^\infty \mathsf{P}(N_1(\lambda) = n) \mathsf{P}\left(\left| \frac{1}{D(t)} \left(\sum_{j=1}^n X_j - c\lambda - C(t) \right) \right| \geq \frac{x}{2} \right) d\mathsf{P}(\Lambda(t) < \lambda) =$$

$$= 2 \int_0^\infty \mathsf{P}\left(\left| \frac{1}{D(t)} \left(\sum_{j=1}^{N_1(\lambda)} X_j - c\lambda - C(t) \right) \right| \geq \frac{x}{2} \right) d\mathsf{P}(\Lambda(t) < \lambda) =$$

$$= 2\mathsf{P}\left(\left| \frac{-R(t) - C(t)}{D(t)} \right| \geq \frac{x}{2} \right).$$

Hence it follows that for any infinitely increasing sequence $\{t_k\}_{k \geq 1}$ we have

$$\lim_{x\to\infty} \sup_k \mathsf{P}(|T(t_k)| \geq x) \leq 2 \lim_{x\to\infty} \sup_k \mathsf{P}\left(\left| \frac{-R(t_k) - C(t_k)}{D(t_k)} \right| \geq \frac{x}{2} \right) = 0,$$

that is, the family $\{T(t)\}$ is weakly compact at infinity.

Let $\{t_k\}_{k \geq 1}$ be an arbitrary infinitely increasing sequence. Almost word-for-word repeating the corresponding part of the proof of Theorem 3.2.1 from (Gnedenko and Korolev,

1996) we show that the sequence $\{N_1(\Lambda(t_k))/D^2(t_k)\}_{k\geq 1}$ contains a weakly convergent subsequence. This means that the family $\{N(t)/D^2(t)\}$ is weakly compact at infinity.

After we established the weak compactness at infinity of the family $\{N(t)/D^2(t)\}$, the weak compactness at infinity of the family $\{\Lambda(t)/D^2(t)\}$ is proved in exactly the same way we used to prove Lemma 2 in (Korolev, 1998).

Now we have everything we need to apply Lemma 2.2. Due to the weak compactness at infinity of the family of r.v.'s $\{V(t)\}$ and condition 3 of Lemma 2.2 the family of r.v.'s $\{((a-c)\Lambda(t)-C(t))/D(t)\}_{t\geq 0}$ is also weakly compact at infinity. Let $\{t_k\}_{k\geq 1}$ be an arbitrary infinitely increasing sequence. Let $l_k(q)$ be the left q-quantile of the r.v. $\Lambda(t_k)$. The weak compactness at infinity of the family $\{\Lambda(t)/D^2(t)\}_{t\geq 0}$ implies

$$I(q) \equiv \sup_k \frac{l_k(q)}{D^2(t_k)} < \infty \tag{2.8}$$

for each $q \in (0,1)$. The weak compactness at infinity of the family $\{((a-c)\Lambda(t) - C(t))/D(t)\}$ implies the boundedness of the sequence $\{((a-c)l_k(q) - C(t_k))/D(t_k)\}_{k\geq 1}$ for each $q \in (0,1)$. But

$$\frac{(a-c)l_k(q) - C(t_k)}{D(t_k)} = D(t_k)\left(\frac{(a-c)l_k(q)}{D^2(t_k)} - \frac{C(t_k)}{D^2(t_k)}\right), \tag{2.9}$$

so to provide the boundedness of the right-hand side of (2.9) in k for each $q \in (0,1)$, the difference $(a-c)l_k(q)/D^2(t_k) - C(t_k)/D^2(t_k)$ must tend to zero as $k \to \infty$ for each $q \in (0,1)$, since $D(t_k) \to \infty$ as $k \to \infty$. But with account of (2.8) this is possible only when (2.3) holds.

Let $v_k(q)$ be the left q-quantile of the r.v.

$$\frac{B(M(t_k))}{D(t_k)} = \frac{\sqrt{(a^2+\sigma^2)\Lambda(t_k)}}{D(t_k)}.$$

For each $q \in (0,1)$ we obviously have

$$v_k(q) = \frac{\sqrt{(a^2+\sigma^2)l_k(q)}}{D(t_k)},$$

whence

$$l_k(q) = \frac{D^2(t_k)v_k^2(q)}{a^2+\sigma^2}.$$

But as we have seen above,

$$\lim_{k\to\infty}\left|\frac{(a-c)l_k(q)}{D^2(t_k)} - \frac{C(t_k)}{D^2(t_k)}\right| = 0.$$

Therefore

$$\lim_{k\to\infty}\left|\frac{(a-c)v_k^2(q)}{a^2+\sigma^2} - \frac{C(t_k)}{D^2(t_k)}\right| = 0,$$

which means that

$$\lim_{k\to\infty}\left|v_k(q)-\sqrt{\frac{(a^2+\sigma^2)|C(t_k)|}{|a-c|D^2(t_k)}}\right|=0$$

irrespective of the value of $q\in(0,1)$. Therefore by virtue of (2.2) and the arbitrariness of the sequence $\{t_k\}_{k\geq 1}$, the triples of r.v.'s $(W(t),U(t),V(t))$ from Lemma 2 in the case under consideration should be sought in the form

$$(W(t),U(t),V(t))=\left(W,\sqrt{\frac{a^2+\sigma^2}{|a-c|}}\cdot\frac{\sqrt{|C(t)|}}{D(t)},V(t)\right)$$

where W is the standard normal r.v. independent of $V(t)$ for each $t\geq 0$. Recall that in accordance with condition 1 of Lemma 2.2 each of these triples must gurantee the representation

$$Z\stackrel{d}{=}\sqrt{\frac{a^2+\sigma^2}{|a-c|}}\cdot\frac{\sqrt{|C(t)|}}{D(t)}\cdot W+V(t) \tag{2.10}$$

for each $t\geq 0$. For convenience denote $k^2(t)=|C(t)|/D^2(t)$. Consider the r.v.'s satisfying (2.10) in more detail. The boundedness of $k(t)$ established above together with the weak compactness at infinity of the family $\{V(t)\}_{t\geq 0}$ implied by Lemma 2.2 allow to extract from an arbitrary infinitely increasing sequence $T=\{t_1,t_2,\ldots\}$ a subsequence T_1 so that $k(t)\to k_0$ and $V(t)\Longrightarrow V_0$ as $t\to\infty$, $t\in T_1$, where k_0 is some number and V_0 is some r.v.. But then, repeating the reasoning we used to prove the transfer theorem (see Theorem 3.1.2 in (Gnedenko and Korolev, 1996)) based on the definition of weak convergence through the convergence of integrals of continuous bounded functions, for any real s we obtain

$$\mathsf{E}e^{isZ}=\mathsf{E}e^{isV(t)}\exp\left\{-\frac{s^2k^2(t)(a^2+\sigma^2)}{2|a-c|}\right\}\to\mathsf{E}e^{isV_0}\exp\left\{-\frac{s^2k_0^2(a^2+\sigma^2)}{2|a-c|}\right\}$$

as $t\to\infty$, $t\in T_1$, whence it follows that the limit pair (k_0,V_0) also satisfies (2.10). In other words, the set of pairs $(k(t),V(t))$ which satisfy (2.10) is closed with respect to the passage to the limit as $t\to\infty$. Let V be a r.v. corresponding to the value $k(t)=k$ (see (2.3)) in representation (2.10). Then for any $t\geq 0$ we have

$$k\cdot\sqrt{\frac{a^2+\sigma^2}{|a-c|}}\cdot W+V\stackrel{d}{=}k(t)\cdot\sqrt{\frac{a^2+\sigma^2}{|a-c|}}\cdot W+V(t), \tag{2.11}$$

where the summands on both sides are independent. Rewriting (2.11) in terms of characteristic functions we obtain

$$\exp\left\{-\frac{s^2k^2(a^2+\sigma^2)}{2|a-c|}\right\}\mathsf{E}e^{isV}=\exp\left\{-\frac{s^2k^2(t)(a^2+\sigma^2)}{2|a-c|}\right\}\mathsf{E}e^{isV(t)} \tag{2.12}$$

for all real s and $t\geq 0$. For the characteristic function of the r.v. $V(t)$ from (2.12) we obtain the representation

$$\mathsf{E}\exp\{isV(t)\}=\exp\left\{-\frac{s^2(a^2+\sigma^2)}{2|a-c|}\left(k^2-k^2(t)\right)\right\}\mathsf{E}e^{isV},\quad s\in\mathbb{R},\ t\geq 0.$$

Finally, relation (2.4) with the r.v. $V(t)$ just described follows from condition 3 of Lemma 2.2.

The "if" part. The assertion of Theorem 2.1 is an almost direct consequence of Lemma 2.2 with $X(t) = -R_0(t)$, $M(t) = \Lambda(t)$, $A(t) = (a-c)t$, $B(t) = \sqrt{t(a^2+\sigma^2)}$. Here the weak compactness at infinity of the family

$$\left\{ \frac{B(M(t))}{D(t)} = \frac{\sqrt{(a^2+\sigma^2)\Lambda(t)}}{D(t)} \right\}_{t\geq 0}$$

required in Lemma 2 follows from (2.3), (2.4), (2.5) and (2.6). Indeed, (2.3) and (2.6) yield the weak compactness an infinity of the family $\{V(t)\}$. Together with (2.5) this means that the family $\{((a-c)\Lambda(t) - C(t))/D(t)\}$ is weakly compact at infinity. This, in its turn, by virtue of representation (2.9) and condition (2.3) yields (2.8) for each $q \in (0,1)$ and hence, the weak compactness at infinity of the family $\{\sqrt{(a^2+\sigma^2)\Lambda(t)}/D(t)\}$. The theorem is proved.

Theorem 1 gives the following criterion of the asymptotic normality of generalized risk processes.

COROLLARY 2.1. *Under the conditions of Theorem 1 a nonrandomly centered and normalized generalized risk process $R(t)$ is asymptotically normal with some asymptotic variance $\delta^2 > 0$:*

$$P\left(\frac{-R(t) - C(t)}{D(t)} < x\right) \Longrightarrow \Phi\left(\frac{x}{\delta}\right) \qquad (t \to \infty),$$

if and only if

$$\limsup_{t\to\infty} \frac{|C(t)|}{D^2(t)} \leq \frac{|a-c|\delta^2}{a^2+\sigma^2}$$

and

$$\lim_{t\to\infty} L_1 \left(P\left(\frac{(a-c)\Lambda(t) - C(t)}{D(t)} < x\right), \ \Phi\left(\frac{\sqrt{|a-c|}D(t)x}{\sqrt{|a-c|\delta^2 D^2(t) - (a^2+\sigma^2)|C(t)|}}\right)\right) = 0.$$

PROOF. This statement follows from Theorem 2.1 and the famous Cramér-Lévy theorem on the decomposability of a normal law only into normal components, according to which any r.v. V satisfying (2.4) should be normal with zero mean and variance

$$\delta^2 - \frac{k^2(a^2+\sigma^2)}{|a-c|}$$

and hence, any r.v. $V(t)$ satisfying (2.6) should inevitably be normal with zero mean and variance

$$\delta^2 - \frac{(a^2+\sigma^2)|C(t)|}{|a-c|D^2(t)}.$$

The corollary is proved.

