SOCIETY OF ACTUARIES

Risk Assessment Applications of Fuzzy Logic

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1 Introduction

This study combines risk assessment (RA) and fuzzy logic (FL), where:

> "Risk assessment is the overall process of risk identification, analysis and evaluation. Identifying risk includes understanding the sources of risk, areas of impact, events and their causes and potential consequences. The goal is to create a comprehensive list of risks, including risks that may be associated with missed opportunities and risks out of the direct control of the organization. A comprehensive review allows a full consideration of potential effects of risk upon the organization." [Gjerdrum and Peter (2011, p. 9)]

> "... Fuzzy logic may be viewed as an attempt at formalization/mechanization of two remarkable human capabilities. First, the capability to converse, reason and make rational decisions in an environment of imprecision, uncertainty, incompleteness of information, conflicting information, partiality of truth and partiality of possibility – in short, in an environment of imperfect information. And second, the capability to perform a wide variety of physical and mental tasks without any measurements and any computations." [Zadeh (2008, p. 2751)]

The motivation for combining RA and FL arises because RA often is hindered by data limitations and ambiguities, such as incomplete or unreliable data, and subjective information owing to a reliance on human experts and their communication of linguistic variables. Since FL models have been shown to be effective tools in such circumstances [Shapiro (2003, 2004); Koissi and Shapiro (2006)], it seems natural to inquire into the RA applications of FL. That this is a fruitful area for exploration was reinforced by CAS (2003a, p. 122), for example, which observed that FL is ideally suited for quantifying operational and strategic risks.

1.1 Purpose of this Study

The purpose of this study was to investigate risk assessment applications of fuzzy logic (RAA-FL). This was accomplished in two phases.

The first phase of the research was a review of the literature, with the intention of identifying FL articles that have relevance from a RA perspective. A major focus was on articles that elaborated on implementation. More than 200 articles were reviewed during this phase. The product of this phase of the study was an annotated bibliography of this literature.

Given the foregoing literature review, the second phase of the research was to document FL methodologies that can be implemented in an actuarial RA context. Since most of the current RAA-FL articles do not have an actuarial perspective, a main task was to identify methodologies that can be repurposed for actuarial use. Here, the goal was to provide sufficient detail so that actuaries will be able to recognize potential opportunities for implementation. To this end, we discuss the conceptual framework underlying RAA-FL models, review empirical studies of RAA-FL, and present specific RAA-FL examples.

Subtopics for the conceptual framework include RA models involving fuzzy sets, fuzzy membership functions, fuzzy arithmetic, fuzzy linear programming, and fuzzy inference systems.

The empirical studies cover the same subtopics as the conceptual framework, but focus on applications.

A major issue is how crisp models, which have fuzzy components that are inadequately accommodated by the model, can be reformulated as fuzzy models. Chapters 4, 5 and 6 address this issue.

The subtopics for the portion of the study that deals with RAA-FL in an actuarial context include each of the areas mentioned in CAS (2003, p. 111): hazard risks, financial risks, operational risks, and strategic risks. This portion of the study provides insights for practitioners.

1.2 The Database

In addition to the publications of actuarial organizations, both here and abroad, books on the topic and web-based databases were used to search for relevant information. Examples of the databases included were ProQuest (which provides access to such journals as Best's Review, Journal of Financial Planning, Geneva Papers on Risk & Insurance, and Journal of Risk and Insurance), ScienceDirect (which provides access to such journals as Insurance: Mathematics and Economics, Journal of Economic Dynamics and Control, Fuzzy Sets and Systems, and Information Sciences), and IEEE Xplore (which provides access to conference papers such as IEEE International Conference on Advanced Management Science, IEEE International Conference on Fuzzy Systems, and Systems, Man, and Cybernetics).

1.3 Summary of the Study

This introduction is followed by an overview of RA, which includes such things as the role of RA in ERM, the conceptualization of RA, and the role of FL as it relates to RA. We turn then to an overview of the FL methodology used in this study. The next three chapters are devoted to FL methodology as it relates to RA issues, and includes a discussion of FL modification of the risk matrix, FL modification of the Analytic Hierarchy Process, and the role of fuzzy optimization as it relates to RA. We turn then to examples of RAA-FL in the areas of operational risk, hazard risk, financial risk and strategic risk. This is followed by closing comments and observations. The study concludes with an annotated bibliography.

2 Risk Assessment

Risk assessment (RA) is a systematic process for identifying and evaluating potential risks and opportunities that could positively or negatively affect the achievement of an enterprise's objectives.[1] While there have been inroads in conducting RA, the recent global financial crisis [SOA, (2008, 2011)] has prompted policy makers and practitioners to question the efficacy of the existing standard for RA. In addition, the RA of emerging risks[2], which includes operational, strategic, reputational, and compliance risks, once under-studied because of the complexity arising from lack of historical data and low probability of occurrence, is now attracting the attention of the actuarial community.

In a nutshell, risk assessment attempts to answer the following fundamental questions: [31010/FDIS (2009, p. 6)]

- What can happen and why (risk identification)?
- What are the consequences?
- What is the probability of their future occurrence?
- Are there any factors that mitigate the consequence of the risk or that reduce the probability of the risk?
- Is the level of risk tolerable or acceptable and does it require further treatment?

2.1 The ERM Context for RA

For the purpose of this report, we take the context for RA to be enterprise risk management (ERM). This is appropriate, since RA represents the core of an effective ERM program, as well as the proposed Own Risk and Solvency Assessment (ORSA) program of the NAIC.[3] A robust ERM or ORSA process also aims at reducing the probability of severe losses for the company, by better understanding and quantifying the risk and uncertainty the firm is facing.

There have been a number of attempts to capture the notion of ERM:

COSO[4] (2004, p. 2) asserted that "Enterprise risk management is a process, effected by an entity's board of directors, management and other personnel, applied in strategy setting and

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[1] Adopted from PricewaterhouseCoopers (2008).
[2] Emerging risks, which are large-scale events or circumstances that arise from global trends, are discussed in PricewaterhouseCoopers (2009).
[3] See Shapella and Stein (2012) for an overview of ORSA.
[4] The Committee of Sponsoring Organizations of the Treadway Commission (COSO) is a joint initiative of the five private sector accounting organizations and is dedicated to providing thought leadership through the development of frameworks and guidance on enterprise risk management, internal control and fraud deterrence. See

across the enterprise, designed to identify potential events that may affect the entity, and manage risk to be within its risk appetite, to provide reasonable assurance regarding the achievement of entity objectives."

The Casualty Actuarial Society [CAS (2003, p. 109)] defined ERM as "The discipline by which an organization in any industry assesses, controls, exploits, finances and monitors risks from all sources for the purpose of increasing the organization's short- and long-term value to its stakeholders."

A definition similar to that of the CAS was adopted by the Society of Actuaries in 2005.

Recently, the Actuarial Standards Board [(2012, p. 12)] postulated that "At the most fundamental level Enterprise risk management can be understood as a control cycle. Within a typical risk management control cycle, risks are identified, risks are evaluated, risk appetites are chosen, risk limits are set, risks are accepted or avoided, risk mitigation activities are performed, and actions are taken when risk limits are breached. Risks are monitored and reported as they are taken and as long as they remain an exposure to the organization. This cycle can be applied to specific risks within a part of an organization or to an aggregation of all risks at the enterprise level."

To put RA in an ERM context, consider a flowchart of the COSO-ERM model as depicted in Figure 2-1,[5] where a "Start", but no "Stop", emphasizes the ongoing nature of the ERM process.

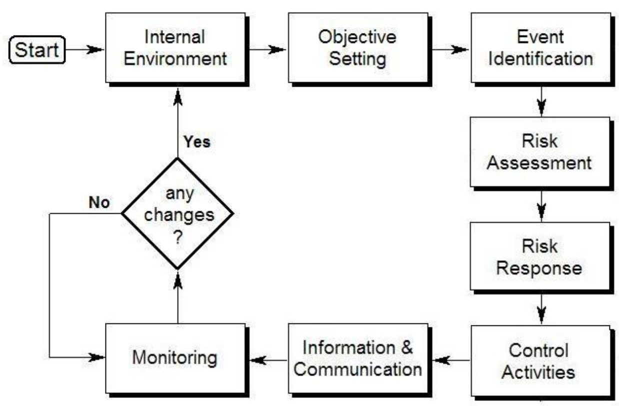

Figure 2-1 The COSO-ERM Model

According to this model, "Enterprise risk management consists of eight interrelated components. These are derived from the way management runs an enterprise and are integrated with the management process. These components are: [COSO (2004, pp. 3-4)]

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http://www.coso.org/default.htm. SEC Commissioner James C. Treadway, Jr., its namesake, was the original chairman of the committee.

[5] Adapted from Cendrowski and Mair (2009, p. 87).

- "Internal Environment – The internal environment encompasses the tone of an organization, and sets the basis for how risk is viewed and addressed.
- "Objective Setting – Objectives must exist before management can identify potential events affecting their achievement.
- "Event Identification – Internal and external events affecting achievement of an entity's objectives must be identified, distinguishing between risks and opportunities.
- "Risk Assessment – Risks are analyzed, considering likelihood and impact, as a basis for determining how they should be managed.
- "Risk Response – Management selects risk responses – avoiding, accepting, reducing, or sharing risk.
- "Control Activities – Policies and procedures are established and implemented to help ensure the risk responses are effectively carried out.
- "Information and Communication – Relevant information is identified, captured, and communicated in a form and timeframe so that people can carry out their responsibilities.
- "Monitoring – The entirety of Enterprise risk management is monitored and modifications made as necessary.

ERM "is not strictly a serial process, where one component affects only the next. It is a multidirectional, iterative process in which almost any component can and does influence another." [COSO (2004, p. 4)]

While there appears to be agreement as to the general nature of the ERM process, there is not universal agreement as to its detail.

Segal (2011, p. 51), for example, views the ERM process cycle as shown in Figure 2-2.

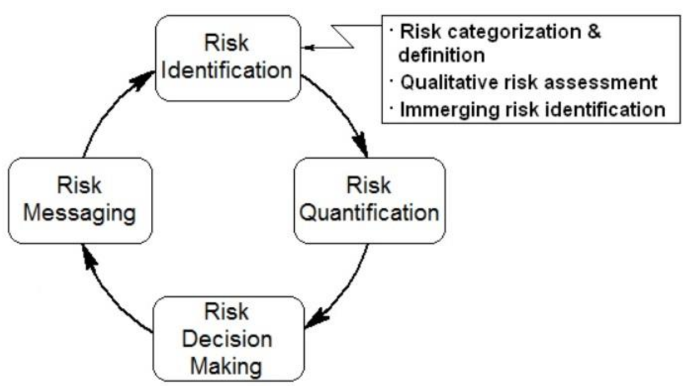

Figure 2-2: ERM process cycle

In the figure, risk quantification includes aggregating projections, analysis of trends, and stress test [Segal (2011, pp. 257-9)], and risk messaging refers to integrating ERM into business performance analysis and incentive compensation (internal risk messaging) and integrating ERM into communications with shareholders, rating agencies and regulators (external risk messaging) [Segal (2011, p. 52)].

Of particular note for our purpose, the risk identification step of Segal (2011, p. 113) includes qualitative risk assessment.

2.2 Definitions of Risk Assessment

There are numerous definitions of RA. Examples include:

> "Qualitative risk assessment. The second component of the risk identification ERM process step, the qualitative risk assessment involves prioritizing the list of potential risks and narrowing them down to the list of key risks. This involves soliciting input from internal personnel regarding the organization's key risks, and a high-level qualitative scoring of each potential key risk's likelihood of occurrence and severity of impact." [Segal (2011, p. 380)]
>
> 31010/FDIS[6] (2009, p. 6) defines risk assessment as "that part of risk management which provides a structured process that identifies how objectives may be affected, and analyses the risk in term of consequences and their probabilities before deciding on whether further treatment is required."
>
> Risk assessment is "[t]he process of identifying, estimating, and prioritizing risks to organizational operations (including mission, functions, image, reputation), organizational assets, individuals, other organizations, and the Nation, resulting from the operation of an information system." [NIST (2012, p. B-9)]
>
> COSO (2004) conceptualizes RA as "Risk Assessment – Risks are analyzed, considering likelihood and impact, as a basis for determining how they should be managed. Risks are assessed on an inherent and a residual basis."

The mention of likelihood (probabilities) and severity (consequences, impact) in the foregoing are a reference to the risk matrix, where risk is taken to be a function of the likelihood of an event and its severity. The risk matrix is the subject of Chapter 4.

NIST (2012, p. ix) notes that "[t]here are no specific requirements with regard to: (i) the formality, rigor, or level of detail that characterizes any particular risk assessment; (ii) the methodologies, tools, and techniques used to conduct such risk assessments; or (iii) the format and content of assessment results and any associated reporting mechanisms. Organizations have maximum flexibility on how risk assessments are conducted and are encouraged to apply the guidance in this document so that the various needs of organizations can be addressed and the risk assessment activities can be integrated into broader organizational risk management processes."

2.3 Conceptualizing RA

For the purpose of this study, we adopt NIST (2012), Appendix L, and conceptualize the tasks involved in conducting the RA step of ERM, as shown in Figure 2-3.

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[6] The standard is intended to reflect current good practices in selection and utilization of risk assessment techniques.

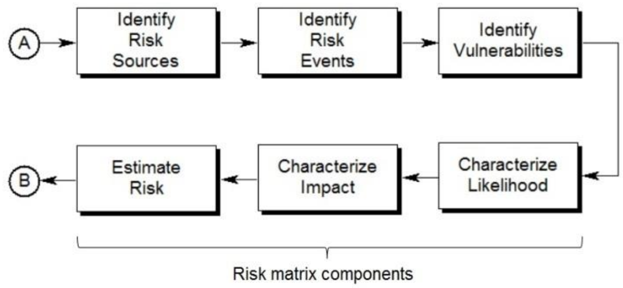

Figure 2-3: A flowchart of the RA step of ERM

Here, A denotes a connection to the previous step, preparing for RA, and B denotes a connection to the subsequent step, communicating and sharing RA results.

As indicated in the figure, the bottom three tasks are the components of the risk matrix, which, as mentioned previously, is the subject of Chapter 4 of this report.

2.4 The Role of FL as it Relates to RA

Fuzzy logic (FL), which was formulated by Zadeh (1965), provides a framework for approximate reasoning and allows qualitative knowledge about a problem to be translated into an executable rule set.

Early on, it was recognized that FL had a potential role in risk assessment.

> "Fuzzy logic has advantages in modeling complex business problems where linguistic variables are used to express the logic rules, the information is subjective, incomplete or unreliable, and the problem spaces are often nonlinear. A fuzzy system is closer to the way people reason and is therefore often used to build expert systems. The fuzzy nature of the rule spaces makes it easy to model multiple, often different or conflicting expert views toward the same model variables. In terms of risk modeling and assessment, fuzzy logic shows potential to be a good approach in dealing with operational risk, where the probability assessment is often based on expert opinion and the risk space is multidimensional and highly nonlinear." [CAS (2003, p. 42)]

More recently, NIST (2012, p. ix) cautions "that risk assessments are often not precise instruments of measurement and reflect: (i) the limitations of the specific assessment methodologies, tools, and techniques employed; (ii) the subjectivity, quality, and trustworthiness of the data used; (iii) the interpretation of assessment results; and (iv) the skills and expertise of those individuals or groups conducting the assessments."

Such comments provide a natural segue to an inquiry into the role of FL as it relates to RA.

In this report, FL is merged with RA to investigate:

- FL modification of the risk matrix
- FL modification of the Analytic Hierarchy Process
- The role of fuzzy optimization as it relates to RA
- Examples of RAA-FL in the areas of hazard risks, financial risks, operational risks, and strategic risks

3 Fuzzy Logic Methodology[7]

> Basically, fuzzy logic is a precise logic of imprecision and uncertainty.
> -- Zadeh (2012, p. 1178)

Before proceeding with the rest of the report, we first present an overview of the fuzzy logic (FL) methodology that will be used. Readers familiar with FL can skip this chapter.

Here, and throughout the report, we generally follow the lead of Zadeh (1965), the founder of FL, and use the term FL in its broad sense, where it is essentially synonymous with fuzzy set theory. According to Zadeh,[8]

> Fuzzy logic, FL, has four principal facets. First, the logical facet, FL, [the logic of approximate reasoning], which is fuzzy logic in its narrow sense. Second, the set-theoretic facet, FLs, which is concerned with classes having unsharp boundaries, that is, with fuzzy sets. Third, the relational facet, FLr, which is concerned with linguistic variables, fuzzy if-then rules and fuzzy relations. It is this facet that underlies almost all applications of fuzzy logic in control, decision analysis, industrial systems, and consumer products. And fourth, the epistemic facet, FLe, which is concerned with knowledge, meaning, and linguistics.

The methodologies of the studies reviewed in this article are based primarily on the first three of these facets, in that they involve linguistic variables and membership functions, fuzzy numbers, fuzzy arithmetic, fuzzy linear programming, and fuzzy inference systems. The rest of this section presents an overview of the conceptual nature of these methodologies.

3.1 Linguistic Variables and Membership Functions

Linguistic variables are the building blocks of FL. They may be defined (Zadeh, 1975, 1981) as variables whose values are expressed as words or sentences. Risk capacity, for example, may be viewed both as a numerical value ranging over the interval [0,100%], and a linguistic variable that can take on values like high, not very high, and so on. Each of these linguistic values may be interpreted as a label of a fuzzy subset of the universe of discourse X = [0,100%], whose base variable, x, is the generic numerical value risk capacity.

A fuzzy set was defined by Zadeh (1965, p. 338) as a class of objects with a continuum of grades of membership.[9] He went on to characterize a fuzzy set (class) A in X by a membership function (MF), $\mu_A(x)$, which associates with each point in X a real number in the interval [0, 1], with the value of $\mu_A(x)$ at x representing the "grade of membership" (GOM) of x in A. It follows that a

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[7] Adapted from Shapiro (2004).
[8] http://www.eecs.berkeley.edu/IPRO/Summary/03abstracts/zadeh.13.html
[9] Readers interested in the history of fuzzy sets will find an excellent overview of that topic in Dubois et al (2000).

fuzzy set is defined as a mapping:[10]

$$\mu : X \rightarrow [0,1] \tag{3.1}$$

An example is shown in Figure 3-1, which is characterized by a membership function (MF), $\mu_{high}(x)$ here, which assigns to each object a grade of membership ranging between zero and one.

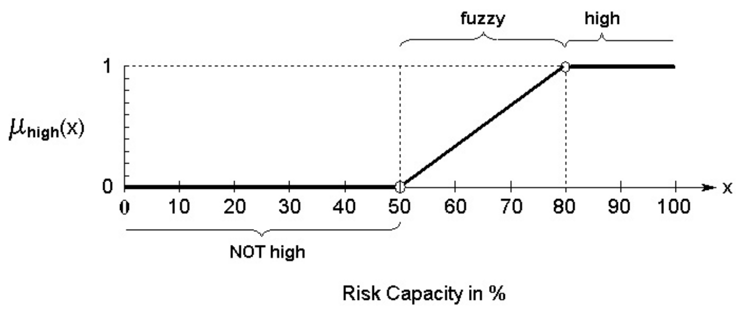

Figure 3-1: (Fuzzy) Set of Clients with High Risk Capacity

In this case, which represents the set of clients with a high risk capacity, individuals with a risk capacity of 50 percent, or less, are assigned a membership grade of zero and those with a risk capacity of 80 percent, or more, are assigned a grade of one. Between those risk capacities, (50%, 80%), the classification assigned to the risk capacity of the client is fuzzy.

If the MF has the shape depicted in Figure 3-1, it is characterized as S-shaped. Figure 3-2 shows examples of four other commonly used classes of MFs: triangular, trapezoidal, Gaussian, and generalized bell.

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[10] While the mapping in the text is the one commonly discussed, a more general formulation was provided by de Figueiredo (2007, p. 174), who envisioned a membership functional of the form $\mu_A(\tilde{A}, J, \cdot) : X \rightarrow [0,1]$, where there is an attribute, or an event characterized by the respective attribute, Ã, and a judgment criterion, J, on the basis of which the membership of x in A is judged.

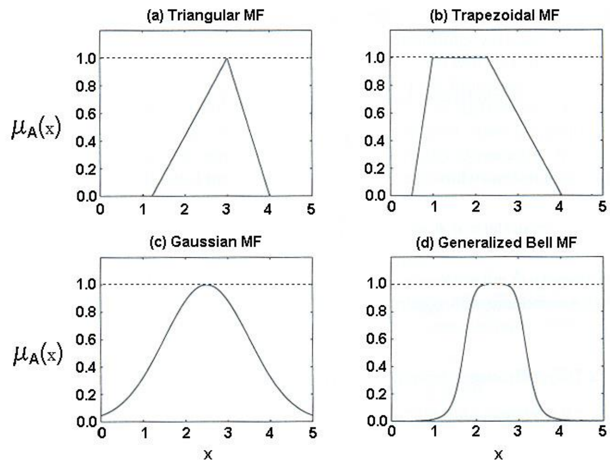

Figure 3-2: Examples of Classes of MFs

Fuzzy sets are implemented by extending many of the basic identities that hold for ordinary sets. Thus, for example, the union of two fuzzy sets, A and B, often is defined as the smallest fuzzy set containing them both, that is, [Hanss (2005, p. 32)]

$$\mu_{A\cup B}(x) = \max[\mu_A(x), \mu_B(x)], x \in X \tag{3.2}$$

and their intersection commonly is defined as the largest fuzzy set that is contained in them both, that is, [Hanss (2005, p. 29)]

$$\mu_{A\cap B}(x) = \min[\mu_A(x), \mu_B(x)], x \in X \tag{3.3}$$

3.2 Methods for Assigning MFs[11]

Methods for assigning MFs to fuzzy variables have been discussed in a number of articles. An excellent overview of the different approaches during the first 25 years following Zadeh's 1965 seminal article are given by Dombi (1990). He segregated the approaches into: heuristically based MFs; MFs based on reliability concerns with respect to the particular problem; MFs based on more theoretical demand, such as axiomatically justified or a probability distribution; MFs related to control, where either one defines the functions and identifies the system parameters, or works with a given system and identifies the MF; and MFs as a model for human concepts. More recent reviews on the topic include those of Bilgic and Turksen (1995) and Smithson and Verkuilen (2006).

A catalogue of methods for the development of MFs appears in Sivanandam et al (2007, chapter 4), where it is noted that the assignment of MFs to fuzzy variables can be done intuitively or by using some algorithms or logical procedures. Among the methods they listed and discussed were: intuition, where the development of the MF is based on the human's own intelligence and

[11] This section is from Shapiro (2013, p. 866).

understanding, and requires the thorough knowledge of the problem and the linguistic variable; inference, which involves the knowledge to perform deductive reasoning, and forms the MF from the facts known and knowledge; and rank ordering, where the polling concept and pairwise comparisons are used to assign membership values by a rank ordering process. They also mention the role of the other soft computing methodologies, neural networks and genetic algorithms, in the MF assignment process.

Additional notable articles that addressed the development of MFs, but were not mentioned in the foregoing, include Chen and Otto (1995), who presented methods for constructing MFs using measurement theory and constrained interpolation, where the former offers a suitable framework for constructing a MF in cases where the membership is based on subjective preferences, and Buckley (2005 §2.8), who showed how to develop triangular-shaped fuzzy MFs based on confidence intervals.

3.3 Fuzzy Numbers

A fuzzy number is a fuzzy set A of the real line P such that [Dubois and Prade (1980, pp. 10, 26)]

A is convex, that is, [Zadeh (1965, p. 345)] for any x_1 and $x_2 \in$ P,

$$\mu_A(\lambda x_1 + (1-\lambda) x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)),\ \lambda \in [0,1] \tag{3.4}$$

A is normalized, that is,

$$\exists x_0 \in R \mid \mu_A(x_0) = 1 \tag{3.5}$$

μ_A is piecewise continuous

Examples of fuzzy numbers are the notions of "around six percent" and "relatively high".

The general characteristic of a fuzzy number [Zadeh (1975) and Dubois and Prade (1980)] often is represented as shown in Figure 3-3, although any of the MF classes depicted in Figure 3-2 can serve as a fuzzy number, depending on the situation.

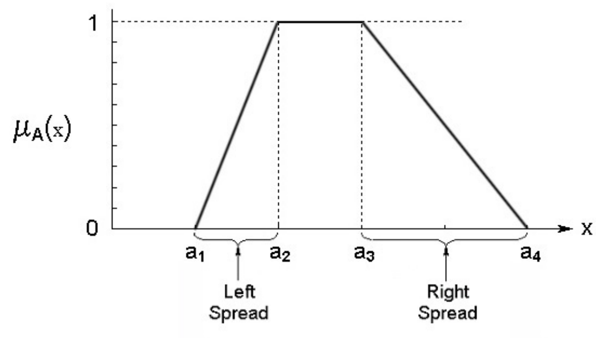

Figure 3-3: Flat Fuzzy Number

This shape of a fuzzy number is referred to as trapezoidal or "flat" and its MF often is denoted as (a_1, a_2, a_3, a_4) or (a_1/a_2, a_3/a_4). A fuzzy number is positive if $a_1 \geq 0$ and negative if $a_4 \leq 0$. The length of the intervals [a_1, a_2] and [a_3, a_4] are known as the left spread and right spread, respectively, and will be referred to as α and β, correspondingly. A triangular fuzzy number (TFN) results when a_2 is equal to a_3, in which case, its MF will be indicated by (a, α, β), where a denotes its central value, which will be referred to as its mode. When the two spreads are equal, the TFN is known as a symmetrical TFN (STFN), and will be symbolized as (a, α).