In other words, an appropriately centered and normalized generalized risk process $R(t)$ is asymptotically normal if and only if so is its controlling process $\Lambda(t)$.

REMARK 2.2. Theorem 2.1 and Corollary 2.1 are actually valid for a more general situation where the generalized risk process (1.3) is generated not necessarily by a standard Poisson process N_1, but by any asymptotically degenerate and asymptotically normal counting process N_1, that is, by any counting process N_1 possessing the properties

$$\frac{N_1(t)}{t} \Longrightarrow \gamma \quad (t \to \infty)$$

and

$$\mathsf{P}\left(\frac{N_1(t) - rt}{p\sqrt{t}} < x\right) \Longrightarrow \Phi(x) \quad (t \to \infty)$$

with some $\gamma > 0$, $r \in \mathbb{R}$ and $p > 0$.

To illustrate to what extent the situation simplifies when we assume that $\mathsf{E}\Lambda(t)$ exists and equals t we present here the following theorem proved in (Bening and Korolev, 1998a).

THEOREM 2.2. *Assume that $c > a$, $\mathsf{E}\Lambda(t) \equiv t$ and $\Lambda(t) \xrightarrow{P} \infty$ as $t \to \infty$. Then one-dimensional distributions of an appropriately centered and normalized generalized risk process weakly converge to the distribution of some r.v. Z as $t \to \infty$, that is,*

$$\frac{R(t) - (c-a)t}{\sqrt{(a^2 + \sigma^2)t}} \Longrightarrow Z \quad (t \to \infty),$$

if and only if there exists a r.v. V such that

1) $\mathsf{P}(Z < x) = \mathsf{E}\Phi\left(x - \dfrac{c-a}{\sqrt{a^2+\sigma^2}} \cdot V\right);$

2) $\dfrac{\Lambda(t) - t}{\sqrt{t}} \Longrightarrow V \quad (t \to \infty).$

This statement is actually a particular case of Theorem 2.1.

REMARK 2.3. Note that in Theorem 2.2 we normalize the process $R(t)$ not by its variance, but by $\sqrt{(a^2 + \sigma^2)t}$ thus not assuming the existence of the variance of the controlling process $\Lambda(t)$.

COROLLARY 2.2. *Under the conditions of Theorem 2, the generalized risk process $R(t)$ ia asymptotically normal*

$$P\left(\frac{R(t) - (c-a)t}{\sqrt{(a^2+\sigma^2)t}} < x\right) \Longrightarrow \Phi(x/\delta) \quad (t \to \infty)$$

with some asymptotic variance δ^2 if and only if $\delta^2 \geq 1$ and

$$\mathsf{P}\left(\frac{\Lambda(t) - t}{\sqrt{t}} < x\right) \Longrightarrow \Phi\left(\frac{x|c-a|}{\sqrt{(\delta^2 - 1)(a^2 + \sigma^2)}}\right), \quad t \to \infty.$$

This statement easily follows from Theorem 2.2 and the Cramér-Lévy theorem on the decomposability of the normal law only into normal components, according to which the r.v. V acting in Theorem 2.2 should be normal itself.

3 Convergence rate estimates for generalized risk processes.

For simplicity of presentation, in Sections 3 – 6 we will assume that $\mathsf{E}\Lambda \equiv t$. From Theorem 2.2 it follows that the distribution of the r.v. $(R(t)-(c-a)t)/\sqrt{(a^2+\sigma^2)t}$ is close to the limiting one if and only if the distributions of the r.v.'s $(\Lambda(t)-t)/\sqrt{t}$ and V are close, or, which is in a certain sense the same, the distributions of the r.v.'s $\Lambda(t)$ and $\sqrt{t}V+t$ are close. However, in general, the latter r.v. may take negative values as well, whereas the controlling process of a Cox process must be positive. Therefore, instead of $\sqrt{t}V+t$ we will consider the "accompanying" process $\Lambda^*(t) = |\sqrt{t}V+t|$, which, as $t\to\infty$, becomes more and more close to $\sqrt{t}V+t$ and hence, to $\Lambda(t)$. Let $N^*(t)$ be the Cox process controlled by $\Lambda^*(t)$ and

$$R^*(t) = c\Lambda^*(t) - \sum_{j=0}^{N^*(t)} X_j. \tag{3.1}$$

The d.f.'s of r.v.'s $(R(t)-(c-a)t)/\sqrt{(a^2+\sigma^2)t}$ and $(R^*(t)-(c-a)t)/\sqrt{(a^2+\sigma^2)t}$ will be denoted $F_t(x)$ and $F_t{}^*(x)$, respectively. Then from the identity

$$F_t(x) = (F_t(x) - F_t{}^*(x)) + F_t{}^*(x) \tag{3.2}$$

it follows that an appropriate convergence rate estimate for the generalized risk process $R(t)$ can be constructed from an appropriate estimate of the accuracy of approximation of the distribution of $R(t)$ by that of $R^*(t)$ and the convergence rate estimate for $R^*(t)$. Consider an estimate of

$$\Delta_t \equiv \sup_x |F_t(x) - F_t^*(x)|.$$

Let $S_1(t) = c\Lambda_1(t) - \sum_{i=1}^{N_1(\Lambda_1(t))} X_i$ and $S_2(t) = c\Lambda_2(t) - \sum_{i=1}^{N_1(\Lambda_2(t))} X_i$ be two generalized risk processes generated by one and the same sequence $\{X_i\}_{i\geq 0}$, but controlled by (in general, different) processes $\Lambda_1(t)$ and $\Lambda_2(t)$, respectively. Denote

$$J_t(\lambda) = \mathsf{P}(\Lambda_1(t) < \lambda) - \mathsf{P}(\Lambda_2(t) < \lambda).$$

In what follows, the k-fold convolution of a distribution density (or a d.f.) $h(x)$ will be denoted $h^{*k}(x)$.

LEMMA 3.1. *Assume that the claims X_i have a finite third moment and a density $p(x)$ satisfying the condition $p(x) \leq A < \infty$. Then*

$$\sup_x |P(S_1(t) < x) - P(S_2(t) < x)| \leq \sup_{\lambda>0} e^{-\lambda}|J_t(\lambda)| +$$

$$+2(cK+1)\int_0^\infty \min\left(1, \frac{1}{\sqrt{\lambda}}\right)|J_t(\lambda)|d\lambda,$$

where

$$K = \sup_k \sqrt{k}\sup_x p^{*k}(x) < \infty.$$

The proof of this lemma can be found in (Bening and Korolev, 1998a)
From Lemma 3.1 it follows that

$$\Delta_t \le \sup_{\lambda\ge 0} e^{-\lambda}|\mathsf{P}(\Lambda(t)<\lambda)-\mathsf{P}(|\sqrt{t}V+t|<\lambda)|+$$

$$+2(cK+1)\int_0^\infty \min\left(1,\frac{1}{\sqrt{\lambda}}\right)|\mathsf{P}(\Lambda(t)<\lambda)-\mathsf{P}(|\sqrt{t}V+t|<\lambda)|\,d\lambda \le$$

$$\le \sup_{\lambda\ge 0} e^{-\lambda}|\mathsf{P}(\Lambda(t)<\lambda)-\mathsf{P}(\sqrt{t}V+t<\lambda)|+\sup_{\lambda\ge 0} e^{-\lambda}\mathsf{P}(\sqrt{t}V+t<-\lambda)+$$

$$+2(cK+1)\left[\int_0^\infty \min\left(1,\frac{1}{\sqrt{\lambda}}\right)|\mathsf{P}(\Lambda(t)<\lambda)-\mathsf{P}(\sqrt{t}V+t<\lambda)|\,d\lambda+\right.$$

$$\left.+\int_0^\infty \min\left(1,\frac{1}{\sqrt{\lambda}}\right)\mathsf{P}(\sqrt{t}V+t<-\lambda)|\,d\lambda\right]=$$

$$=\sup_{y\ge-\sqrt{t}} e^{-(\sqrt{t}y+t)}\left|\mathsf{P}\left(\frac{\Lambda(t)-t}{\sqrt{t}}<y\right)-\mathsf{P}(V<y)\right|+\sup_{\lambda\ge 0} e^{-\lambda}\mathsf{P}\left(V<-\frac{\lambda+t}{\sqrt{t}}\right)+$$

$$+2(cK+1)\left[\int_{-\sqrt{t}}^\infty \min\left(\sqrt{t},\frac{1}{\sqrt{\frac{u}{\sqrt{t}}+1}}\right)\left|\mathsf{P}\left(\frac{\Lambda(t)-t}{\sqrt{t}}<u\right)-\mathsf{P}(V<u)\right|du+\right.$$

$$\left.+\int_0^\infty \min\left(1,\frac{1}{\sqrt{\lambda}}\right)\mathsf{P}\left(V<-\frac{\lambda+t}{\sqrt{t}}\right)d\lambda\right]. \tag{3.3}$$

Consider the third summand on the right-hand side of (3.3). It represents the mean metric with the weight $w(t,u)=\min\{\sqrt{t},(u/\sqrt{t}+1)^{-1/2}\}\mathbf{1}(u>-\sqrt{t})$ characterizing the distance between the limit and pre-limit d.f.'s of the controlling process. It is easy to see that at $u<0$, the derivative with respect to t of the function $(u/\sqrt{t}+1)^{-1/2}$ is negative. Therefore, for these u the function $w(t,u)$ does not exceed its value at the point t at which $\sqrt{t}=(u/\sqrt{t}+1)^{-1/2}$. This value is equal to $\frac{1}{2}\left(|u|+\sqrt{u^2+4}\right)\le|u|+1$. Obviously, for negative u the inequality $w(t,u)\le 1$ holds so that $w(t,u)\le|u|+1$. Therefore, the properties of the third summand on the right-hand side of (3.3) appear to be similar to those of the well-investigated difference pseudomoments (see, e. g., (Zolotarev, 1997)). Thus, continuing (3.3), we obtain

$$\Delta_t \le \sup_{y\ge-\sqrt{t}}\left|\mathsf{P}\left(\frac{\Lambda(t)-t}{\sqrt{t}}<y\right)-\mathsf{P}(V<y)\right|+\mathsf{P}(V<-\sqrt{t})+$$

$$+2(cK+1)\left[\int_{-\sqrt{t}}^\infty (|u|+1)\left|\mathsf{P}\left(\frac{\Lambda(t)-t}{\sqrt{t}}<u\right)-\mathsf{P}(V<u)\right|du+\right.$$

$$+\int_0^\infty \min\left(1,\frac{1}{\sqrt{\lambda}}\right)\mathsf{P}\left(V<-\frac{\lambda+t}{\sqrt{t}}\right)d\lambda\Bigg] \equiv \omega(t). \tag{3.4}$$

Note that the second and the fourth summands in $\omega(t)$ do not depend on the pre-limit controlling process. For instance, if V is the standard normal r.v., then

$$\mathsf{P}(V<-\sqrt{t}) \leq \frac{\phi(\sqrt{t})}{\sqrt{t}} = \frac{1}{2\mathsf{P}it}\exp\left\{-\frac{t}{2}\right\},$$

$$\int_0^\infty \min\left(1,\frac{1}{\sqrt{\lambda}}\right)\mathsf{P}\left(V<-\frac{\lambda+t}{\sqrt{t}}\right)d\lambda \leq \int_0^\infty \min\left(1,\frac{1}{\sqrt{\lambda}}\right)\frac{\sqrt{t}}{\lambda+t}\phi\left(\frac{\lambda+t}{\sqrt{t}}\right)d\lambda \leq$$

$$\leq \exp\left\{-\frac{t}{2}\right\}\left(\frac{1}{2\sqrt{\pi t}}+\sqrt{\frac{t\pi}{8}}\right),$$

that is, as t grows, the summands in $\omega(t)$ mentioned above decrease at least as slow as exponentially.

Denote $\beta_3 = \mathsf{E}|X_1|^3$,

$$L_3 = C_0\frac{\beta_3}{\sigma^3},$$

where C_0 is the absolute constant in the Berry-Esseen inequality. It is known that $C_0 \leq 0.7655$ (see (Shiganov, 1986) or (Zolotarev, 1997)).

LEMMA 3.2. *Assume that $\beta_3 < \infty$. Then*

$$\sup_x\left|\mathsf{P}\left(\frac{1}{\sqrt{\lambda(\sigma^2+a^2)}}\left(\sum_{j=1}^{N_1(\lambda)}X_j - a\lambda\right)<x\right)-\Phi(x)\right| \leq \frac{L_3}{\sqrt{\lambda}}.$$

For the proof of this lemma see (Michel, 1986) or (Korolev and Shorgin, 1997).

THEOREM 3.1. *Let $\beta_3<\infty$, $E|V|<\infty$. Then for all $t\geq 1$ such that $P(V=-\sqrt{t})=0$ we have*

$$\sup_x\left|F_t(x)-E\Phi\left(x-\frac{c-a}{\sqrt{\sigma^2+a^2}}\cdot V\right)\right| \leq \frac{1}{\sqrt{t}}\inf_{0<\varepsilon<1}\left\{\frac{L_3}{\sqrt{(1-\varepsilon)}}+Q(\varepsilon)E|V|\right\}+\omega(t), \tag{3.5}$$

with $\omega(t)$ defined in (3.4) and

$$Q(\varepsilon) = \max\left\{\frac{1}{\varepsilon},\ \frac{1}{\sqrt{2\pi e(1-\varepsilon)}(1+\sqrt{1-\varepsilon})}\right\}.$$

The proof of this theorem based on Lemmas 3.1 and 3.2 was published in (Bening and Korolev, 1998a).

COROLLARY 3.1. *Under the conditions of Theorem 3.1 we have*

$$\sup_x \left| F_t(x) - \mathsf{E}\Phi\left(x - \frac{c-a}{\sqrt{\sigma^2+a^2}} \cdot V\right)\right| \le \frac{1.32}{\sqrt{t}}\left(\frac{\beta_3}{(a^2+\sigma^2)^{3/2}} + \mathsf{E}|V|\right) + \omega(t).$$

To prove this result it suffices to evaluate the right-hand side of (3.5) with ε being the root of the equation $Q(\varepsilon)\sqrt{1-\varepsilon} = 0.7655$.

As an illustration, consider the discrete time case $t = 0,\ 1,\ 2, \ldots$ and define the controlling process $\Lambda(t)$ as

$$\Lambda(0) = 0, \quad \Lambda(t) = \sum_{i=1}^{t} \Lambda_i, \tag{3.6}$$

where $\Lambda_1, \Lambda_2, \ldots$ are independent identically distributed nonnegative r.v.'s with $\mathsf{E}\Lambda_i = 1$, $i \ge 1$. This representation is typical for the situations where the controlling processes are homogeneous and have independent increments.

In addition, assume that $\delta^3 = \mathsf{E}|\Lambda_1 - 1|^3 < \infty$ and denote $\gamma^2 = \mathsf{D}\Lambda_1$. In this case, obviously, we have $\mathsf{P}(V < v) = \Phi(v/\gamma)$. Consider the first and the third summands in $\omega(t)$ (see (3.4)). By the Berry-Esseen inequality we have

$$\sup_{y \ge -\sqrt{t}} \left| \mathsf{P}\left(\frac{\Lambda(t)-t}{\sqrt{t}} < y\right) - \mathsf{P}(V < y)\right| \le \frac{C_0\delta^3}{\gamma^3\sqrt{t}} \tag{3.7}$$

Using the nonuniform estimate of the rate of convergence in the central limit theorem (see (Nagaev, 1965) and (Paditz, 1989)) we obtain

$$\int_{-\sqrt{t}}^{\infty} (|u|+1)\left|\mathsf{P}\left(\frac{\Lambda(t)-t}{\sqrt{t}} < u\right) - \mathsf{P}(V < u)\right| du \ \le$$

$$\le \ \frac{64\delta^3}{\gamma^3\sqrt{t}} \int_0^{\infty} \frac{u+1}{u^3+1} du \ = \ \frac{256\pi}{3\sqrt{3}} \cdot \frac{\delta^3}{\gamma^3\sqrt{t}}. \tag{3.8}$$

With the account of the estimates for the second and fourth summands in $\omega(t)$, from (3.7) and (3.8) by Theorem 3.1 we finally obtain that in the situation under consideration

$$\sup_x \left| F_t(x) - \mathsf{E}\Phi\left(x - \frac{(c-a)V}{\sqrt{\sigma^2+a^2}}\right)\right| = \sup_x \left| F_t(x) - \Phi\left(\frac{x}{\sqrt{1+\frac{\gamma^2(c-a)^2}{\sigma^2+a^2}}}\right)\right| = O\left(\frac{1}{\sqrt{t}}\right).$$

4 Exponential bounds for the probabilities of negative values of generalized risk processes.

Let u be the initial capital of the insurance company. Within the framework of the *dynamic insurance model*, a ruin is understood as the event consisting in that there exists *some* $t > 0$ at which the surplus of the company becomes negative. We will consider the calculation and statistical estimation of the probability of this event (ruin probability) for a generalized risk process in the concluding section of this paper.

At the same time, when the surplus becomes negative, actually there may be no ruin since there is a possibity for the insurer to take a credit. Therefore, it may be of considerable interest to know the probability of the surplus to be negative at some *fixed* time. It is quite resonable to identify these probabilities with ruin probabilities within the so-called *static* insurance models which concern the insurance activities within an unchanged (fixed) portfolio (see, e. g., (Rotar' and Bening, 1994)). Indeed, the number of insurance payments $N(t) = N_1(\Lambda(t))$ *within a certain portfolio* should be in some sense proportional to $\Lambda(t)$ which is interpreted as the "size" of the insurance porfolio. In this situation it is reasonable not to consider t as a "physical" time, but to assume simply that the process $\Lambda(t)$ is parametrized by its location characteristic, say, expectation, and, hence, to investigate the behavior of the probabilities of negative values of a generalized risk process (which are ruin probabilities in the static model) depending on the mean value of $\Lambda(t)$, that is, on the mean size of the portfolio under the condition that this size (infinitely) increases.

We will look for the bounds for the ruin probability in the static model, that is, for $\mathsf{P}(u + R_2(t) < 0)$, which characterize the decrease rate of this probability as u and t grow.

Whatever $x \in \mathbb{R}$ is, we have

$$\mathsf{P}\left(u + R_2(t) < x\right) = \mathsf{P}\left(\sum_{i=1}^{N_1(\Lambda(t))} X_i > u + c\Lambda(t) - x\right) =$$

$$= \int_0^\infty \mathsf{P}\left(\sum_{i=1}^{N_1(\lambda)} X_i > u + c\lambda - x\right) d\mathsf{P}(\Lambda(t) < \lambda). \tag{4.1}$$

In the paper (Rotar', 1972) the following analog of the Bernstein inequality for Poisson random sums.

LEMMA 4.1. *Let $|X_i| < M < \infty$ with probability one, $i \geq 1$. Then for any $y > 0$*

$$\mathsf{P}\left(\frac{\sum_{i=1}^{N_1(\lambda)} X_i - a\lambda}{\sqrt{\lambda(a^2+\sigma^2)}} > y\right) \leq \begin{cases} \exp\left\{-\dfrac{y^2}{4}\right\}, & \text{если} \quad y \leq \dfrac{\sqrt{\lambda(a^2+\sigma^2)}}{M}, \\[2ex] \exp\left\{-\dfrac{y\sqrt{\lambda(a^2+\sigma^2)}}{4M}\right\}, & \text{если} \quad y > \dfrac{\sqrt{\lambda(a^2+\sigma^2)}}{M}. \end{cases}$$

Assuming henceforth that the claims are uniformly bounded, that is, $0 \leq X_i < M < \infty$, $i \geq 1$, and continuing (4.1) with regard to the requirement of positiveness of y in Lemma 4.1, we obtain

$$\mathsf{P}(u + R_2(t) < x) \leq \mathsf{P}\left(\Lambda(t) < \frac{x-u}{c-a}\right) +$$

$$+ \int_{(x-u)/(c-a)}^{\infty} \mathsf{P}\left(\frac{\sum_{i=1}^{N_1(\lambda)} X_i - a\lambda}{\sqrt{\lambda(a^2+\sigma^2)}} > \frac{(c-a)\lambda + u - x}{\sqrt{\lambda(a^2+\sigma^2)}}\right) d\mathsf{P}(\Lambda(t) < \lambda) \leq$$

$$\leq \mathsf{P}\left(\Lambda(t) < \frac{x-u}{c-a}\right) + \int\limits_{(x-u)/(c-a)}^{(x-u)/\left(c-a-\frac{a^2+\sigma^2}{M}\right)} \exp\left\{-\frac{((c-a)\lambda+u-x)^2}{4\lambda(a^2+\sigma^2)}\right\} d\mathsf{P}(\Lambda(t)<\lambda) +$$

$$+ \int\limits_{(x-u)/\left(c-a-\frac{a^2+\sigma^2}{M}\right)}^{\infty} \exp\left\{-\frac{(c-a)\lambda+u-x}{4M}\right\} d\mathsf{P}(\Lambda(t)<\lambda) \leq$$

$$\leq \mathsf{P}\left(\Lambda(t) \leq \frac{x-u}{c-a-\frac{a^2+\sigma^2}{M}}\right) + \int\limits_{(x-u)/\left(c-a-\frac{a^2+\sigma^2}{M}\right)}^{\infty} \exp\left\{-\frac{(c-a)\lambda+u-x}{4M}\right\} d\mathsf{P}(\Lambda(t)<\lambda). \tag{4.2}$$

Put $x = 0$ in (4.2) and denote $\psi(t,u) = \mathsf{P}(u + R_2(t) < 0)$. Then from (4.2) we obtain the following result

THEOREM 4.1. *If* $0 \leq X_i < M < \infty$ *a. s.,* $i \geq 1$, *then*

$$\psi(t,u) \leq \exp\left\{-\frac{u}{4M}\right\} \int\limits_0^{\infty} \exp\left\{-\frac{(c-a)\lambda}{4M}\right\} d\mathsf{P}(\Lambda(t)<\lambda). \tag{4.3}$$

For illustration, consider the discrete time case $t = n = 1, 2, \ldots$ and assume that the controlling process $\Lambda(n)$ is representable in the form (3.6) in which $\Lambda_1, \Lambda_2, \ldots$ are independent identically distributed nonnegative r.v.'s. With the account of the abovesaid, $\psi(n,u)$ can be interpreted as the ruin probability of an insurance company working with n homogeneous portfolios with the total initial capital u. From Theorem 4.1 we easily obtain the following estimate

COROLLARY 4.1. *In the discrete time case* $t = n = 1, 2, \ldots$, *if the controlling process* $\Lambda(n)$ *is representable in the form (3.6), then under the conditions of Theorem 4.1 we have*

$$\psi(n,u) \leq \exp\left\{-\frac{u}{4M}\right\} \Delta^n,$$

where

$$0 < \Delta = \mathsf{E}\exp\left\{-\frac{c-a}{4M}\Lambda_1\right\} < 1.$$

From the discrete time case return to the general situation.