3.4 α-cuts

A useful concept insofar as linguistic variables and fuzzy numbers is the α-cut. Let A be a fuzzy set in the universe X. The (crisp) set of elements that belong to the fuzzy set A at least to the degree α is called the α-cut or α-level set and is defined by:

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\} \tag{3.6}$$

An example of an α-cut is depicted in Figure 3-4[12]

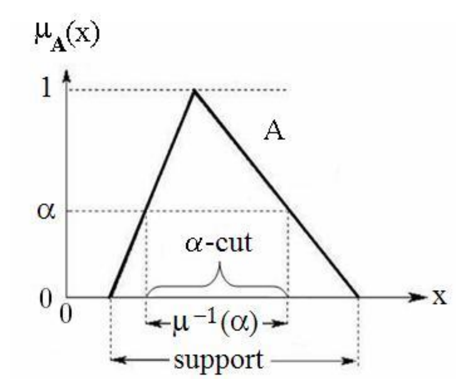

Figure 3-4: Alpha Cut

[12] Adapted from Sinha and Gupta (2000), Figure 7.13, and Liu (2012), Figure 6.8.

As indicated, the essence of the α-cut is that it limits the domain under consideration to the set of elements with degree of membership of at least α. Thus, while the support of fuzzy set A is its entire base, its α-cut only includes a portion of the base.

By the Representation Theorem of fuzzy set theory, each fuzzy subset is uniquely determined by its α-cuts. [Krätschmer (2001, p. 3)]

Two common uses of α-cuts are as a cropping tool and as part of a methodology for performing fuzzy arithmetic. First, following Zadeh (1968, p. 97) and Dubois and Prade (1980, p. 19), the α-cut can be implemented as a cropping tool. In this case, when values outside the α-cut are considered to have a level of membership that is too insignificant to be relevant, they can be excluded from consideration, that is, cut out. Second, as discussed by Buckley (2005, p. 16) and Hanss (2005, p. 20), among others, α-cuts and interval arithmetic[13] can be used to perform arithmetic on fuzzy numbers. Essentially, the α-cuts of fuzzy numbers are treated as intervals, subjected to interval arithmetic, and then the outcomes are merged to form a fuzzy number.

3.5 Fuzzy Arithmetic

As one would anticipate, fuzzy arithmetic can be applied to the fuzzy numbers. Using the extension principle (Zadeh, 1975), the nonfuzzy arithmetic operations can be extended to incorporate fuzzy sets and fuzzy numbers[14]. Briefly, if * is a binary operation such as addition (+), min ($\wedge$), or max ($\vee$), the fuzzy number z, defined by z = x * y, is given as a fuzzy set by

$$\mu_z(w) = \vee_{u,v}[\mu_x(u) \wedge \mu_y(v)], \quad u, v, w \in R \tag{3.7}$$

subject to the constraint that w = u * v, where μ_x, μ_y, and μ_z denote the membership functions of x, y, and z, respectively, and $\vee_{u,v}$ and $\wedge_{u,v}$ denote the supremum and infimum over u,v, respectively.[15]

A simple application of the extension principle is the sum of the fuzzy numbers A and B, denoted by A $\oplus$ B = C, which has the membership function:

$$\mu_C(z) = \max\{\min[\mu_A(x), \mu_B(y)] : x + y = z\} \tag{3.8}$$

[13] Moore (1966) provides a treatise on interval arithmetic.

[14] Fuzzy arithmetic is related to interval arithmetic or categorical calculus, where the operations use intervals, consisting of the range of numbers bounded by the interval endpoints, as the basic data objects. The primary difference between the two is that interval arithmetic involves crisp (rather than overlapping) boundaries at the extremes of each interval and it provides no intrinsic measure (like membership functions) of the degree to which a value belongs to a given interval. Babad and Berliner (1995) discussed the use of interval arithmetic in an insurance context.

[15] See Zimmermann (1996), Chapter 5, for a discussion of the extension principle.

An alternative approach is to use α-cuts and interval arithmetic, as mentioned in the previous section.

The general nature of the fuzzy arithmetic operations is depicted in Figure 3-5 for A = (-1,1,3) and B = (1,3,5) [16].

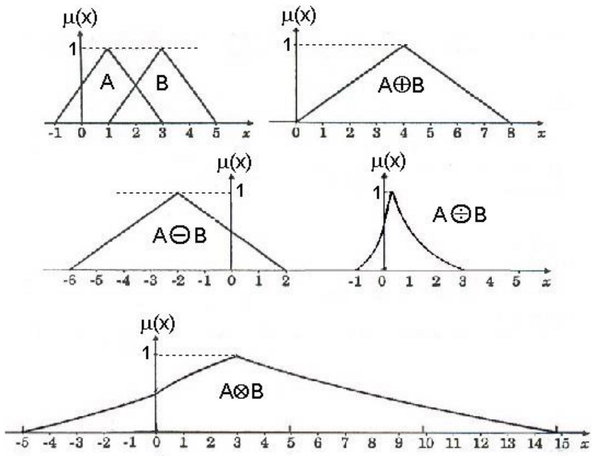

Figure 3-5: Fuzzy Arithmetic Operations

The first row shows the two membership functions A and B and their sum; the second row shows their difference and their quotient; and the third row shows their product.

For illustration purpose, consider the addition operation, and the two values z=0 and z=1.

For z=0, $\mu_C(0) = \max \{\min [\mu_A(x), \mu_B(y)]: x+y=0\}$
x+y = 0 in the feasible region $\Rightarrow$ (x = -1 and y=1) or ($\mu_A(-1)$=0 and ($\mu_A(1)$=0).
Thus $\mu_C(0) = \max \{\min [0, 0]\}= 0$

For z=1, $\mu_C(1) = \max \{\min [\mu_A(x), \mu_B(y)]: x+y=1\}$
x+y = 1 => y= - x+1 in the feasible region, and one observes the following:
(x = -1, y = 2) gives ($\mu_A(-1)$, $\mu_B(2)$) = (0, ½), then min [$\mu_A(-1)$, $\mu_B(2)$]=0
(x = -0.5, y = 1.5) will give ($\mu_A(-0.5)$, $\mu_B(1.5)$) = (¼, ¼), then min [$\mu_A(-0.5)$, $\mu_B(1.5)$] =1/4
(x = 0, y = 1) will give ($\mu_A(0)$, $\mu_B(1)$) = (½, 0), then min [$\mu_A(0)$, $\mu_B(1)$] =0

The maximum value is ¼ which corresponds to the value $\mu_C(z)$ for z = 1, as displayed on the graph.

[16] This figure is similar to Musilek and Gupta (2000, p. 157) Figure 18, after correcting for an apparent discrepancy in their multiplication and division representations.

3.6 Fuzzy Linear Programming

Many of the fuzzy logic studies in insurance involve decision making, and most of these studies rely on the framework established by Bellman and Zadeh (1970). The essential notion is that, given a non-fuzzy space of options, X, a fuzzy goal, G, and a fuzzy constraint, C, then G and C combine to form a decision, D, which is a fuzzy set resulting from the intersection of G and C. Assuming the goals and constraints enter into the expression for D in exactly the same way, a simple representation of the relationship between G, C and D is given in Figure 3-6.

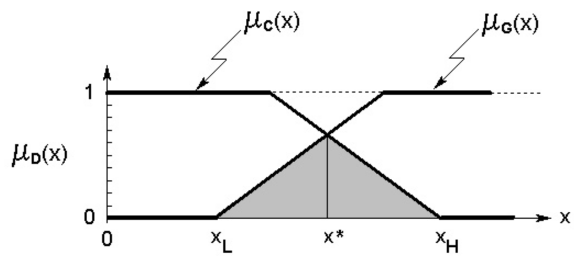

Figure 3-6: Decision Making

As indicated, the decision involves the fuzzy intersection of the goal and constraint MFs, and the set of possible options in the interval x_L to x_H. If the optimal decision is the option with the highest degree of membership in the decision set, the crisp solution to this problem would be

$$x^* = \arg[\max_x \min\{\mu_G(x), \mu_C(x)\}] \tag{3.9}$$

Here, we focus on the role of fuzzy linear programming (LP) in decision making. Like its crisp counterpart, fuzzy LP might involve finding an **x** such that [Zimmermann (1996, p. 289)]

$$\begin{aligned} C &= \sum_{ij} c_{ij}\, x_{ij} \,\tilde{\leq}\, C_0 \\ z_i &= \sum_j a_{ij}\, x_{ij} \,\tilde{\geq}\, b_i \\ x_{ij} &\geq 0 \end{aligned} \tag{3.10}$$

where C_0 is the aspiration level for the objective function, "~" over a symbol denotes the fuzzy version of that symbol, and the coefficients a_{ij}, b_i, and c_{ij} are not necessarily crisp numbers.

This fuzzy LP problem can be resolved by reformulating it as a crisp LP problem. The essence of one approach[17] to doing this is depicted in Figure 3-7.

[17] Adapted from Brockett and Xia (1995), pp. 34-38.

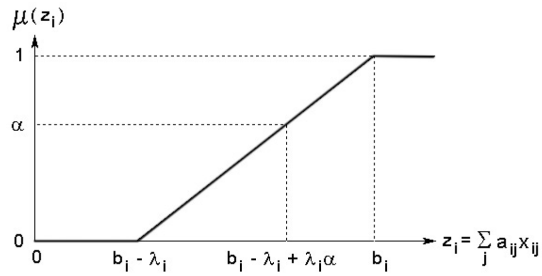

Figure 3-7: Equivalent Crisp Constraint

As indicated, z_i is a fuzzy number, whose membership function is zero for $z_i \le b_i - \lambda_i$, one for $z_i \ge b_i$, and linearly increasing in the interval. Zimmermann refers to λ as a tolerance interval. Using an α–cut to provide a minimum acceptable satisfaction level, that is, $\mu(z_i) \ge \alpha$ is an acceptable constraint, we see from the diagram that an equivalent constraint is $z_i \ge b_i - \lambda_i + \lambda_i\,\alpha$. Similarly, $C \le C_0 + \lambda - \lambda\,\alpha$.

Thus, given the values of λ, the equivalent crisp programming problem becomes one of maximizing α subject to the equivalent constraints, that is:

$$\text{Maximize:} \quad \alpha \tag{3.11}$$

$$\begin{aligned} \text{Subject to:} \quad & z_i - \lambda_i\,\alpha \ge b_i - \lambda_i\,; \\ & C + \lambda\,\alpha \le C_0 + \lambda\,;\ \text{and} \\ & 0 \le \alpha \le 1. \end{aligned}$$

Fuzzy linear programming is used in Chapter 6.

3.7 Fuzzy Inference Systems

The fuzzy inference system (FIS) is a popular methodology for implementing FL. FISs are also known as fuzzy rule based systems, fuzzy expert systems (FES), fuzzy models, fuzzy associative memories (FAM), or fuzzy logic controllers when used as controllers [Jang et al. (1997, p. 73)]. The essence of the system can be represented as shown in Figure 3-8.[18]

[18] Adapted from Peña-Reyes and Sipper (1999), Figure 2.

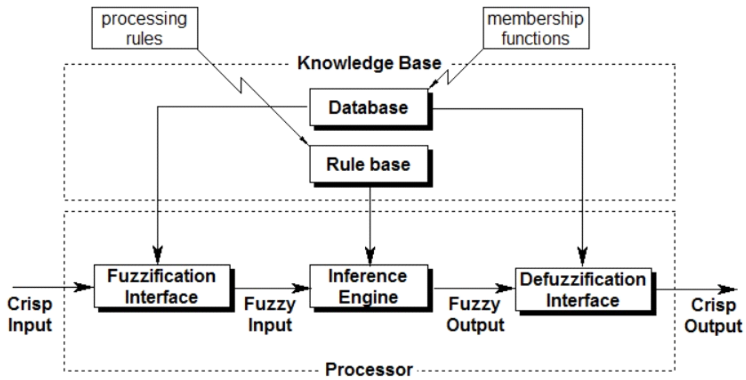

Figure 3-8: Fuzzy Inference System (FIS)

As indicated in the figure, the FIS can be envisioned as involving a knowledge base and a processing stage. The knowledge base provides MFs and fuzzy rules needed for the process. In the processing stage, numerical crisp variables are the input of the system.[19] These variables are passed through a fuzzification stage where they are transformed to linguistic variables, which become the fuzzy input for the inference engine. This fuzzy input is transformed by the rules of the inference engine to fuzzy output. These linguistic results are then changed by a defuzzification stage into numerical values that become the output of the system.

The operations t-norms (triangular-norms) and t-conorms (its dual) are used in FISs to combine the incoming signals and weights and to aggregate their products. The simplest examples of the t-norm and the t-conorm are the min-operator and max-operator, respectively. Frees and Valdez (1998) show that many copulas can serve as t-norms.