COROLLARY 4.1. *Under the conditions of Theorem 3, for any* $t > 0$

$$\psi(t,u) \leq \exp\left\{-\frac{u}{4M}\right\}.$$

To emphasize the character of the dependence of the right-hand side of (4.3) on t, recall that according to Theorem 2.2, the distribution of the risk process $R(t)$ is close to limiting as $t \to \infty$ if and only if the distribution of the r.v. $(\Lambda(t) - t)/\sqrt{t}$ is close to that of the r.v.

V or, which is in some sense the same, if the distribution of the r.v. $\Lambda(t)$ is close to that of $\sqrt{t}V+t$. Continuing (4.3) with the help of the identity

$$\mathsf{P}(\Lambda(t)<\lambda)=\mathsf{P}(\sqrt{t}V+t<\lambda)+[\mathsf{P}(\Lambda(t)<\lambda)-\mathsf{P}(\sqrt{t}V+t<\lambda)]$$

and integrating by parts, we obtain the following statement.

COROLLARY 4.2. *Under the conditions of Theorem 4.1 we have*

$$\psi(t,u)\le\exp\left\{-\frac{u+(c-a)t}{4M}\right\}\left(\int\limits_{-\sqrt{t}}^{\infty}\exp\left\{-\frac{(c-a)\sqrt{t}v}{4M}\right\}dP(V<v)+Q(t,u)\right),$$

where

$$Q(t,u)\ \le\ \mathsf{P}(V<-\sqrt{t})\exp\left\{\frac{(c-a)t}{4M}\right\}+$$
$$+\frac{(c-a)\sqrt{t}}{4M}\int\limits_{-\sqrt{t}}^{\infty}\exp\left\{-\frac{(c-a)\sqrt{t}v}{4M}\right\}\left|\mathsf{P}\left(\frac{\Lambda(t)-t}{\sqrt{t}}<v\right)-\mathsf{P}(V<v)\right|dv$$

For more details see (Bening and Korolev, 1998a). It is clear that if $\Lambda(t)=\sqrt{t}V+t$, где $\mathsf{P}(V\ge-\sqrt{t})=1$, then $Q(t,u)=0$.

5 Asymptotic expansions for distributions of generalized risk processes.

We shall say that a r.v. Y satisfies the Cramér (C) condition if

$$\limsup_{|s|\to\infty}|\mathsf{E}\exp\{isY\}|<1. \tag{5.1}$$

The following statement is well known (see, e. g., (Petrov, 1975)). Denote $S_n=X_1+\cdots+X_n$. The standard normal density will be denoted $\phi(x)$. The characteristic function (ch.f.) of a r.v. X will be denoted $f_X(s)$, $s\in\mathbb{R}$.

PROPOSITION 5.1. *Let, in addition to the conditions introduced in Sect. 1, the r.v.'s $\{X_j\}_{j\ge1}$ satisfy the Cramér condition (5.1) and $\mathsf{E}|X_1|^k<\infty$ for some integer $k\ge3$. Then*

$$\sup_x\left|P\left(\frac{S_n-an}{\sigma\sqrt{n}}<x\right)-\Phi(x)-\sum_{j=1}^{k-2}\frac{Q_j(x)}{n^{j/2}}\right|\ =\ o\left(n^{-(k-2)/2}\right),$$

where the functions $Q_j(x)$ are defined by the formulas

$$Q_j(x)=-\phi(x)\sum H_{j+2l-1}(x)\prod_{m=1}^{j}\frac{1}{k_m!}\left(\frac{\gamma_{m+2}}{(m+2)!\sigma^{m+2}}\right)^{k_m},\quad j=1,\dots,k-2. \tag{5.2}$$

Here the summation is carried out by all integer nonnegative solutions $(k_1, \ldots, k_j)$ *of the equation* $k_1 + 2k_2 + \ldots + jk_j = j$, $l = k_1 + \ldots + k_j$; γ_{m+2} *is the* $(m+2)$*th semiinvariant of the r.v.* X_1 *and* $H_m(x)$ *are the Chebyshev–Hermite polynomials of power* m, *that is,*

$$H_m(x)\phi(x) = (-1)^m \phi^{(m)}(x).$$

In particular, if we denote $\alpha_l = \mathsf{E}X_1^l$, $l = 1, 2, \ldots$, then

$$Q_1(x) = -\phi(x)(x^2 - 1)\frac{\alpha_3}{6\sigma^3}, \tag{5.3}$$

$$Q_2(x) = -\phi(x)\left[(x^3 - 3x)\frac{\alpha_4 - 3\sigma^4}{24\sigma^4} + (x^5 - 10x^3 + 15x)\frac{\alpha_3^2}{72\sigma^6}\right]. \tag{5.4}$$

Now we will present some analogs of this statement for generalized risk processes. Without essential restriction of the applicability of our results, we will assume that $\Lambda(t) = \sqrt{t}V + t$ where V is a r.v. with $\mathsf{E}V = 0$.

THEOREM 5.1. *Assume that the r.v.* V *is representable as* $V = V_0 - EV_0$ *with* V_0 *being a nonnegative r.v. satisfying the condition: there satisfies a* $\gamma > 0$ *such that for any* $h \geq 0$

$$Ee^{hV_0} \leq e^{\gamma h^2}. \tag{5.5}$$

Let $E|X_1|^k < \infty$ *for some integer* $k \geq 3$ *and the r.v.* X_1 *satisfies the Cramér condition (5.1). Then for any* $t \geq (EV_0)^2$

$$\sup_x \left| P\left(\frac{R_2(t) - (c-a)t}{\sqrt{t(a^2+\sigma^2)}} < x\right) - E\Phi\left(x - \frac{c-a}{\sqrt{a^2+\sigma^2}} \cdot V\right) - \sum_{j=1}^{k-2} \frac{w_j(x)}{t^{j/2}}\right| = o(t^{-\frac{k-2}{2}}),$$

where

$$w_j(x) = -\frac{\sigma}{\alpha_2} \sum_{\substack{l+m=j\\ l,m\geq 0}} \sum_{n=0}^{m} \int \int_{-\infty}^{(cz - x\sqrt{\alpha_2})/\sigma} \overline{P}_l(-D_y) P_{mn}(-D_y) \phi\left(\frac{\sigma y - az}{\sqrt{\alpha_2}}\right) dy z^n dP(V < z),$$

$$j = 0, \ldots, k-2;$$

D_y *is the operator of formal differentiation with respect to* y. *The polynomials* $\overline{P}_l(\cdot)$ *are defined by the relation*

$$\overline{P}_l(it) = \sum \prod_{m=1}^{j} \frac{1}{k_m!}\left[\frac{(it)^{m+2}\alpha_{m+2}}{(m+2)!\sigma^{m+2}}\right]^{k_m}, \quad j = 1, \ldots, k-2,$$

where the summation is carried out by all nonnegative solutions $k_1, \ldots, k_j$ *of the equation* $k_1 + 2k_2 + \ldots + jk_j = j$, $\alpha_{m+2} = EX_1^{m+2}$, $\overline{P}_0(x) \equiv 1$, *the polynomials* $P_{md}(\cdot)$ *are defined by the formal equality*

$$\exp\left\{\sum_{l=1}^{k-2} v \frac{x^{l+1}\alpha_{l+1}}{(l+1)!\sigma^{l+1}} t^{-l/2}\right\} = \sum_{m=0}^{\infty} t^{-m/2} \sum_{d=0}^{m} v^d P_{md}(x). \tag{5.6}$$

The proof of Theorem 5.1 is similar to that of Theorem 3 from the paper (Bening and Korolev, 1998a).

REMARK 5.1. It is easy to see that

$$P_{00}(x) \equiv 1, \; P_{j0} \equiv 0, \; j = 1, \ldots, k-2;$$
$$P_{11}(x) = \frac{x^2\alpha_2}{2\sigma^2}, \; P_{21}(x) = \frac{x^3\alpha_3}{6\sigma^3}, \; P_{22}(x) = \frac{x^4\alpha_2^2}{8\sigma^4},$$
$$P_{31}(x) = \frac{x^4\alpha_4}{24\sigma^4}, \; P_{32}(x) = \frac{x^5\alpha_2\alpha_3}{12\sigma^5}, \; P_{33}(x) = \frac{x^6\alpha_2^3}{48\sigma^6},$$

$$\overline{P}_1(x) = \frac{x^3\alpha_3}{6\sigma^3}, \; \overline{P}_2(x) = \frac{x^4\alpha_4}{24\sigma^4} + \frac{x^6\alpha_3^2}{72\sigma^6}.$$

REMARK 5.2. The first two functions w_1 and w_2 have the form

$$w_1(x) = \int \phi\left(x - z\frac{c-a}{\sqrt{a^2+\sigma^2}}\right)\left[\frac{\alpha_3}{6\sigma\alpha_2}H_2\left(x - z\frac{c-a}{\sqrt{a^2+\sigma^2}}\right) - \right.$$
$$\left. -\frac{z}{2}H_1\left(x - z\frac{c-a}{\sqrt{a^2+\sigma^2}}\right)\right] d\mathsf{P}(V<z);$$
$$w_2(x) = -\int \phi\left(\frac{\sigma x - az}{\sqrt{a^2+\sigma^2}}\right)\left[\frac{\alpha_4}{24\sigma^2\alpha_2}H_3\left(x - z\frac{c-a}{\sqrt{a^2+\sigma^2}}\right) + \right.$$
$$+\frac{\alpha_3}{72\sigma^4\alpha_2}H_5\left(x - z\frac{c-a}{\sqrt{a^2+\sigma^2}}\right) - z\left(\frac{\alpha_3}{12\sigma^3}H_4\left(x - z\frac{c-a}{\sqrt{a^2+\sigma^2}}\right) + \right.$$
$$\left.\left. +\frac{\alpha_3}{6\sigma\alpha_2}H_2\left(x - z\frac{c-a}{\sqrt{a^2+\sigma^2}}\right)\right) + \frac{z^2\alpha_2^3}{8\sigma^2}H_3\left(x - z\frac{c-a}{\sqrt{a^2+\sigma^2}}\right)\right] d\mathsf{P}(V<z),$$

where $H_m(\cdot)$ are the Chebyshev–Hermite polynomials.

REMARK 5.3. Condition (5.5) holds, e. g., if $V_0 = |\xi|$ with ξ being a bounded r.v.. It also holds for the normal r.v. ξ. However, this condition does not hold if V_0 is a Poisson or exponential r.v.. The r.v. V_0 must have all moments.