The Mamdani FIS, a representation of which is shown in Figure 3-9, has been the most commonly mentioned FIS in the insurance literature.

[19] In practice, input and output scaling factors are often used to normalize the crisp inputs and outputs. Also, the numerical input can be crisp or fuzzy. In this latter event, of course, the input does not have to be fuzzified.

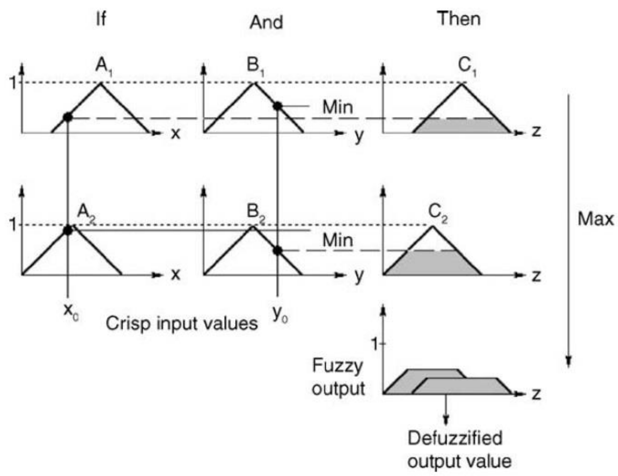

Figure 3-9: The Mamdani FIS

In this case, there are two crisp inputs, x_0 and y_0, and three sets of membership functions, A_j, B_j and C_j, j=1,2, each set of which represent the rule A_j and $B_j \Rightarrow C_j$, where the conjunction "and" is interpreted to mean the fuzzy intersection. The minimum of the fuzzy inputs in the first two columns gives the levels of the firing (shown by the dashed lines) and their impact on the inference results (shown by the shaded areas in the third column). Taking the union of the shaded areas of the first two rows of column three results in the fuzzy set show in the third row, which represents the overall conclusion.

Defuzzification converts the fuzzy overall conclusion into a numerical value that is a best estimate in some sense. A common tactic in insurance articles is to use the center of gravity (COG) approach, which defines the numerical value of the output to be the abscissa of the center of gravity of the union. In practice, this is computed as $\Sigma_j\, w_j\, x_j$, where the weight w_j is the relative value of the membership function at x_j, that is, $w_j = \mu(x_j) / \Sigma_j\, \mu(x_j)$.

The Mamdani FIS is used to model the risk matrix in Chapter 4.

3.8 Comment

This chapter provided a cursory review of the FL methodologies discussed in this report. Readers who prefer a more extensive introduction to the topic, with an insurance perspective, are referred to Ostaszewski (1993) and Shapiro (2004). Those who are interested in a comprehensive introduction to the topic are referred to Zimmermann (1996) and DuBois and Prade (1997). Readers interested in a grand tour of fuzzy logic are urged to read the collection of Zadeh’s papers contained in Yager *et. al.* (1987), Klir and Yuan (1996), and Ruan and Huang (2000), and the overview of fuzzy logic in Zadeh (2012).

4 Using Fuzzy Logic to Model the Risk Matrix

4.1 Introduction

Generally, the risk associated with an event consists of two major components or inputs: the likelihood of the occurrence of the event and the severity of the consequence of the event. See, for example, Hopkin (2010, Chapter 13), Segal (2011, Chapter 4), and Sweeting (2011, Chapter 15). Given these two components, one of the common mechanisms used to characterize and rank risks is the risk matrix.[20] Essentially, the likelihood and severity are scored according to their influence on the risk and then the risk matrix aggregates these scores into an overall risk score or ranking.

Traditionally, the risk matrix was regarded as a quantification tool, in that it was developed from crisp (non-fuzzy) inputs. Apparently, it was first discussed by Hussey (1978), in the context of strategic portfolio analysis. Its rationale was later described by Kaplan and Garrick (1981, p. 13), who envisioned risk analysis as the answers to the following three questions:

(i) What can happen? (i.e., What can go wrong?)
(ii) How likely is it that that will happen? and
(iii) If it does happen, what are the consequences?

It became apparent, however, that not all risks can be quantified. For example, as noted by Sweeting (2011, p. 401), potential losses might be difficult to assess with any degree of certainty, as with reputational risk, the negative impact of poor publicity on future sales, and many types of operational risk relating to issues such as fraud and business continuity. As a response to these types of situations, the traditional risk matrix was modified to accommodate a qualitative risk assessment (RA) database,[21] which transformed it into a fuzzy risk matrix.

One of the first articles to address this topic of fuzzy risk matrices was Karwowski and Mital (1986). In that paper, a fuzzy set theoretic approach to risk analysis was proposed and linguistic variables were introduced to analyze situations involving approximate reasoning methods.

In this chapter, we present a fuzzy risk matrix example. We begin with a brief overview of risk maps, a simple form of risk matrix. We then turn our attention to a more explicit form of the risk matrix, one which depicts the mapping of the likelihood and severity inputs onto a risk-based

---

[20] Not everyone agrees that the risk matrix is a sound approach. Wall (2011), for example, contends that the theoretical basis of the risk matrix is superficial and the validity of the qualitative information it employs is highly suspect. However, such concerns notwithstanding, we view the popularity of the risk matrix concept as sufficient motivation for this chapter.

[21] While beyond the scope of this study, Segal (2011, pp. 129-153) gives an overview of a four-step process for developing a qualitative risk assessment database, based on participant identification, advance communication, qualitative risk assessment surveys, and consensus meeting.

grid. Since the likelihood and severity inputs often involve imprecise or vague judgments, we next discuss how the fuzzy inputs and output can be represented by membership functions (MFs) and how those MFs can be processed to form risk scores. We then discuss how the risk scores can be merged to form a fuzzy risk matrix. The chapter concludes with a comment on the fuzzy risk matrix approach.

4.2 Risk Maps

An example of a simple form of a risk matrix, often referred to as a risk map, is shown in Figure 4-1. [Institute of Management Accountants (2007, p. 13)] Its essential feature is that it is primarily a mapping of impact (severity) and likelihood onto a common space.

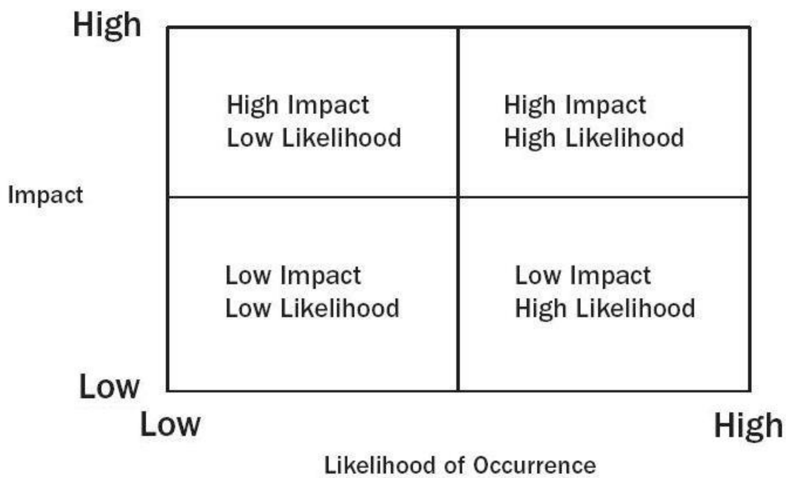

Figure 4-1: Risk map

For example, this particular risk map maps low likelihood and high impact into the upper left quadrant.

Figure 4-2 shows how Sweeting (2011, p. 402) envisioned implementing a risk map. The context here is pension scheme operational risk, and in this diagram the likelihood and impact (severity) of three such risks[22] are mapped onto a two-dimensional chart. Sweeting scored both likelihood and impact from one (unlikely/low impact) to five (very likely/high impact), as depicted in the figure.

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[22] In his Figure 15.6, Sweeting (2011), showed twelve such risk, including the three in Figure 4-2.

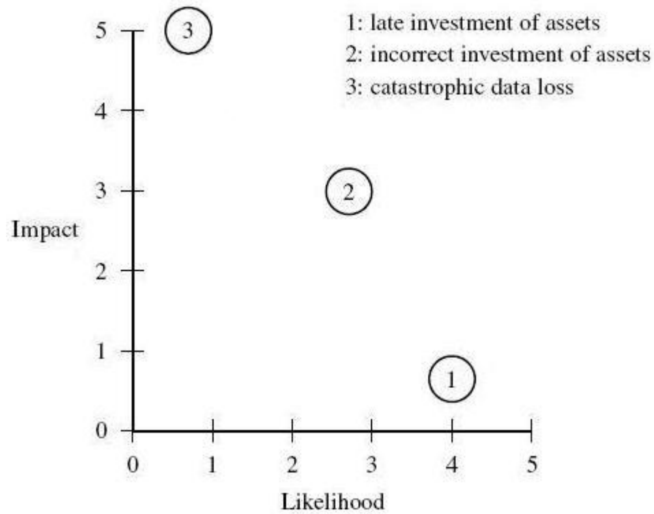

Figure 4-2: Risk Map

4.3 Risk-based Grid

While the foregoing risk map shows the relative importance of the likelihood and impact based on their coordinates in the risk space, a more explicit form of the risk matrix is one which shows the likelihood and severity mapped onto a risk-based grid. An example of this, based on the work of Wu et al (2013, p. 28), is shown in Figure 4-3.

Severity	1	2	3	4	5
5	MH 5	MH	MH	H	H 25
4	M	M	MH	MH	H
3	M	M 8	M	MH	MH
2	L	L 7	L	M	MH
1	L 1	L	L	M	MH

Likelihood

Risk categories:
1. low (L)
2. medium (M)
3. medium high (MH)
4. high (H)

Severity categories:
1. negligible
2. low
3. moderate
4. high
5. catastrophic

Likelihood categories:
1. very low
2. low
3. moderate
4. high
5. very high

Figure 4-3: Risk matrix example

As indicated in the figure, there are 5 categories of likelihood[23] and 5 categories of severity, for a total of 25 rules. Also, in the figure, rules 1, 5, 7, 8, and 25 are labeled in the bottom right of their respective cells. These rules can be expressed in the IF-THEN format, examples of which are: [~Wu et al (2013, p. 30)]

$$\begin{aligned}&\text{Rule 5: IF Likelihood is "Very Low" and Severity is "Catastrophic"}\\&\quad\text{THEN the risk is "Medium high".}\end{aligned} \tag{4.1}$$

Rule 7: IF Likelihood is "Low" and Severity is "Low"
THEN the risk is "Low";
Rule 8: IF Likelihood is "Low" and Severity is "Moderate"
THEN the risk is "Medium";

4.4 Formulating the Categories as MFs

The risk matrix of Figure 4-3 establishes the dual-input and single-output logic system for calculating the risk index, based on the likelihood and severity input variables and the risk variable. However, the figure involves crisp values, and, as mentioned earlier, in practice the likelihood and severity inputs often involve imprecise or vague judgments, that is, fuzzy values. To accommodate this, the system can be reformulated as a fuzzy risk matrix by replacing the crisp inputs and output with fuzzy inputs and output, which we represent by MFs.

Methods for assigning MFs were discussed Chapter 3. The type used in any particular situation will depend on the perceived specific nature of the input and output variables.

Here, by way of example, we follow Wu et al (2013, p. 30), and use the triangular type of membership function. Figure 4-4 shows these membership functions for the likelihood, severity, and risk, discussed with respect to the risk assessment matrix of Figure 4-3.

---

[23] To provide a rough guideline for the experts who are providing qualitative data for this type of analysis, it is common, in order to promote consistency, to provide them with likelihood categories and stipulated ranges of probabilities. In Department of Defense (1993, p. A-5) and Molland (2008, p. 823), for example, the qualitative assessment of the likelihood of an event, and its range of probabilities, x, is given as: frequent ($x > 10^{-1}$), probable ($10^{-1} > x > 10^{-2}$), occasional ($10^{-2} > x > 10^{-3}$), remote ($10^{-3} > x > 10^{-6}$) or improbable ($x < 10^{-6}$). Another example of an approach to this problem is to have the experts imagine, and then provide scores for, a credible worst-case scenario, which tends to ensure a reasonable level of consistency in scoring, yet is not overly prescriptive. See Segal (2011, pp. 135-6).