Now consider the discrete-time case $t = n = 1, 2, \ldots$ and assume that the controlling process $\Lambda(n)$ has the form (3.6) where $\{\Lambda_i\}$ are independent identically distributed r.v.'s, $\Lambda_1 \geq 0$, $i \geq 1$. As we have noted above, this representation corresponds, for instance, to the case where $\Lambda(t)$ is a homogeneous process with independent increments and the generalized risk process is observed at equidistant time instants so that Λ_i are the increments of the controlling process $\Lambda(t)$ on (unit) intervals between observations. Assume that $\mathsf{E}\Lambda_1 = 1$ so that $\mathsf{E}\Lambda(n) = n$. Denote

$$\nu_l = \mathsf{E}(\Lambda_1 - 1)^l, \quad l = 1, 2, \ldots$$

Define the formal "semiinvariants" $æ_j$ by the equality

$$\log \mathsf{E}\exp\{(\Lambda_1 - 1)(f_{X_1}(s) - 1 - isc)\} = \sum_{j=2}^{\infty} \frac{æ_j}{j!}(is)^j.$$

In particular,

$$æ_2 = \nu_2(\alpha_1 - c)^2, \; æ_3 = 3\nu_2\alpha_2(\alpha_1 - c) + \nu_3(\alpha_1 - c)^3,$$
$$æ_4 = 3\nu_2\alpha_2^2 + \nu_4(\alpha_1 - c)^4 + 6(\alpha_1 - c)^2\alpha_2\nu_3 - 3\nu_2^2(\alpha_1 - c)^4.$$

THEOREM 5.2. *Assume that there exists a $\gamma > 0$ such that for any $h \geq 0$ the r.v. Λ_1 satisfies the inequality*

$$\mathsf{E}e^{h\Lambda_1} \leq e^{\gamma h}. \tag{5.7}$$

Let $\mathsf{E}|X_1|^k < \infty$ for some integer $k \geq 3$. Assume that X_1 satisfies the Cramér condition (5.1). Then

$$\sup_x \left| P\left(\frac{S(n) - (a-c)n}{\sqrt{n\alpha_2}}\right) - \Phi\left(x\sqrt{\frac{\alpha_2}{æ_2 + \alpha_2}}\right) - \sum_{j=1}^{k-2} \frac{v_j(x)}{n^{j/2}}\right| = o(n^{-\frac{k-2}{2}}),$$

where

$$v_j(x) = -\frac{\alpha_2}{æ_2 + \alpha_2}\overline{Q}_j\left(-\frac{\sigma x}{\sqrt{æ_2 + \alpha_2}}\right), \quad j = 1, \ldots, k-2,$$

and the functions $\overline{Q}_j$ are defined by the formulas (5.2) with the semiinvariants γ_{m+2} replaced by $\alpha_{m+2} + æ_{m+2}$.

REMARK 5.4. We do not assume that the r.v.'s Λ_i satisfy the Cramér condition (5.1). Therefore they may be lattice.

REMARK 5.5. Condition (5.7) is stronger than (5.5). This condition is satisfied by any bounded r.v. Λ_1, for example, by a binomial r.v., whereas Poisson or exponential r.v.'s do not satisfy this condition. Also note that the r.v. Λ_1 must have all moments.

REMARK 5.6. The first two functions $v_j(x)$ have the following forms:

$$v_1(x) = -\frac{æ_3 + \alpha_3}{6\sigma(æ_2 + \alpha_2}H_2\left(x\sqrt{\frac{\alpha_2}{æ_2 + \alpha_2}}\right)\phi\left(x\sqrt{\frac{\alpha_2}{æ_2 + \alpha_2}}\right),$$

$$v_2(x) = -\frac{æ_4 + \alpha_4}{24\sigma^2(æ_2 + \alpha_2)}H_3\left(x\sqrt{\frac{\alpha_2}{æ_2 + \alpha_2}}\right)\phi\left(x\sqrt{\frac{\alpha_2}{æ_2 + \alpha_2}}\right) -$$
$$\frac{(æ_3 + \alpha_3)^2}{72\sigma^4(æ_2 + \alpha_2)}H_5\left(x\sqrt{\frac{\alpha_2}{æ_2 + \alpha_2}}\right)\phi\left(x\sqrt{\frac{\alpha_2}{æ_2 + \alpha_2}}\right),$$

where $H_m(\cdot)$ are Chebyshev–Hermite polynomials.

6 Asymptotic expansions for the quantiles of generalized risk processes.

In this section we will give the asymptotic expansions up to the terms of order $o(t^{-1/2})$ for the quantiles of generalized risk processes. The following statement will play the main role in our constructions. Let $\{Z(t)\}_{t\geq 0}$ be a random process. Assume that for each $t \geq 0$ the distribution of the r.v. $Z(t)$ is continuous. For $\beta \in (0,1)$ and $t \geq 0$, the quantile of order β of the r.v. $Z(t)$ will be denoted $u_\beta(t)$:

$$\mathsf{P}(Z(t) < u_\beta(t)) = \beta.$$

LEMMA 6.1. *Assume that for the one-dimensional distribution function of the random process $Z(t)$ the asymptotic expansion of the form*

$$P(Z(t) < x) = G_0(x) + t^{-1/2}G_1(x) + t^{-1}G_2(x) + o(t^{-1})$$

is valid with the functions $G_0''(x)$, $G_1'(x)$ and $G_2(x)$ being continuous and $G_0'(x) > 0$. Then for any $\beta \in (0,1)$ the quantiles $u_\beta(t)$ of $Z(t)$ can be expanded as

$$u_\beta(t) = u_\beta - \frac{G_1(u_\beta)}{G_0'(u_\beta)}t^{-1/2} +$$

$$+ \frac{G_0'(u_\beta)G_1(u_\beta)G_1'(u_\beta) - (G_0'(u_\beta))^2 G_2(u_\beta) - \frac{1}{2}G_1^2(u_\beta)G_0''(u_\beta)}{(G_0'(u_\beta))^3}t^{-1} + o(t^{-1}),$$

where u_β is the β-quantile of G_0, $G_0(u_\beta) = \beta$.

This statement was proved in (Bening and Korolev, 1998b).

REMARK 6.1. If we denote

$$u_\beta^*(t) = u_\beta - \frac{G_1(u_\beta)}{G_0'(u_\beta)}t^{-1/2} +$$

$$+ \frac{G_0'(u_\beta)G_1(u_\beta)G_1'(u_\beta) - (G_0'(u_\beta))^2 G_2(u_\beta) - \frac{1}{2}G_1^2(u_\beta)G_0''(u_\beta)}{(G_0'(u_\beta))^3}t^{-1},$$

where $G_0(u_\beta) = \beta$, then it can be shown that

$$\mathsf{P}(Z(t) < u_\beta^*(t)) = \beta + o(t^{-1}).$$

Assume that the controlling process has the form $\Lambda(t) = \sqrt{t}V + t$, $t \geq 0$, where $V = V_0 - \mathsf{E}V_0$, and V_0 is a nonnegative r.v.. For $\beta \in (0,1)$, the β-quantile of the r.v. $R(t)$ with this controlling process will be denoted $v_\beta(t)$.

THEOREM 6.1. *Let the r.v. V_0 satisfy condition (5.5). Assume that $\mathsf{E}|X_1|^4 < \infty$ and X_1 satisfies the Cramér condition (5.1). Then, as $t \to \infty$, we have*

$$v_\beta(t) = -\frac{w_1(v_\beta)}{w_0'(v_\beta)} + \sqrt{t\alpha_2}v_\beta + (c-a)t +$$

$$+ \frac{1}{\sqrt{t}}\left[\frac{w_0'(v_\beta)w_1(v_\beta)w_1'(v_\beta) - (w_0'(v_\beta))^2 w_2(v_\beta) - \frac{1}{2}w_1^2(v_\beta)w_0''(v_\beta)}{(w_0'(v_\beta))^3}\right] + o(t^{-1/2}),$$

where v_β is the β-quantile of the d.f.

$$w_0(x) = E\Phi\left(x - \frac{c-a}{\sqrt{\alpha_2}} \cdot V\right),$$

and the form of the functions $w_1(x)$ and $w_2(x)$ was presented in Remark 5.2.

This statement directly follows from Lemma 6.1 and Theorem 5.1 with the account of the obvious relation $v_\beta(t) = \sqrt{t\alpha_2}\hat{v}_\beta(t) + (c-a)t$, where $\hat{v}_\beta(t)$ is the β-quantile of the r.v. $(R(t) - (c-a)t)/\sqrt{t\alpha_2}$.

REMARK 6.2. In practice, Theorem 6.1 is applicable for $t \geq (\mathsf{E}V_0)^2$.

Now consider the discrete-time case $t = n = 1, 2, \ldots$ and assume that the controlling process is represented as (3.6) where $\Lambda_i \geq 0$, are independent and identically distributed and $\mathsf{E}\Lambda_1 = 1$. For $\beta \in (0,1)$ the β-quantile of the r.v. $R(n)$ with this controlling process will be denoted $\overline{v}_\beta(n)$.

THEOREM 6.2. *Assume that for any $h \geq 0$ the r.v. Λ_1 satisfies the inequality (5.7). Let $E|X_1|^4 < \infty$. Assume that X_1 satisfies the Cramér condition (5.1). Then, as $n \to \infty$,*

$$\overline{v}_\beta(n) = -\frac{æ_3 + \alpha_3}{6\sigma\sqrt{æ_2 + \alpha_2}}(u_\beta^2 - 1) + \sqrt{n(æ_2 + \alpha_2)}u_\beta + n(c-a) +$$

$$+ \frac{1}{24\sigma^2\sqrt{n(æ_2 + \alpha_2)}}\left[(æ_4 + \alpha_4)(u_\beta^3 - 3u_\beta) - \frac{(æ_3 + \alpha_3)^2(u_\beta^5 - 6u_\beta^3 + 5u_\beta)}{3(æ_2 + \alpha_2)} + \right.$$

$$\left. \frac{(æ_3 + \alpha_3)^2}{3\sigma^2}(u_\beta^5 - 10u_\beta^3 + 15u_\beta)\right] + o(n^{-1/2}),$$

where u_β is the β-quantile of the standard normal distribution and the form of the "semi-invariants" $æ_j$ was presented just before Theorem 5.2.

Theorem 6.2 directly follows from Lemma 6.1 and Theorem 5.2.

Theorems 6.1 and 6.2 make it possible to construct asymptotic approximations for the minimum admissible surplus of an insurance company within the model of a generalized risk process. Namely, if u is the initial capital of the insurance company, then its surplus at time t is $R(t) + u$. If the confidence level is $\gamma \in (0,1)$, then the minimum admissible bound is the function $m_\gamma(t,u)$ such that $\mathsf{P}(R(t) + u \geq m_\gamma(t,u)) = \gamma$. This means that

$$m_\gamma(t,u) = v_{1-\gamma}(t) + u$$

within the assumptions of Theorem 6.1 or

$$m_\gamma(n,u) = \overline{v}_{1-\gamma}(n) + u$$

within the assumptions of Theorem 6.2.