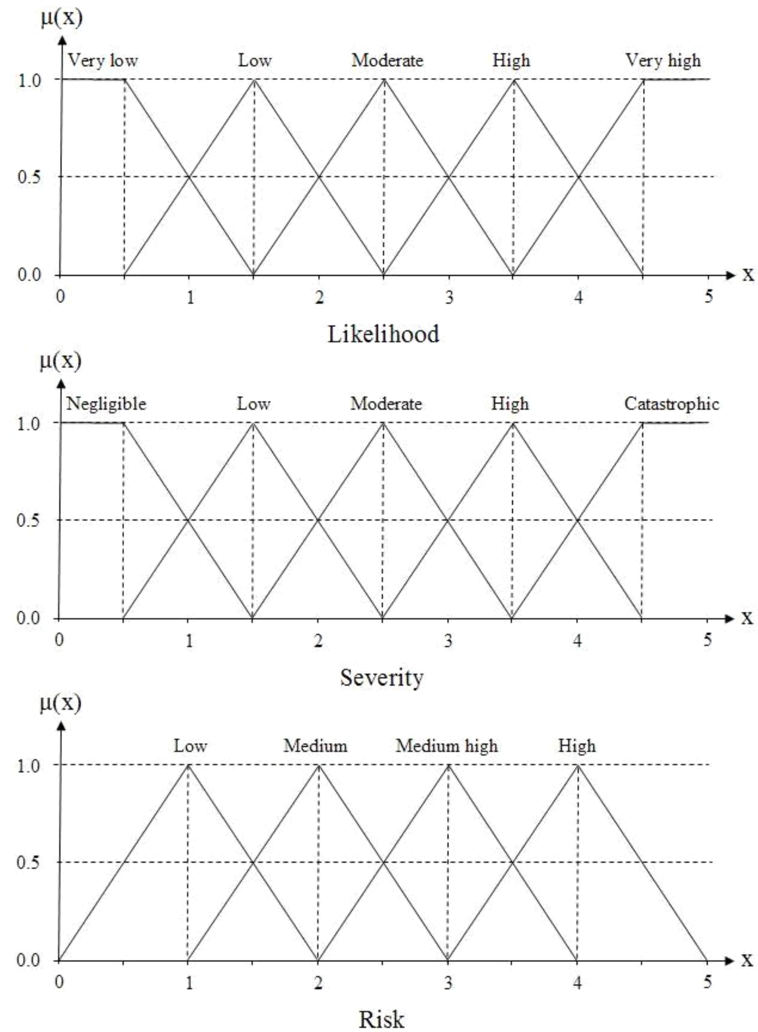

Figure 4-4: Likelihood, Severity and Risk MFs

As far as implementation of these MFs is concerned, consider again Rule 8:

IF Likelihood is "Low" AND Severity is "Moderate", THEN the risk is "Medium".

In terms of Figure 4-3, the MFs under this rule can be represented as shown in Figure 4-5.

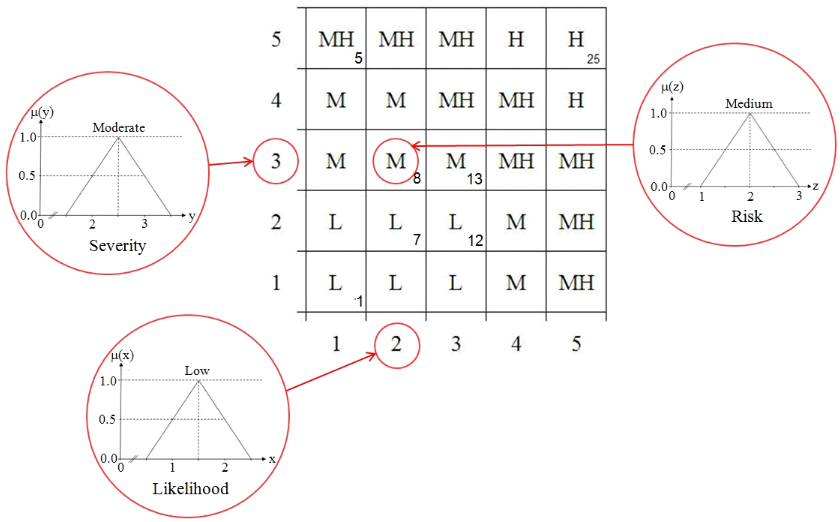

Figure 4-5: Membership functions for fuzzy Rule 8

Then, the fuzzy version of Rule 8, itself, can be represented as shown in Figure 4-6.

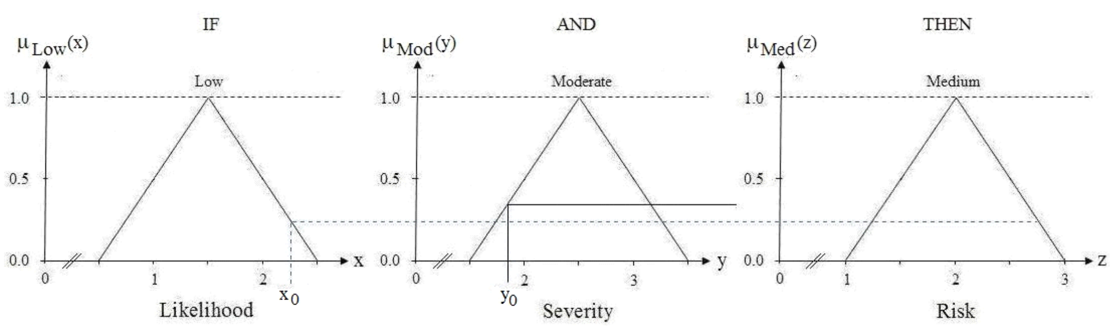

Figure 4-6: Risk matrix fuzzy Rule 8

Now, suppose, as is indicated in the figure, that a likelihood and severity input pair take the values x_0 and y_0, respectively. The grade of membership (GOM) of the likelihood input, $\mu_{Low}(x_0)$, in the category of low values, indicated by the horizontal dashed line, is less than the GOM of the severity input, $\mu_{Mod}(y_0)$, in the category of moderate values, indicated by the horizontal solid line. As a consequence, the GOM imputed to the risk output in the medium category is $\mu_{Med}(z) = \min\{\mu_{Low}(x_0), \mu_{Mod}(y_0)\}$, which is why the horizontal dashed line is extended to the risk MF.

4.5 Using an FIS to Model the RM

Figure 4-6 and its discussion provide a natural segue into using the Mamdani fuzzy inference system (FIS) of Chapter 3 to model the fuzzy risk matrix. To see this, consider the representation shown in Figure 4-7.[24]

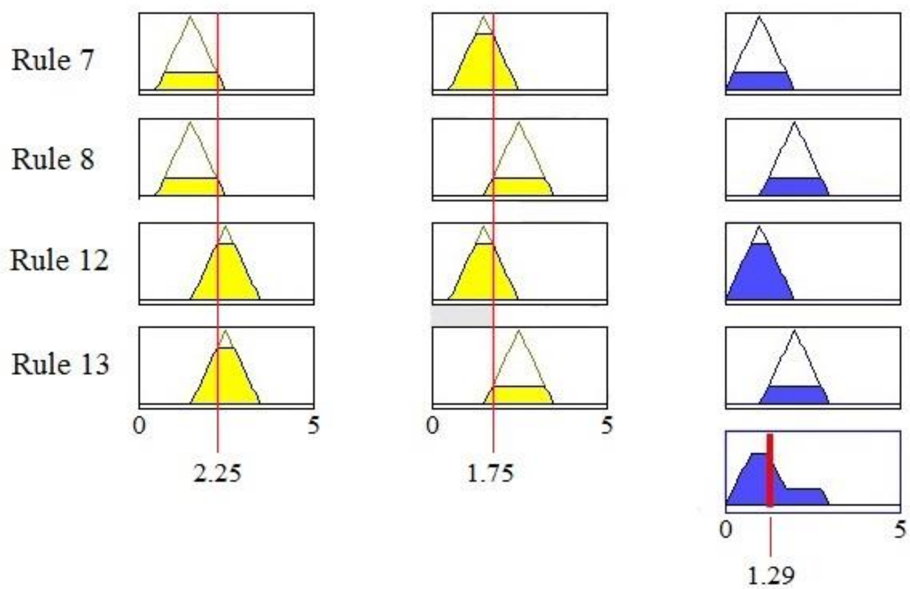

Figure 4-7: Mamdani FIS example involving rules 7, 8, 12, and 13.

In this case, there are three sets of membership functions, $\mu_{F_j}(x)$, $\mu_{S_j}(y)$, and $\mu_{R_j}(z)$, j=1, 2, 3, 4, representing the likelihood, severity and risk, respectively, and two crisp inputs, $\hat{l} = 2.25$ and $\hat{s} = 1.75$.[25] Only the MFs of the rules that are affected by the inputs are shown in the figure. Each membership set represents the rule $\mu_{R_j}(z) = \min\left\{\mu_{F_j}(\hat{l}),\ \mu_{S_j}(\hat{s})\right\}$, where, as before, the minimum of the fuzzy inputs in the first two columns gives the levels of the firing and their impact on the inference results is shown by the shaded areas in the third column. Taking the union of the shaded areas of the first four rows of column three results in the fuzzy set show in the fifth row, which represents the overall conclusion.

As discussed in chapter 3, defuzzification is used to convert the fuzzy overall conclusion into a numerical value that is a best estimate in some sense. Here, as indicated in the figure, defuzzification leads to a risk index of 1.29.

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[24] Generated using the Rule Viewer of the Fuzzy Logic ToolBox of Matlab.
[25] These values were arbitrarily chosen for expository purposes. However, they need not be arbitrary. Wu et al (2013, p. 27), for example, referred to such values as the likelihood index (LI) and severity index (SI), respectively, because they were based on weights, developed using the Analytic Hierarchy Process (see Chapter 5), applied to the likelihood and severity MFs.

4.6 The Fuzzy Risk Matrix

The FIS of the previous section provides the basis for the conceptualization of a fuzzy risk matrix. To see this, we use the FIS to construct the three-dimensional plot that represents the mapping from all combinations of the two fuzzy inputs (likelihood and severity) to the fuzzy output (risk), as shown in Figure 4-8.[26]

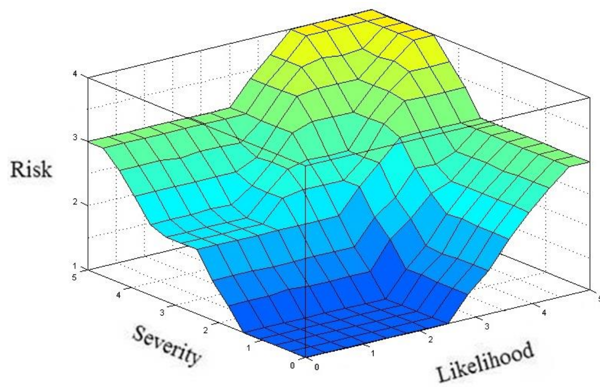

Figure 4-8: Fuzzy risk matrix

As the figure indicates, this fuzzy risk surface, since it is a granulated version of the risk matrix, rather than just four discrete rankings, better reflects the differences associated with the risk category.

4.7 Comments

This chapter presented a fuzzy logic version of the risk matrix, a commonly used risk assessment tool. The value-added by the fuzzy logic was that verbal expressions and linguistic variables, which are commonly associated with imprecise or vague judgments, were easily accommodated into the process.

Further insight into the mechanics of fuzzy risk matrix applications may be obtained by working through the examples in Elsayed (2009), Markowski and Mannan (2008) and Hu et al (2007).

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[26] Generated using the Surface Viewer of the Fuzzy Logic ToolBox of Matlab.

5 Fuzzy Logic Modifications of the Analytic Hierarchy Process

5.1 Introduction

The Analytic Hierarchy Process (AHP) [Saaty (1980, 1999, 2008)] is a theory of measurement through pair-wise comparisons that relies on judgment to derive priority scales. During implementation of the AHP, one constructs hierarchies, then makes judgments or performs measurements on pairs of elements with respect to a criterion to derive preference scales, which are then synthesized throughout the structure to select the preferred alternative.

One of the areas[27] where the AHP finds application is in the subjective phases of RA, where it is used to structure and prioritize diverse risk factors, including the judgments of experts. Since FL has been shown to be an effective tool for accommodating human experts and their communication of linguistic variables, there has been research aimed at modeling the fuzziness in the AHP (FAHP), and recently the focus of some of that modeling has been in RA.

The examples of FAHP in RA generally relate to engineering topics. Zeng et al (2007) and Nieto-Morote and Ruz-Vila (2011), for example, presented a FAHP-based RA methodology to cope with the multitude of risks associated with complicated construction projects, where FL and the AHP were used to deal with subjective judgments and to structure the large number of risks, respectively. In a safety context, Shi et al (2012) use the FAHP to model RA associated with falling from height on construction projects, Fera and Macchiaroli (2010) used FAHP to develop a new RA model to address safety management of small and medium enterprises, and An et al (2011) used FAHP to develop a RA system for evaluating both qualitative and quantitative risk data and information associated with the safety management of railway systems. Another application area was offshore drilling, where Miri Lavasani et al (2011) used FAHP to estimate the weights required for grouping non-commensurate risk sources associated with the RA of oil and gas offshore wells, and Zhang et al (2012) use FAHP to develop a RA model of relief wells to cope with potential accidents during onshore and offshore drilling.

The literature discusses more than one FAHP model, which raises the question as to which are the prominent models and what are their characteristics. In response to this question, we examine the models underlying three of the most influential FAHP articles. The chapter proceeds as follows. It begins with a brief overview of the AHP and its limitations when confronted with a fuzzy environment. This is followed with a discussion of FL modifications of the AHP.

[27] Surveys of other areas of AHP applications can be found in Vargas (1990), Vaidya and Kumar (2006), Subramanian and Ramanathan (2012) and Saaty and Vargas (2012).

5.2 The Hierarchical Structure

We start with a discussion of the hierarchical structure since it is key to the study of the AHP. A simple representation of a hierarchical structure is the K × n version depicted Figure 5-1.[28]

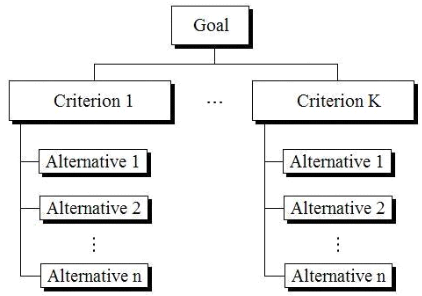

Figure 5-1: A simple hierarchical structure

As indicated, this hierarchy consists of three levels: [Saaty and Vargas (2012, p. 2)]

The goal of the decision at the top level,

The criteria by which the alternatives will be evaluated, in the second level, and

The alternatives, which are located in the third level.

This structure makes it possible to evaluate the importance of the elements in a given level with respect to elements in a higher level.

5.2.1 A likelihood score example

As an example of the use of a hierarchical structure, consider the task of developing the likelihood of a particular risk (the goal), given the three factors upon which it depends (the alternatives), and four criteria upon which those factors depend (the criteria). We are to compute the weights to be associated with the three factors. This situation is depicted in Figure 5-2.

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[28] Adapted from Buckley (1985b, p. 238) Figure 1.

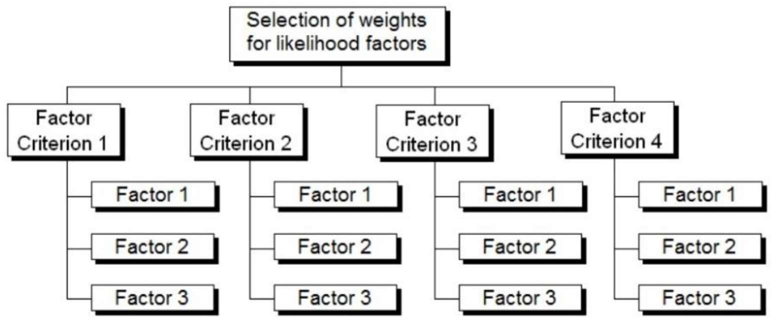

Figure 5-2: Influence factor hierachical structure

Most of the examples in this chapter will be based on this scenario.

5.2.2 **Our notation**

The relationships implicit in Figure 5-1, and the weights associated with its components, are captured in the following notation.

A_i = the i-th alternative, i = 1, 2, ..., n
C_k = the k-th criterion, k = 1, 2, ..., K
G = the goal
$w_i^{A|G}$ = the weight associated with A_i, with respect to G
$w_{ij}^{A|G}$ = the relative preference of A_i over A_j, with respect to G

$$\equiv \frac{w_i^{A|G}}{w_j^{A|G}}, \text{ i, j = 1, 2, ..., n}$$

$w_k^{C|G}$ = the weight associated with C_k, with respect to G
$w_{kj}^{C|G}$ = the relative preference of C_k over C_j, with respect to G

$$\equiv \frac{w_k^{C|G}}{w_j^{C|G}}, \text{ k, j = 1, 2, ..., K}$$

$w_i^{A|C_k}$ = the weight associated with A_i, with respect to C_k
$w_{ij}^{A|C_k}$ = the relative preference of A_i over A_j, with respect to C_k

$$\equiv \frac{w_i^{A|C_k}}{w_j^{A|C_k}}, \text{ i, j = 1, 2, ..., n; k = 1, 2, ..., K}$$

In some instances, when several decision-makers express their opinion on the relative significance of a pair of factors, there may be multiple estimates for the comparison ratios. Conversely, there may be situations where there are no estimates for certain ratios (missing data). These cases can be accommodated with an array of the form

$$\underline{w}_{ij} = (w_{ij1}, w_{ij2}, \cdots, w_{ijn_{ij}})^T \tag{5.1}$$

where n_{ij}, in the last subscript, is defined as

$n_{ij} = 0$ is associated with an empty cell,
$n_{ij} = 1$ indicates a single comparison, and
$n_{ij} > 1$ indicates a cell where there are multiple comparisons.

Let "^" indicate a perceived[29] value, and following Saaty, let "a" and "c" denote the base symbols for the perceived weights associated with the alternatives and criteria, respectively. Then

$$a_{ij}^{A|G} = \hat{w}_{ij}^{A|G} \approx w_{ij}^{A|G} \tag{5.2}$$

$$a_{ij}^{A|C_k} = \hat{w}_{ij}^{A|C_k} \approx w_{ij}^{A|C_k} \tag{5.3}$$

$$c_{kj}^{C|G} = \hat{w}_{kj}^{C|G} \approx w_{kj}^{C|G} \tag{5.4}$$

In what follows, except for emphasis, the superscript on the c will be suppressed, since it is redundant.

5.3 An Overview of the AHP

Given the hierarchical structure of the previous section, this section provides a brief overview of the salient features of the AHP relative to that structure.

We begin with the flowchart of the AHP shown in Figure 5-3.

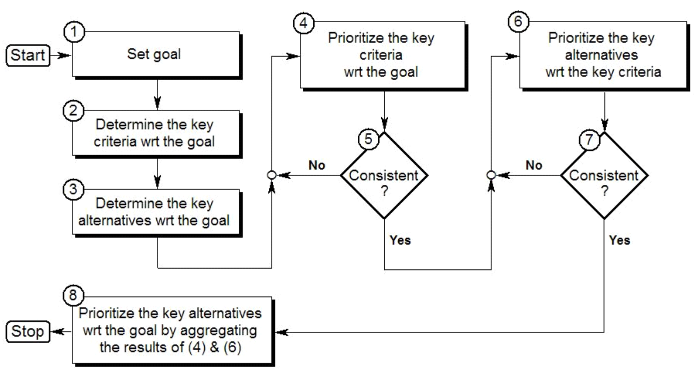

Figure 5-3: Steps of the AHP

As indicated, in the first step, the goal is set. This is followed by the determination of the key criteria with respect to the goal and the key alternatives with respect to the goal. Having accomplished that, the key criteria are prioritized relative to the goal, and the consistency of their prioritization is validated. If the consistency is inadequate, the prioritization is iteratively redone and the consistency revalidated until either the consistency is validated, or the AHP attempt is abandoned.[30] Similarly, the alternatives are prioritized relative to each criterion. Finally,

[29] The term "perceived" denotes a value that may be based on incomplete or unreliable data and/or vague and subjective information.
[30] The current version of Figure 5-3 does not explicitly provide an option to abandon the AHP attempt.

assuming consistency, the key alternatives are prioritized with respect to the goal, using aggregation, in the manner discussed at the end of this section.

5.3.1 **The Essence of the AHP**

The essence of the AHP is based on the following idealized situation.[31]

As far as assumptions underlying the AHP model, it is assumed that: [Kumar and Maiti (2012, p. 9947), Adamcsek (2008, p. 7)]]

The decision-making can be modeled in a linear top-to-bottom form as a hierarchy,

The dependencies among elements can only be between the levels of the hierarchy,

The upper level in the hierarchy does not depend on the lower levels, and

The elements of a given level in a hierarchy are independent of each other.

Starting with the comparison judgments related to the criteria, the pairwise relative preference of K criteria items is modeled by a K × K preference matrix C, a representation of which is as follows:

$$C = \left[c_{kj}^{C|G} \right]_{K \times K} = \begin{bmatrix} 1 & c_{12}^{C|G} & \cdots & c_{1K}^{C|G} \\ c_{21}^{C|G} & 1 & \cdots & c_{2K}^{C|G} \\ \vdots & \vdots & \ddots & \vdots \\ c_{K1}^{C|G} & c_{K2}^{C|G} & \cdots & 1 \end{bmatrix}, \qquad (5.5)$$

where each cell $c_{kj}^{C|G}$ reflects how many more times, relative to the goal (G), criterion k is preferred to criterion j. To assist in this classification, Saaty (1980, Table 3-1), provided a table of relative intensities, which ranged from a minimum of 1, where the activities are "equally important", to 3, where there is "weak importance of one over the other," to 5, where "experience and judgment strongly favour one activity over another", to a maximum of 9, where the preference of "one activity over another is of the highest possible order of affirmation".

The construction of this matrix is a two-step process. First, the cells $c_{kj}^{C|G}$, j > k, are filled in. Then, under the assumption of a consistent preference matrix, that is, $c_{jk}^{C|G} \cdot c_{kj}^{C|G} = 1$ (reciprocal) and $c_{kj}^{C|G} = c_{ki}^{C|G} \cdot c_{ij}^{C|G}$, $\forall$i, j, k (product-transitive), the cells $c_{kj}^{C|G}$, j < k, are filled in. $c_{kk}^{C|G} = 1, \forall k$.

[31] Adapted from Dubois (2011, pp. 18-19).

Since C is a consistent K × K matrix, its largest eigenvalue is λ = K and there exists a corresponding eigenvector $\underline{w}^{C|G} = (w_1^{C|G}, w_2^{C|G}, \cdots, w_K^{C|G})^T$ with $c_{kj}^{C|G} \approx w_k^{C|G} / w_j^{C|G}$, $\forall$k, j, yielding the relative importance of the weights.

Given the complete preference matrix, either its principal eigenvector (discussed in §4) or one of its approximations (discussed in §5) is used as the vector of priorities.

Then, to verify that the preference matrix is sufficiently consistent, first compute λ_{max}, the principal eigenvalue, as: [Saaty and Vargas (2012, pp. 26-7)]

$$\lambda_{max} = \frac{1}{K}\sum_{k=1}^{K}\frac{(CW)_k}{w_k} = \frac{1}{K}\sum_{k=1}^{K}\frac{\sum_{j=1}^{K} c_{kj}^{C|G} w_j^{C|G}}{w_k^{C|G}} \tag{5.6}$$

Given λ_{max}, the consistency index, CI, is computed as

$$CI = \frac{\lambda_{max} - K}{K - 1} \tag{5.7}$$

The final stage is to calculate a Consistency Ratio (CR) to measure how consistent the judgments have been relative to large samples of purely random judgments. Specifically, the CR of the preference matrix is computed as

$$CR = \frac{CI}{RI} \tag{5.8}$$

where the random index, RI, is a simulated random pairwise comparison for different size matrices, and is given in the following table.

Table 5-1 Saaty's random index

n	1	2	3	4	5	6	7	8	9	10
RI	0.00	0.00	0.58	0.90	1.12	1.24	1.32	1.41	1.46	1.49

Source: Saaty (1980)

According to Saaty, a CR < 10% implies consistency, while if it is not less than 10% the judgments need to be revised.[32]

Similarly, the preference matrixes for the alternatives, relative to each of the criterion, $A = \left[a_{ij}^{A|C_k}\right]_{n\times n}$, k=1, 2, ..., K, are constructed and checked for consistency.

Given these two local values, the criteria preferences with respect to the goal, and the alternatives preferences with respect to the criteria, the global result, the alternatives preferences with respect to the goal, comes from their aggregation:

$$\begin{bmatrix} w_1^{A|C_1} & w_1^{A|C_2} & \cdots & w_1^{A|C_K} \\ w_2^{A|C_1} & w_2^{A|C_2} & \cdots & w_2^{A|C_K} \\ \vdots & \vdots & \ddots & \vdots \\ w_n^{A|C_1} & w_n^{A|C_2} & \cdots & w_n^{A|C_K} \end{bmatrix} \begin{bmatrix} w_1^{C|G} \\ w_2^{C|G} \\ \vdots \\ w_K^{C|G} \end{bmatrix} = \begin{bmatrix} w_1^{A|G} \\ w_2^{A|G} \\ \vdots \\ w_n^{A|G} \end{bmatrix} \tag{5.9}$$

5.4 The Eigenvector Method for Determining the Weights[33]

Let

$$W = [w_{ij}]_{n\times n} = \left[\frac{w_i}{w_j}\right] \tag{5.10}$$

be an n × n consistent pairwise comparison matrix, where $w_1, w_2, ..., w_n$ are weights and $\sum_{i=1}^{n} w_i = 1$. Then $w_{ij} = \dfrac{1}{w_{ji}}$ and $w_{ij} = \dfrac{w_i}{w_j} = \dfrac{w_i}{w_k}\dfrac{w_k}{w_j} = w_{ik}w_{kj}\ \forall i, j, k.$

If W is known, but $\underline{w} = (w_1, w_2, ..., w_n)^T$, the vector of weights, is not, the latter can be recovered using the eigenvalue method.