7 Statistical estimation of ruin probabilities for generalized risk processes.

7.1 Nonparametric estimator for ruin probability.

Let $u > 0$ be the initial capital of an insurance company. It is easy to see that the ruin probability for the generalized risk process

$$\psi(u) = \mathsf{P}(u + \inf_{t>0} R(t) < 0)$$

coincides with ruin probability for the classical risk process

$$\psi_0(u) = \mathsf{P}(u + \inf_{t>0} R_0(t) < 0),$$

since $R(t)$ differs from $R_0(t)$ only by a random (in general, inhomogeneous) compression of time which does not change the amplitudes of trajectories. We will essentially use this fact.

There are many analytical methods of calculating the bounds of ruin probabilities. All of them essentially use the information on the behavior of the tails of the distributions of claims. However, in real practice, this behavior *is unknown* since the statistical inference concerning the distribution of claims is made on the basis of a *finite* sample of observed payments resulting in that it is impossible to obtain the complete information on the behavior of the tails of their distribution at those values of arguments which exceed the greatest observation and are less than the smallest observation. Therefore, it is very important to have a possibility to directly estimate the ruin probability statistically.

The problem of statistical estimation of ruin probability for a generalized risk process (as well as for a classical risk process) from its pre-history up to some time t_0 has an important peculiarity. Namely, the number $N(t_0)$ of insurance payments up to this time is random. Therefore, in practice, the ruin probability should be estimated from the sample $X_1, X_2, \ldots, X_{N(t_0)}$ of *random* size. As this is so, the class of possible distributions of the r.v. $N(t_0)$ is very wide even under the restrictions introduced above and by no means reduces to Poisson laws. For example, if $\Lambda(t_0)$ has the gamma-distribution, then the distribution of $N(t_0)$ is negative binomial.

A very important step in the direction of statistical estimation of ruin probability was made by Croux and Veraverbecke (Croux and Veraverbecke, 1990). However, their nice results seem to be hardly applicable in practice, since, first, they are based on samples with *nonrandom* sizes and, second, they cannot be used for the construction of (asymptotic) confidence intervals because the asymptotic distribution (more precisely, its variance) of the estimator proposed in (Croux and Veraverbecke, 1990) depends on the *unknown* distribution of claims.

Our aim is to construct practically applicable point and interval estimators of ruin probability for generalized risk processess.

Based on the principle of expected nonruin, assume that $c > a$. As we have seen above, $\psi(u) = \psi_0(u)$. Therefore, we can use the well-known representation of the ruin probability for the classical risk process

$$\psi_0(u) = \left(1 - \frac{a}{c}\right) \sum_{k=1}^{\infty} \left(\frac{a}{c}\right)^k \left(1 - G^{*k}(u)\right), \tag{7.1}$$

where

$$G(x) = \frac{1}{a} \int_0^x (1 - F(y))dy,$$

$F(y) = \mathsf{P}(X_1 < y)$, and the symbol G^{*k}, as above, denotes the k-fold convolution of the d.f. G with itself: $G^{*k}(x) = (G^{*(k-1)} * G)(x)$, $k \geq 1$, and G^{*0} is the d.f. with a single unit jump at zero. Relation (7.1) is called the Pollaczek–Khinchin formula or Beekman's convolution formula (see, e. g., (Beekman, 1968), (Asmussen, 1987), (Croux and Veraverbecke, 1990)).

Assume that the parameters c and a are known. At first we formally assume that at our disposal we have a sample $X_1, \ldots, X_n$, where $n \geq 1$ is some *nonrandom* integer number. For this situation, in (Croux and Veraverbecke, 1990) the following nonparametric estimator for $\psi_0(u)$ was proposed. Since

$$\psi_0(u) = \frac{a}{c} - \left(1 - \frac{a}{c}\right) \overline{\psi}_0(u), \tag{7.2}$$

where

$$\overline{\psi}_0(u) = \sum_{k=1}^{\infty} \left(\frac{a}{c}\right)^k G^{*k}(u), \tag{7.3}$$

it suffices to construct an estimator for $\overline{\psi}_0(u)$.

Let $A \subseteq \mathbb{R}^1$. The symbol $\mathbf{1}_A(x)$ will denote the indicator function of a set A: $\mathbf{1}_A(x) = 1$, if $x \in A$, and $\mathbf{1}_A(x) = 0$, if $x \notin A$. Let $Y_1, Y_2, \ldots$ be independent identically distributed r.v.'s with the d.f. $G(x)$. Then

$$G^{*k}(u) = \mathsf{P}(Y_1 + \ldots + Y_k < u) =$$

$$= \frac{1}{a^k} \int_0^{\infty} \cdots \int_0^{\infty} \mathbf{1}_{(-\infty,u)}(y_1 + \ldots + y_k) \prod_{j=1}^{k} (1 - F(y_j)) dy_1 \cdots dy_k. \tag{7.4}$$

Let $F_n(x)$ be the empirical distribution function constructed from the sample $X_1, \ldots, X_n$, that is,

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{(-\infty,x)}(X_i).$$

Then, replacing F in (7.4) by F_n, we obtain an estimator for $G^{*k}(u)$ in the form of the U-statistic

$$U_{n,k} = \left(C_n^k\right)^{-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} h_k(X_{i_1}, \ldots, X_{i_k})$$

with the symmetric kernel

$$h_k(x_1 \ldots, x_k) = \frac{1}{a^k} \int_0^{\infty} \cdots \int_0^{\infty} \mathbf{1}_{(-\infty,u)}(y_1 + \ldots + y_k) \prod_{j=1}^{k} \mathbf{1}_{[y_j,\infty)}(x_j) dy_1 \cdots dy_k.$$

Let $m(n)$ be some integer, $1 \le m(n) \le n$. By virtue of (7.2) and (7.3), as an estimator for $\psi_0(u)$ under a nonrandom sample size n we formally take

$$\psi_n(u) = \frac{a}{c} - \left(1 - \frac{a}{c}\right)\overline{\psi}_n(u), \tag{7.5}$$

where

$$\overline{\psi}_n(u) = \sum_{k=1}^{m(n)} \left(\frac{a}{c}\right)^k U_{n,k}. \tag{7.6}$$

Now it is clear that, based on the sample $X_1, \ldots, X_{N(t_0)}$, as an estimator for $\psi(u) = \psi_0(u)$ we should take

$$\psi_{N(t_0)}(u) = \frac{a}{c} - \left(1 - \frac{a}{c}\right)\overline{\psi}_{N(t_0)}(u), \tag{7.7}$$

where

$$\overline{\psi}_{N(t_0)}(u) = \sum_{k=1}^{m(N(t_0))} \left(\frac{a}{c}\right)^k U_{N(t_0),k}. \tag{7.8}$$

7.2 The asymptotic properties of the estimator of ruin probability.

Denote

$$\sigma^2 = \left(1 - \frac{a}{c}\right)^2 \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \left(\frac{a}{c}\right)^{k+m} km\sigma_{k,m}, \tag{7.9}$$

where

$$\sigma_{k,m} = \mathsf{E}h_k(X_1)h_m(X_1) - G^{*k}(u)G^{*m}(u), \tag{7.10}$$

$$h_j(x) = \frac{1}{a}\int_0^{\infty} G^{*(j-1)}(u-y)\mathbf{1}_{[y,\infty)}(x)dy. \tag{7.11}$$

The following statement plays the main role in this subsection.

THEOREM 7.1. *Let the estimator $\psi_{N(t)}(u)$ be defined by the relations (7.7) and (7.8) with the function $m(n)$ infinitely increasing as $n \to \infty$ so that*

$$\lim_{n\to\infty} \frac{\log n}{m(n)} = 0. \tag{7.12}$$

Assume that $\Lambda(t) \xrightarrow{P} \infty$ as $t \to \infty$. Let $d(t) > 0$ be a function infinitely increasing as $t \to \infty$. Then the r.v. $\sigma^{-1}\sqrt{d(t)}(\psi_{N(t)}(u) - \psi(u))$ has a limit distribution as $t \to \infty$:

$$\sigma^{-1}\sqrt{d(t)}(\psi_{N(t)}(u) - \psi(u)) \Longrightarrow Z \quad (t \to \infty), \tag{7.13}$$

if and only if there exists a nonnegative r.v. Y such that

$$\frac{\Lambda(t)}{d(t)} \Longrightarrow Y \quad (t \to \infty). \tag{7.14}$$

As this is so,

$$P(Z < x) = E\Phi(x\sqrt{Y}), \quad x \in \mathbb{R}^1. \tag{7.15}$$

The limit distribution (7.15) is the same for any $s > 0$.

The proof of Theorem 7.1 will be forestalled by some lemmas.

LEMMA 7.1. *Let $N(t)$ be a Cox process controlled by a process $\Lambda(t)$. Let $d(t) > 0$ be a function such that $d(t) \to \infty$ ($t \to \infty$). Then one-dimensional distributions of the normalized Cox process weakly converge to the distribution of some r.v. Y :*

$$\frac{N(t)}{d(t)} \Longrightarrow Y \ (t \to \infty),$$

if and only if one-dimensional distributions of the normalized controlling process $\Lambda(t)$ weakly converge to the same distribution:

$$\frac{\Lambda(t)}{d(t)} \Longrightarrow Y \ (t \to \infty).$$

For the proof see (Gnedenko and Korolev, 1996) and (Korolev, 1998).

LEMMA 7.2. *Let the estimator $\psi_n(u)$ be defined by relations (7.5) and (7.6) with the function $m(n) \le n$ infinitely increasing so that (7.12) holds. Then the r.v. $\sigma^{-1}\sqrt{n}(\psi_n(u) - \psi_0(u))$ is asymptotically normal:*

$$\mathsf{P}\left(\sigma^{-1}\sqrt{n}(\psi_n(u) - \psi_0(u)) < x\right) \Longrightarrow \Phi(x) \quad (n \to \infty).$$

For the proof see (Croux and Veraverbecke, 1990).

Consider the r.v.'s $N_1, N_2, \ldots, X_1, X_2, \ldots$, defined on the same measurable space $(\Omega, \mathcal{A})$. Assume that for each $n \ge 1$ the r.v.'s $N_n, X_1, X_2, \ldots$ are independent and N_n takes only natural values. Consider a family of probability measures $\{\mathsf{P}_\theta,\ \theta \in \Theta\}$, each of which is defined on $\mathcal{A}$. Let $T_n = T_n(X_1, \ldots, X_n)$ be some statistic. For each $n \ge 1$ define the r.v. T_{N_n} by putting $T_{N_n(\omega)} = T_{N_n(\omega)}\left(X_1(\omega), \ldots, X_{N_n(\omega)}(\omega)\right)$ for each elementary outcome $\omega \in \Omega$. We shall say that the statistic T_n is asymptotically normal if there exist functions $\delta(\theta)$ and $t(\theta)$ such that for each $\theta \in \Theta$ we have

$$\mathsf{P}_\theta\left(\delta(\theta)\sqrt{n}(T_n - t(\theta)) < x\right) \Longrightarrow \Phi(x) \quad (n \to \infty). \tag{7.16}$$

LEMMA 7.3. *Let $\{d_n\}_{n\ge1}$ be an infinitely increasing sequence of positive numbers. Assume that $N_n \xrightarrow{P} \infty$ as $n \to \infty$. Let a statistic T_n be asymptotically normal (7.16). Then for each $\theta \in \Theta$ there exists a d.f. $F(x, \theta)$ such that*

$$P_\theta\left(\delta(\theta)\sqrt{d_n}(T_{N_n} - t(\theta)) < x\right) \Longrightarrow F(x, \theta) \quad (n \to \infty),$$

if and only if there exists a family of d.f.'s $\mathcal{H} = \{H(x, \theta) :\ \theta \in \Theta\}$ satisfying the conditions

$$H(x, \theta) = 0, \quad x < 0, \quad \theta \in \Theta;$$

$$F(x,\theta) = \int_0^\infty \Phi(x\sqrt{y})d_y H(y,\theta), \quad x \in \mathbb{R}^1, \quad \theta \in \Theta;$$

$$P_\theta(N_n < d_n x) \Longrightarrow H(x,\theta), \quad n \to \infty, \quad \theta \in \Theta.$$

Moreover, if the d.f.'s of the r.v.'s N_n do not depend on θ, then so do the d.f.'s $H(x,\theta)$, that is, the family $\mathcal{H}$ consists of a single element.