We begin by taking the matrix product of the matrix W with the vector $\underline{w}$ to obtain:

32 Saaty counsel that "… improving the consistency of a judgment matrix does not necessarily improve the validity of the outcome. Validity is the goal in decision-making, not consistency, which can be successively improved by manipulating the judgments as the answer gets farther and farther from reality." [Saaty and Tran (2007)]

33 Adapted from Saaty (1980, pp. 49-51, §7-5, pp. 258-9) and Adamcsek (2008, Chapter 3).

$$W\,\underline{w} = \begin{bmatrix} 1 & \frac{w_1}{w_2} & \cdots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & 1 & \cdots & \frac{w_2}{w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \cdots & 1 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} nw_1 \\ nw_2 \\ nw_3 \\ \vdots \\ nw_n \end{pmatrix} = n\,\underline{w} \tag{5.11}$$

This is an eigenvalue problem of the form $W\underline{w} = \lambda\underline{w}$, where λ is an eigenvalue, which can be solved for $\underline{w}$.[34]

In this instance, since each row of W is a constant multiple of its first row, its rank is one and all its eigenvalues, save one, are equal to zero. Moreover, since the sum of the eigenvalues of a positive ($w_{ij} > 0, \forall$ i,j) matrix is equal to its trace (the sum of its diagonal elements), the non zero eigenvalue has a value of n, the order of the matrix. Since $W\underline{w} = n\underline{w}$, $\underline{w}$ is said to be the eigenvector of W corresponding to the maximum eigenvalue n.

In practice, in contrast to $W=[w_{ij}]$, where $w_{ij} = w_{ik}w_{kj}$, the preference matrices, C and A, generally do not have this product-transitive characteristic, since the judgment of experts have some degree of inconsistency. In the case of the criteria, for example, w_{ij} is estimated by c_{ij}, and the eigenvalue problem for the inconsistent case is:

$$A\underline{w} = \lambda_{\max}\underline{w} \tag{5.12}$$

where λ_{max} will be close to n (actually greater than or equal to n [Saaty (1980, p. 181)]) and the other eigenvalues will be close to zero. The estimates of the weights can be found by normalizing the eigenvector corresponding to the largest eigenvalue in the above matrix equation.

5.5 Alternate Vectors of Priorities

In the previous section, vectors of priorities were constructed from the pair-wise comparison matrix by first computing the principal eigenvector, and then normalizing it. For those instances when this approach is not feasible, Saaty (1980, pp. 19-21, 231-3) suggested various approximations that can be used. Given $C = \left[c_{ij}^{C|G}\right]$, three of those methods that are used in subsequent discussions are:

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[34] Essentially [Saaty (1980, pp. 258-9)], this reduces to the problem of finding the λ's that are the roots of $|W - \lambda I|=0$.

(1) Normalized arithmetic mean. Sum the elements in each row and normalize by dividing each sum by the total of all the sums, so that the results add up to unity:

$$w_i = \frac{\sum_{j=1}^{K} c_{ij}}{\sum_{i=1}^{K}\sum_{j=1}^{K} c_{ij}}, \quad i = 1, 2, ..., K \tag{5.13}$$

The first entry of the resulting vector is the priority of the first activity, the second of the second activity, and so on.

(2) Normalized geometric mean. Multiply the n elements in each row and take the nth root. Normalize the resulting numbers. Thus,

$$w_i = \frac{\left(\prod_{j=1}^{K} c_{ij}\right)^{1/K}}{\sum_{i=1}^{K}\left(\prod_{j=1}^{K} c_{ij}\right)^{1/K}}, \quad i = 1, 2, ..., K \tag{5.14}$$

(3) Logarithmic least squares model. The vector of priority is estimated by the normalized vector that minimizes:

$$\sum_{i=1}^{K}\sum_{\substack{j=1 \\ j\neq i}}^{K}\left(\ln c_{ij} - \ln\left(\frac{w_i^{C|G}}{w_j^{C|G}}\right)\right)^2 \tag{5.15}$$

which turns out to be the same as the normalized geometric mean in certain instances.[35]

5.6 Examples of Application

This section presents three examples of how the foregoing models can be implemented: the first shows how the normalized geometric mean can be used to develop a vector of priorities; the second discusses the development and validation of λ_{max}; and the third develops weights for the likelihood factors.

5.6.1 An example of the normalized geometric mean

Buckley et al (2001) presented the following informative example with respect to the development of a vector of priorities, based on the geometric mean approach.

Let

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[35] See page 45.

$$\begin{bmatrix} 1 & a & b \\ a^{-1} & 1 & c \\ b^{-1} & c^{-1} & 1 \end{bmatrix}$$

be the positive, reciprocal, matrix, and

$w^T = (w_1, w_2, w_3)$

be the unique, positive, normalized, eigenvector corresponding to λ_{max}.

Then, the development of w^T can be depicted as: [Saaty (1980)]

$$\begin{bmatrix} 1 & a & b \\ a^{-1} & 1 & c \\ b^{-1} & c^{-1} & 1 \end{bmatrix} \Rightarrow \frac{\begin{matrix} a^{1/3} b^{1/3} \\ a^{-1/3} c^{1/3} \\ b^{-1/3} c^{-1/3} \end{matrix}}{T} \Rightarrow \begin{matrix} a^{1/3} b^{1/3} T^{-1} \\ a^{-1/3} c^{1/3} T^{-1} \\ b^{-1/3} c^{-1/3} T^{-1} \end{matrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

where $T = a^{1/3} b^{1/3} + a^{-1/3} c^{1/3} + b^{-1/3} c^{-1/3}$.

5.6.2 Determining and validating λ_{max}

Continuing with the likelihood score example, suppose the pairwise comparisons of the criteria, with respect to the goal, are as given by Table 5-2, cells B2 to E5. Thus, for example, cell D2, which is equal to 1, indicates that C_1 is equally important as C_3, while cell D3, which is equal to 3, indicates that C_2 is weakly preferred to C_3. We want to compute the vector of priorities of the criteria, with respect to the goal, and the principal eigenvalue, λ_{max}.

Table 5-2: Pairwise comparison of the criteria and development of the eigenvalue

	A	B	C	D	E	F	G	H	I
1	G	C_1	C_2	C_3	C_4	Total	k	$w_k^{C\|G}$	
2	C_1	1.00	1.17	1.00	0.39	3.56	1	0.19	4.52
3	C_2	0.85	1.00	3.00	1.33	6.19	2	0.33	4.23
4	C_3	1.00	0.33	1.00	0.50	2.83	3	0.15	4.10
5	C_4	2.56	0.75	2.00	1.00	6.31	4	0.33	4.08
6									
7	Total					18.90		1.00	
8	λ_{max}								4.23

Adapted from Lootsma (1980) Figure 3.

Following Lootsma (1980), we use the normalized arithmetic mean of (5.13) to develop the vector of priorities of the criteria, with respect to the goal. We begin by summing each of the columns, B to E. Thus, for example, the sum of B2 to E2 is F2. Then, the vector of priorities, H2 to H5, is obtained as the weighted average of the totals in column F.

Finally, the eigenvector, I2 to I5, and their average, the principal eigenvalue, $\lambda_{max} = 4.23$, shown in cell I8, are obtained by applying (5.6).

From (5.7), the consistency index is

$$CI = \frac{\lambda_{max} - K}{K - 1} = \frac{4.23 - 4}{3} = 0.02875$$

and from (5.8), the consistency ratio is

$$CR = \frac{CI}{RI} = \frac{0.02875}{.58} = 0.05 < 0.10,$$

which validates that the pairwise comparisons of the criteria are consistent.

5.6.3 The weights associated with the likelihood factors

The overall approach for determining the weights associated with the likelihood factors is summarized in (5.9) and implemented in Table 5-3. The cell values for B2 to E4, which represent $w_k^{F|C_j}$, j=1, ..., 4, and k=1, 2, 3, are taken from cells F2 to F4 in Table 5-5 to Table 5-8 in Appendix 5-A. F2 to F5 are from Table 5-2, column H. The scores (relative weights) associated with each of the likelihood factors are given in column G, and are 0.41, 0.26, 0.33.

Table 5-3: Final score (weight) associated with each of the likelihood factors

	A	B	C	D	E	F	G
1	k	$w_k^{F\|C_1}$	$w_k^{F\|C_2}$	$w_k^{F\|C_3}$	$w_k^{F\|C_4}$	$w_k^{C\|G}$	$w_k^{F\|G}$
2	1	0.25	0.43	0.60	0.40	0.19	0.41
3	2	0.25	0.14	0.20	0.40	0.33	0.26
4	3	0.50	0.43	0.20	0.20	0.15	0.33
5	4					0.33	
6							
7	Total	1.00	1.00	1.00	1.00	1.00	1.00

5.7 Perceived Limitations of the AHP

In spite of the popularity of the AHP, there have been criticisms of the process. The perceived limitations are of two types: limitations associated with the AHP as a methodology, and limitations due to the uncertainty associated with the parameters.

Limitation regarding the validity of the AHP as a methodology include the following:

The aggregation method of Saaty's AHP suffers rank reversal (an alternative chosen as the best out of a set, fails to be chosen when another, perhaps unimportant, alternative is excluded from the set).[36] [Watson and Freeling (1982 and 1983), Dyer (1990a and 1990b)]

Similarly, the addition of indifferent criteria (for which all alternatives perform equally) causes a significant alteration of the aggregated priorities of alternatives. [Pérez et al (2006)]

In practice, pairwise comparison data do not provide consistent matrices. [Dubois (2011, p. 19)]

There are concerns about such things as measurement issues and an absolute scale with no degrees of freedom. [Bouyssou et al (2001, pp. 103 and 115)]

Concerns of the type expressed by Bouyssou et al (2001) provide a natural segue into the use of a fuzzy modification of the AHP. Advocates of this regard the crisp inputs and crisp relative intensities as limitations of the AHP process. Their comments regarding the appropriateness of fuzzy versions of the foregoing include the following:

It gives decision makers the opportunity to express their - essentially fuzzy - opinions in fuzzy numbers. [van Laarhoven and Pedrycz (1983)]

Decision makers prefer natural language expression [Lee et al. (2013, p. 349)]

It is more reliable to consider interval judgments than fixed-value judgments [Jia et al. (2013)]

Crisp values are not capable of reflecting a person's vague thoughts [Kutlu and Ekmekçioğlu (2012, p. 62)]

Asking for precise pairwise comparison is debatable, because these are arguably imprecisely known. [Dubois (2011, p. 19)]

The next section discusses fuzzy AHP models.

5.8 Fuzzy AHP (FAHP) Models

Given the foregoing as background, we turn now to a description of the models underlying three of the most influential fuzzy AHP (FAHP) articles, based on Google Scholar citations, van

[36] Saaty (1991, p. 918) argues that this is not a deficiency of the process. He observes that, given relative measurements, adding alternatives can logically create a new ranking which has no relationship to the old ranking.

Laarhoven and Pedrycz (1983), Buckley (1985) and Chang (1996).[37] The articles are discussed in chronological order.

The FL methodology underlying the FAHP is discussed in Chapter 3.

5.8.1 A fuzzy dataset

To put the fuzziness issue in context, consider the nature of the dataset. Continuing the example of §5.6.3, Table 4 shows fuzzy pairwise criteria comparisons with respect to the goal.

Table 5-4: Pairwise comparisons of the fuzzy criteria wrt the goal

	A	B	C	D	E
1	G	C_1	C_2	C_3	C_4
2	C_1	(1,1,1)	(0.86, 1.17, 1.56)	(0.67, 1, 1.5)	(0.33, 0.39, 0.49)
3	C_2	(0.64, 0.85, 1.16)	(1,1,1)	(2.5, 3, 3.5)	(0.95, 1.33, 1.83)
4	C_3	(0.67, 1, 1.49)	(0.29, 0.33, 0.40)	(1,1,1)	(0.4, 0.5, 0.67)
5	C_4	(2.04, 2.56, 3.03)	(0.55, 0.75, 1.05)	(1.49, 2, 2.5)	(1,1,1)

Chang (1996) Table 2, after averaging the cell values in van Laarhoven and Pedrycz (1983) Table 1

For examples, cell B2 denotes the crisp number 1, while cell C2 denotes the fuzzy triangular number (0.86, 1.17, 1.56).[38] The modes of this table reflect the comparable values in Table 5-2.

Similarly, there are fuzzy versions of the likelihood factors with respect to the criteria.