This lemma is a slight generalization of Theorem 3 from (Korolev, 1995).

The PROOF of Theorem 7.1. Let $\{t_1, t_2, \ldots\}$ be an arbitrary infinitely increasing sequence. Put $N_n = N(t_n)$, $n \geq 1$. By Lemma 2.1 the conditions $\Lambda(t) \xrightarrow{P} \infty$ and $N(t) \xrightarrow{P} \infty$ are equivalent as $t \to \infty$. Therefore, since by Lemma 7.2, the estimator $\psi_n(u)$ is asymptotically normal, then by Lemma 7.3, for convergence (7.13) in which t runs along the sequence $\{t_1, t_2, \ldots\}$, it is necessary and sufficient that there exists a r.v. $Y \geq 0$ such that

$$\frac{N_n}{d(t_n)} \Longrightarrow Y \quad (n \to \infty). \tag{7.17}$$

But by Lemma 7.1, convergence (7.17) takes place if and only if

$$\frac{\Lambda(t_n)}{d(t_n)} \Longrightarrow Y \quad (n \to \infty). \tag{7.18}$$

Since the family of scale mixtures of normal laws (7.15) is identifiable, the distribution of the r.v. Y does not depend on the choice of the sequence $\{t_1, t_2, \ldots\}$. The arbitrariness of the sequence $\{t_n\}_{n\geq 1}$ implies that (7.18) is equivalent to (7.14). The theorem is proved.

COROLLARY 7.1. *Under the conditions of Theorem 7.1, the estimator $\psi_{N(t)}(u)$ is asymptotically normal as $t \to \infty$:*

$$P(\sigma^{-1}\sqrt{d(t)}(\psi_{N(t)}(u) - \psi(u)) < x) \Longrightarrow \Phi(x) \quad (t \to \infty)$$

if and only if

$$\frac{\Lambda(t)}{d(t)} \xrightarrow{P} 1 \quad (t \to \infty). \tag{7.19}$$

This statement is a direct consequence of Theorem 7.1 with the account of the inedtifiability of scale mixtures of normal laws.

From Theorem 7.1 we can derive some conclusions concerning the consistency and asymptotic unbiasedness of the estimator $\psi_{N(t)}(u)$.

COROLLARY 7.2. *Let the conditions of Theorem 7.1 hold and let there exist an infinitely icreasing function $d(t)$ and a r.v. Y such that convergence (7.14) takes place. Then the estimator $\psi_{N(t)}(u)$ is consistent.*

PROOF. The d.f. of the r.v. $\sigma^{-1}\sqrt{d(t)}(\psi_{N(t)}(u) - \psi(u))$ will be denoted $\Psi_t(x)$. For an arbitrary $\epsilon > 0$ we have

$$\mathsf{P}\left(\left|\psi_{N(t)}(u) - \psi(u)\right| > \epsilon\right) = \mathsf{P}\left(\sigma^{-1}\sqrt{d(t)}\left|\psi_{N(t)}(u) - \psi(u)\right| > \epsilon\sigma^{-1}\sqrt{d(t)}\right) =$$

$$= \Psi_t\left(-\epsilon\sigma^{-1}\sqrt{d(t)}\right) + 1 - \Psi_t\left(\epsilon\sigma^{-1}\sqrt{d(t)}\right). \tag{7.20}$$

But in accordance with Theorem 7.1, under the conditions of Corollary 7.2, the family of d.f.'s $\{\Psi_t(x)\}_{t>0}$ is weakly compact due to convergence (7.13). This means that for any $\delta > 0$ there exists an $R_\delta > 0$ such that, whatever $t > 0$ is, for any $R \geq R_\delta$ the inequality $\Psi_t(-R) + 1 - \Psi_t(R) < \delta$ holds. This also holds for $t \geq t_\epsilon = \inf\left\{t: \epsilon\sigma^{-1}\sqrt{d(t)} > R\right\}$. Thus, from (7.20) it follows that for arbitrary $\epsilon > 0$ and $\delta > 0$ there exists a $t_0 = t_0(\epsilon, \delta)$ such that for all $t \geq t_0$ we have

$$\mathsf{P}\left(\left|\psi_{N(t)}(u) - \psi(u)\right| > \epsilon\right) < \delta,$$

which means the consistency of the estimator $\psi_{N(t)}(u)$. The corollary is proved.

A distinguishing feature of the situation under consideration is that in the case of non-degenerate mixing r.v. Y the asymptotic distribution of the estimator $\psi_{N(t)}(u)$ has heavier tails than the normal law.

As an illustration, consider the case where $\Lambda(t)$ has either geometric distribution with parameter $1/t$ or the exponential distribution with the same expectation. In both these cases the distribution of the r.v. Y in (7.14) is standard exponential, and hence, the distribution of the limit r.v. Z in (7.13) has the form

$$\mathsf{P}(Z < x) = \int_0^\infty \Phi(\sqrt{y}x)e^{-y}dy = \frac{1}{2}\left(1 + \frac{x}{\sqrt{2+x^2}}\right). \tag{7.21}$$

This distribution does not possess moments of any positive order! It is easy to see that for $\frac{1}{2} < \beta < 1$, the β-quantile of this distribution is equal to $\sqrt{2}(2\beta-1)/\sqrt{1-(2\beta-1)^2}$. Therefore, say, the difference between the quantiles of orders 0.975 and 0.025 of this distribution appears to be almost 2.2 times greater than the corresponding characteristic of the normal distribution with the same scale parameter. This example clearly illustrates how imortant it is to take account of the randomness of the size of the sample from which the ruin probability is estimated. Otherwise a considerable error is possible in the determination of the real accuracy of the estimators or in their reliability (it is easy to see that the confidence probability of the "95% normal" interval calculated by distribution (7.21) turns out to be less than 0.82).

At the same time, if $N(t) = N_1(t)$, that is, if $\Lambda(t) \equiv t$ which corresponds to the classical risk process, then, as it follows from Corollary 7.1 with $d(t) \equiv t$, the statistic $\psi_{N(t)}(u)$ is asymptotically normal. In other words, in this situation the estimator for the ruin probability constructed from the sample $(X_1, \ldots, X_{N_1(n)})$ is asymptotically (as $n \to \infty$) equivalent to the estimator $\psi_n(u)$ defined by the relations (7.5) and (7.6).

Due to the peculiarity of the limit laws mentioned above (the heavy tails which may result in the absence of moments, in particular, of the expectation) it is not quite reasonable to consider the asymptotic unbiasedness of the estimator $\psi_{N(t)}(u)$ in terms of moments. Nevertheless, the following statement is valid. As usual, med$X$ denotes the median of a r.v. X.

COROLLARY 7.3. *Let the conditions of Theorem 7.1 hold and let there exist an infinitely increasing function $d(t)$ and a r.v. Y such that convergence (7.14) takes place. Then the* *estimator $\psi_{N(t)}(u)$ is asymptotically unbiased in the sense that*

$$\lim_{t\to\infty} \text{med}\psi_{N(t)}(u) = \psi(u). \tag{7.22}$$

Moreover,

$$\text{med}\psi_{N(t)}(u) - \psi(u) = o\left((d(t))^{-1/2}\right). \tag{7.23}$$

PROOF. Under the conditions of Corollary 7.3, according to Theorem 7.1 convergence (7.13) takes place, whence it easily follows that

$$\lim_{t\to\infty} \text{med}\left(\sigma^{-1}\sqrt{d(t)}\left(\psi_{N(t)}(u) - \psi(u)\right)\right) = \text{med}Z = 0,$$

implying

$$\sigma^{-1}\sqrt{d(t)}\left(\text{med}\psi_{N(t)}(u) - \psi(u)\right) \to 0, \tag{7.24}$$

which is possible only if (7.22) holds, since the function $d(t)$ infinitely increases. Further, relation actually (7.24) means the validity of (7.23). The corollary is proved.

7.3 Nonparametric estimation of the asymptotic variance.

Unfortunately, Theorem 7.1 cannot be used directly for the construction of asymptotic confidence intervals for the ruin probability since in relation (7.15) the asymptotic variance σ^2 defined by relations (7.9) – (7.11) is unknown being dependent on the unknown d.f. $F(x)$ (moreover, as a rule, the mixing distribution of the r.v. Y is unknown as well). So, to construct a confidence interval for $\psi(u)$ we should find a consistent estimator of the asymptotic variance σ^2. In this subsection we concentrate our attention on the construction of a nonparametric estimator of σ^2 and the investigation of its asymptotic properties.

As in Subsection 7.2, at first we formally assume that at our disposal is a sample $X_1, \ldots, X_n$ of independent identically distributed claims. Let $\{k(n)\}_{n\geq 1}$ be a sequence of natural numbers satisfying the conditions $1 \leq k(n) \leq n$, $k(n) \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{(k(n))^{3/2}}{\sqrt{n}} \left(\frac{u}{c}\right)^{k(n)} = 0. \tag{7.25}$$

As an estimator of σ^2, consider the statistic

$$\sigma_n^2 = \left(1 - \frac{a}{c}\right)^2 \sum_{r=1}^{k(n)} \sum_{l=1}^{k(n)} \left(\frac{a}{c}\right)^{r+l} rl\overline{\sigma}_{r,l}, \tag{7.26}$$

where

$$\overline{\sigma}_{r,l} = \frac{1}{n}\sum_{i=1}^{n} \overline{h}_r(X_i)\overline{h}_l(X_i) - U_{n,r} \cdot U_{n,l}, \tag{7.27}$$

$$\overline{h}_j(x) = \frac{1}{a}\int_0^{\infty} U_{n,j-1}(u-y)\mathbf{1}_{(y,\infty)}(x)dy, \tag{7.28}$$

$$U_{n,k} = U_{n,k}(u) = \frac{1}{C_n^k} \sum_{1 \le i_1 < \ldots < i_k \le n} h_k(X_{i_1}, \ldots, X_{i_k}), \tag{7.29}$$

$$h_k(x_1, \ldots, x_k) = \frac{1}{a^k} \int_0^\infty \cdots \int_0^\infty \mathbf{1}_{(0,u)}(y_1 + \ldots + y_k) \prod_{j=1}^{k} \mathbf{1}_{[y_j,\infty)}(x_j) dy_1 \ldots dy_k. \tag{7.30}$$

LEMMA 7.4. *Let the sequence $\{k(n)\}_{n\ge1}$ of natural numbers satisfy the conditions $1 \le k(n) \le n$, $k(n) \to \infty$ and (7.25). Then the estimator σ_n^2 is consistent:*

$$\sigma_n^2 \xrightarrow{P} \sigma^2 \quad (n \to \infty) \tag{7.31}$$

The proof of this statement essentially uses the properties of U-statistics and is rather cumbersome. For the sake of brevity we omit this proof. In full detail, it will be published in our forthcoming paper (Bening and Korolev, 1999b).

REMARK 7.1. Condition (7.25) means that if $s < c$, then $k(n)$ can increase rather rapidly, say, $k(n)$ can equal n. If $s = c$, then $k(n) = o(n^{1/3})$. And if $s > c$, which is typical, then $k(n)$ can increase rather slowly, say, as $\log\log n$. Furthermore, if we assume that there exists the second moment $\alpha_2 = \mathsf{E}X_1^2$, then condition (7.25) can be replaced by a more accurate condition

$$\lim_{n\to\infty} \frac{(k(n))^{3/2}}{\sqrt{n}} \left(\frac{\min\{u, \alpha_2\}}{c}\right)^{k(n)} = 0.$$

LEMMA 7.5. *Let $\{\xi_n\}_{n\ge1}$ and $\{\eta_n\}_{n\ge1}$ be sequences of r.v.'s such that $\xi_n \Longrightarrow \xi$ and $\eta_n \xrightarrow{P} b$ as $n \to \infty$, where $b \in \mathbb{R}^1$ and ξ is a r.v.. Then $\xi_n \cdot \eta_n \Longrightarrow b \cdot \xi$ $(n \to \infty)$.*

For the proof see, e. g., (Cramér, 1974), Sect. 20.6.

Although the following statement is auxiliar, due to its importance we formulate it as a theorem.

THEOREM 7.2. *Let the estimator $\psi_n(u)$ be defined by relations (7.5) and (7.9) with the function $m(n) \le n$ infinitely increasing as $n \to \infty$ so that (7.12) holds. Let the estimator σ_n^2 be defined by relations (7.26) – (7.30) with the sequence $\{k(n)\}_{n\ge1}$ of natural numbers satisfying the conditions $1 \le k(n) \le n$, $k(n) \to \infty$ and (7.25). Then the r.v. $\sigma_n^{-1}\sqrt{n}(\psi_n(u) - \psi_0(u))$ is asymptotically normal:*

$$\mathsf{P}\left(\sigma_n^{-1}\sqrt{n}(\psi_n(u) - \psi_0(u)) < x\right) \Longrightarrow \Phi(x) \quad (n \to \infty).$$

Proof. Set

$$\xi_n = \sigma^{-1}\sqrt{n}(\psi_n(u) - \psi_0(u)), \quad \eta_n = \sigma_n^{-1}\sigma.$$

By Lemma 7.2, $\mathsf{P}(\xi_n < x) \Longrightarrow \Phi(x)$ and by Lemma 7.4, $\eta_n \xrightarrow{P} 1$ as $n \to \infty$. Now the desired result follows from Lemma 7.5. The theorem is proved.

Now turn to the construction of an estimator of the ruin probability of the generalized risk process given a trajectory of this process up to some time t_0. From the above reasoning it is clear that, as an estimator of the asymptotic variance σ^2 we should take the statistic $\sigma^2_{N(t_0)}$ by putting

$$\sigma^2_{N(t)} = \left(1 - \frac{a}{c}\right)^2 \sum_{r=1}^{k(N(t))} \sum_{l=1}^{k(N(t))} \left(\frac{a}{c}\right)^{r+l} rl\overline{\sigma}_{r,l}, \tag{7.32}$$

where

$$\overline{\sigma}_{r,l} = \frac{1}{N(t)} \sum_{i=1}^{N(t)} \overline{h}_r(X_i)\overline{h}_l(X_i) - U_{N(t),r} \cdot U_{N(t),l}, \tag{7.33}$$

$$\overline{h}_j(x) = \frac{1}{a} \int_0^\infty U_{N(t),j-1}(u-y)\mathbf{1}_{(y,\infty)}(x)dy, \tag{7.34}$$

$$U_{N(t),k} = U_{N(t),k}(u) = \frac{1}{C^k_{N(t)}} \sum_{1 \le i_1 < \ldots < i_k \le N(t)} h_k(X_{i_1}, \ldots, X_{i_k}), \tag{7.35}$$

$$h_k(x_1, \ldots, x_k) = \frac{1}{a^k} \int_0^\infty \cdots \int_0^\infty \mathbf{1}_{(0,u)}(y_1 + \ldots + y_k) \prod_{j=1}^k \mathbf{1}_{[y_j,\infty)}(x_j)dy_1 \ldots dy_k. \tag{7.36}$$

To formulate a statement concerning the consistency of the estimator $\sigma^2_{N(t)}$ as $t \to \infty$, we will require one more auxiliary statement.

LEMMA 7.6. *Let $\{\xi_j\}_{j\ge1}$ be a sequence of r.v.'s weakly convergent to some r.v. ξ as $j \to \infty$. Let $\{N_n\}_{n\ge1}$ be a sequence of integer positive r.v.'s such that for each $n \ge 1$, the r.v. N_n is independent of the sequence $\{\xi_j\}_{j\ge1}$ u $N_n \xrightarrow{P} \infty$ as $n \to \infty$. Then*

$$\xi_{N_n} \Longrightarrow \xi \quad (n \to \infty).$$

The PROOF of this result can be found, e. g., in (Kruglov and Korolev, 1990) (see Theorem 5.1.1 there).

Now we can formulate the following statement on the consistency of the estimator $\sigma^2_{N(t)}$.

LEMMA 7.7. *Let the estimator $\sigma^2_{N(t)}$ be defined by relations (7.22) – (7.26) in which the sequence $\{k(n)\}_{n\ge1}$ satisfies the conditions $1 \le k(n) \le n$, $k(n) \to \infty$ and (7.25). Assume that $\Lambda(t) \xrightarrow{P} \infty$ as $t \to \infty$. Then*

$$\sigma^{-1}_{N(t)} \cdot \sigma \xrightarrow{P} 1 \quad (t \to \infty).$$

PROOF. By Lemma 2.1, from the condition $\Lambda(t) \xrightarrow{P} \infty$ it follows that $N(t) \xrightarrow{P} \infty$. Now the desired result follows from Lemmas 7.4 and 7.6. The lemma is proved.

7.4 Confidence intervals for the ruin probability for generalized risk process.

In this subsection, the main role is played by the following statement.

THEOREM 7.3. *Let the estimator $\psi_{N(t)}(u)$ be defined by relations (7.7) and (7.8) with the function $m(n)$ infinitely increasing as $n \to \infty$ so that condition (7.18) holds. Let the estimator $\sigma^2_{N(t)}$ be defined by the relations (7.22) – (7.26) with the sequence $\{k(n)\}_{n\geq 1}$ of natural numbers satisfying the conditions $1 \leq k(n) \leq n$, $k(n) \to \infty$ and (7.25). Assume that $\Lambda(t) \xrightarrow{P} \infty$ as $t \to \infty$. Then*

$$\mathsf{P}\left(\sigma^{-1}_{N(t)}\sqrt{N(t)}(\psi_{N(t)}(u) - \psi(u)) < x\right) \Longrightarrow \Phi(x) \quad (t \to \infty). \tag{7.37}$$

This statement is a direct consequence of Theorem 7.2 and Lemma 7.6.

For $\delta \in (0,1)$, the δ-quantile of the standard normal distribution will be denoted u_δ. For large t, from (7.37) it follows that

$$\mathsf{P}\left(\sigma^{-1}_{N(t)}\sqrt{N(t)}(\psi_{N(t)}(u) - \psi(u)) < x\right) \approx \Phi(x). \tag{7.38}$$

Therefore, for $\gamma \in (0,1)$, an approximate $100\gamma\%$ confidence interval for the ruin probability $\psi(u)$ will have the form

$$\psi_{N(t)}(u) - \frac{u_{(\gamma+1)/2}\sigma_{N(t)}}{\sqrt{N(t)}} \leq \psi(u) \leq \psi_{N(t)}(u) + \frac{u_{(\gamma+1)/2}\sigma_{N(t)}}{\sqrt{N(t)}}. \tag{7.39}$$

Along with Theorem 7.3, we can formulate one more asymptotic result.

THEOREM 7.4. *Let the conditions of Theorem 7.3 hold and let there exist an infinitely increasing function $d(t)$ and a r.v. Y such that convergence (7.14) takes place. Then*

$$\mathsf{P}\left(\sigma^{-1}_{N(t)}\sqrt{d(t)}(\psi_{N(t)}(u) - \psi(u)) < x\right) \Longrightarrow \mathsf{E}\Phi(x\sqrt{Y}) \quad (t \to \infty). \tag{7.40}$$

PROOF. By Theorem 7.1 we have

$$\mathsf{P}\left(\sigma^{-1}\sqrt{d(t)}(\psi_{N(t)}(u) - \psi(u)) < x\right) \Longrightarrow \mathsf{E}\Phi(x\sqrt{Y}) \quad (t \to \infty). \tag{7.41}$$

Set

$$\eta_t = \frac{\sigma^{-1}_{N(t)}}{\sigma^{-1}}, \quad \xi_t = \sigma^{-1}\sqrt{d(t)}(\psi_{N(t)}(u) - \psi(u)).$$

Now the desired result follows from Lemma 7.5 with the account of (7.41) and Lemma 7.7. The theorem is proved.

If the δ-quantile of the d.f. $\Psi(x) = \mathsf{E}\Phi(x\sqrt{Y})$ is denoted w_δ, then along with (7.39), when the distribution of the r.v. Y is known, from (7.40) for $\gamma \in (0,1)$ we can construct one more approximate $100\gamma\%$ confidence interval for the ruin probability $\psi(u)$:

$$\psi_{N(t)}(u) - \frac{w_{(\gamma+1)/2}\sigma_{N(t)}}{\sqrt{d(t)}} \leq \psi(u) \leq \psi_{N(t)}(u) + \frac{w_{(\gamma+1)/2}\sigma_{N(t)}}{\sqrt{d(t)}}. \tag{7.42}$$

In particular, if $N(t)$ is the standard Poisson process which corresponds to $\Lambda(t) \equiv t$, then it is quite reasonable to set $d(t) \equiv t$. In this case, as is easily seen, $\mathsf{P}(Y = 1) = 1$ so that $\Psi(x) \equiv \Phi(x)$, and the approximate $100\gamma\%$ confidence interval (7.42) takes the form

$$\psi_{N(t)}(u) - \frac{u_{(\gamma+1)/2}\sigma_{N(t)}}{\sqrt{t}} \leq \psi(u) \leq \psi_{N(t)}(u) + \frac{u_{(\gamma+1)/2}\sigma_{N(t)}}{\sqrt{t}}.$$

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