5.8.2 The van Laarhoven and Pedrycz (1983) FAHP model

van Laarhoven and Pedrycz (1983) were the first to develop a FAHP. The main features of their approach were the following:

- Triangular fuzzy numbers (TFNs) were used to extend the AHP to FAHP
- Multiple decision-makers were accommodated
- Logarithmic least squares were used to derive
  - Fuzzy weights and
  - Fuzzy performance scores
- Approximate fuzzy multiplication was used

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[37] Although, as mentioned earlier, there have been surveys of the AHP articles, only a small portion of those studies were devoted to FAHP. While not a survey, per se, the bibliography of Dubois et al (2000) cites a number of FAHP articles.

[38] As originally proposed by van Laarhoven and Pedrycz (1983, p. 237), there were multiple pairwise comparisons in some of the cells, reflecting the opinions of more than one expert. Cell C2, for example, contained the values $(\frac{2}{3}, 1, \frac{3}{2}), (\frac{2}{5}, \frac{1}{2}, \frac{2}{3}), (\frac{3}{2}, 2, \frac{5}{2})$. In practice, such cells are averaged before implementing, using either the arithmetic mean, as is done here, or the geometric mean.

Following Saaty (1980, p. 231), as extended by Lootsma (1981), van Laarhoven and Pedrycz (1983) modeled their fuzzy version of the AHP using triangular membership functions and logarithmic regression.[39] The general structure of their comparison matrix for the criteria took the following form[40]:

$$\tilde{C} = \left(\underline{\tilde{c}}_{ij}\right)_{K\times K} = \begin{pmatrix} (1,1,1) & \underline{\tilde{c}}_{12} & \cdots & \underline{\tilde{c}}_{1K} \\ \underline{\tilde{c}}_{21} & (1,1,1) & \cdots & \underline{\tilde{c}}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\tilde{c}}_{K1} & \underline{\tilde{c}}_{K2} & \cdots & (1,1,1) \end{pmatrix} \tag{5.16}$$

where $\underline{\tilde{c}}_{ij} = \left(\tilde{c}_{ij1}, \cdots, \tilde{c}_{ijn_{ij}}\right)^T$, and $\tilde{c}_{ijt} = (l_{ijt}, m_{ijt}, u_{ijt}) = \tilde{c}_{jit}^{-1} = \left(\frac{1}{u_{jit}}, \frac{1}{m_{jit}}, \frac{1}{l_{jit}}\right)$ i, j = 1, 2, ..., K, $i \neq j$, t = 0, ..., n_{ij}. $n_{ij} = 0$ is associated with an empty cell, and $n_{ij} > 1$ indicates a cell where there are multiple comparisons, which occurs when several decision-makers express their opinion on the relative significance of a pair of factors.

It follows that for the fuzzy weight vector, $\underline{w}$, the fuzzy logarithmic least squares model to be minimized is [Boender et al (1989, p. 135), Wang et al (2006, p. 3057)]

$$\begin{aligned} J &= \sum_{i=1}^{K} \sum_{\substack{j=1 \\ j\neq i}}^{K} \sum_{t=1}^{n_{ij}} \left(\ln \tilde{c}_{ijt} - \ln\left(\frac{\tilde{w}_i^{C|G}}{\tilde{w}_j^{C|G}} \right) \right)^2 \\ &= \sum_{i=1}^{K} \sum_{\substack{j=1 \\ j\neq i}}^{K} \sum_{t=1}^{n_{ij}} \left((\ln c_{ijt}^L - \ln w_i^L + \ln w_j^U)^2 + (\ln c_{ijt}^M - \ln w_i^M + \ln w_j^M)^2 + (\ln c_{ijt}^U - \ln w_i^U + \ln w_j^L)^2 \right) \end{aligned} \tag{5.17}$$

where L and U denote the lower and upper extremes of a TFN and M denotes the mode.

Setting $l_i = \ln w_i^L$, $m_i = \ln w_i^M, u_i = \ln w_i^U$, van Laarhoven and Pedrycz gave the normalized result as:

$$\left(\frac{\exp(l_i)}{\sum_{i=1}^{K} \exp(u_i)}, \frac{\exp(m_i)}{\sum_{i=1}^{K} \exp(m_i)}, \frac{\exp(u_i)}{\sum_{i=1}^{K} \exp(l_i)} \right), \quad i = 1, \ldots, K, \tag{5.18}$$

which they used as an estimate for w_i.

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[39] Lootsma (1981) had shown that logarithmic regression can accommodate the case of multiple estimates for the comparison ratios and situations where there were no estimates for certain ratios (missing data). [Buckley (1985, p. 242)]

[40] Adapted from Wang et al (2006, p. 3056)

They then use their modified TFN multiplication (5.19) to aggregate the local weights in order to approximate the global TFN weights for the alternatives.

5.8.2.1 The output, given the Van Laarhoven and Pedrycz model

Figure 5-4, which is based on the van Laarhoven and Pedrycz (1983) model, shows a simple representation of the essence of the consequence of using a fuzzy version of the AHP[41] to compute the weights associated with the likelihood factors, $w_i^{F|G}$.

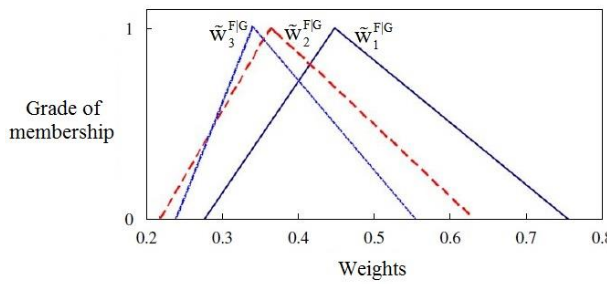

Figure 5-4: Global fuzzy weights

The fuzzy values are not exactly comparable with the previous crisp case because van Laarhoven and Pedrycz assumed that some of the cells were empty. Nonetheless, the consequence of incorporating the fuzziness inherent in the problem is clearly evident.

5.8.2.2 Limitations of the Van Laarhoven and Pedrycz study

The limitations of the Van Laarhoven and Pedrycz study include:

- The formula they used for multiplication

$$(l_1,\ m_1,\ u_1) \otimes (l_2,\ m_2,\ u_2) \sim (l_1 l_2,\ m_1 m_2,\ u_1 u_2), \qquad (5.19)$$

  which results in a triangular fuzzy number, is only an approximation.[42]

- A change in the priorities may cause rank reversal when replicating existing judgments on a single comparison. [Zhu (2012)]

- The methodology used to normalize the local fuzzy weights was problematic [Wang et al (2006)]

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[41] This figure does not reflect the Boender et al (1989) or Wang et al (2006) modifications of the van Laarhoven and Pedrycz (1983) FAHP.
[42] See Figure 7-2 and related discussion.

- Uncertainty of local fuzzy weights for incomplete fuzzy comparison matrices [Wang et al (2006)]

5.8.3 **The Buckley (1985) FAHP model**

The main features of Buckley (1985b) were the following:

Trapezoidal FNs were used to extend the AHP to FAHP
The geometric mean method was used to derive
Fuzzy weights and
Performance scores
Fuzzy multiplication and the fuzzy K-th root was used, based on
α-cuts and interval arithmetic

Buckley's method was to substitute the fuzzy ratios $\tilde{a}_{ij}$ and $\tilde{c}_{ij}$ into the solution of the normal equations. He chose the geometric mean procedure because it resulted from the log least squares method, and he wanted a method that extends easily to fuzzy positive reciprocal matrices.

For $\tilde{C} = [\tilde{c}_{ij}^{C|G}]$, the geometric mean procedure takes the form: [Buckley (1985b, p. 237)]

$$\tilde{r}_i = (\tilde{c}_{i1} \otimes \cdots \otimes \tilde{c}_{iK})^{1/K},\ i = 1, ..., K \tag{5.20}$$

and

$$\tilde{w}_i = \tilde{r}_i \otimes (\tilde{r}_1 \oplus \cdots \oplus \tilde{r}_K)^{-1},\ i = 1, ..., K \tag{5.21}$$

where $\oplus$ and $\otimes$ represent fuzzy addition and multiplication, respectively.

Based on Buckley (1985a; 1985b, p. 237) and assuming the trapezoidal fuzzy number, $\tilde{c}_{ij} = (\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij})$, the increasing and decreasing portion of the MF for $\tilde{w}_i^{C|G}$ was developed as:

$$f_i(y) = \left[\prod_{j=1}^{K} ((\beta_{ij} - \alpha_{ij})y + \alpha_{ij}) \right]^{1/K},\ i = 1, ..., K \tag{5.22}$$

$$g_i(y) = \left[\prod_{j=1}^{K} ((\gamma_{ij} - \delta_{ij})y + \delta_{ij}) \right]^{1/K},\ i = 1, ..., K \tag{5.23}$$

for $0 \le y \le 1$, respectively.

Then, he defined

$$\alpha_i = \left[\prod_{j=1}^{K} \alpha_{ij}\right]^{1/K} \text{ and } \alpha = \sum_{i=1}^{K} \alpha_i . \tag{5.24}$$

Similarly, he defined β_i and β, γ_i and γ, δ_i and δ .

Finally, let

$$f(y) = \sum_{i=1}^{K} f_i(y) \text{ and } g(y) = \sum_{i=1}^{K} g_i(y). \tag{5.25}$$

Then, the fuzzy weights $\tilde{w}_i^{C|G}$ are determined by

$$\left(\frac{\alpha_i}{\delta}, \frac{\beta_i}{\gamma}, \frac{\gamma_i}{\beta}, \frac{\delta_i}{\alpha}\right), \tag{5.26}$$

where the graph of the MF for $\tilde{w}_i^{C|G}$ is

zero to the left of $\alpha_i \delta^{-1}$,

$x = f_i(y)/g(y)$ on the interval $[\alpha_i \delta^{-1}, \beta_i \gamma^{-1}]$,

a horizontal line from $(\beta_i \gamma^{-1}, 1)$ to $(\gamma_i \beta^{-1}, 1)$,

$x = g_i(y)/f(y)$ on the interval $[\gamma_i \beta^{-1}, \delta_i \alpha^{-1}]$, and

zero to the right of $\delta_i \alpha^{-1}$.

If necessary, $\tilde{w}_i^{C|G}$ can then be multiplied by a normalizing constant so that its support lies in the interval [0, 1].

Similarly, the weights, $\tilde{w}_{ij}^{A|C_k}$ and $\tilde{w}_i^{A|C_k}$, k = 1, ..., K, can be developed for the alternatives.

Then, the final fuzzy weight for the alternatives, relative to the goal, is:

$$\tilde{w}_i^{A|G} = (\tilde{w}_i^{A|C_1} \otimes \tilde{w}_1^{C|G}) \oplus \cdots \oplus (\tilde{w}_i^{A|C_K} \otimes \tilde{w}_K^{C|G}). \tag{5.27}$$

These values can now be normalized.

Buckley (1985b, p. 240-1) extends this analysis to the case involving multiple experts.

5.8.3.1 Notable features of the Buckley model

When there is just one expert opinion per cell and the judgments are consistent, the log least squares estimate of $w_k^{F|G}$, k=1, 2, 3, is the same as that under the Buckley geometric row mean procedure.

However, when there is more than one expert opinion per cell for some cells, the geometric row mean procedure averages their values by using their geometric average, rather than the arithmetic average as in the log least squares approach.

Finally, if the positive reciprocal matrix is perfectly consistent, then the geometric row mean procedure gives the same weights as the eigenvector method.

5.8.3.2 Limitations of the Buckley study

If there is not perfect consistency, the geometric row procedure can give different weights compared to the eigenvector method. [Csutora and Buckley (2001)]

5.8.4 **The Chang (1996) FAHP model**

The main features of the Chang (1996) approach were the following:

- TFNs were used to extend the AHP to FAHP
- Arithmetic means were used to determine the priority vector
- The final ranking was done using crisp numbers

Chang (1996) used the arithmetic mean algorithm to find fuzzy priorities for the comparison matrices, whose elements were represented by triangular fuzzy numbers.
Given the criteria comparison matrix, whose elements were TFNs, $\tilde{c}_{ij}$, he applied the fuzzy counterpart of the arithmetic means, which he interpreted to be:

$$\tilde{S}_i = \sum_{j=1}^{K} \tilde{c}_{ij} \otimes \left[\sum_{k=1}^{K} \sum_{j=1}^{K} \tilde{c}_{kj} \right]^{-1} \tag{5.28}$$

and which he called the fuzzy synthetic extent with respect to the i-th object.[43]

He went on to interpret this value as

---

[43] The term "extent analysis" refers to an analysis of the extent to which an object satisfies a goal.

$$\tilde{S}_i = \left(\frac{\sum_{j=1}^{K} l_{ij}}{\sum_{k=1}^{K}\sum_{j=1}^{K} u_{kj}}, \frac{\sum_{j=1}^{K} m_{ij}}{\sum_{k=1}^{K}\sum_{j=1}^{K} m_{kj}}, \frac{\sum_{j=1}^{K} u_{ij}}{\sum_{k=1}^{K}\sum_{j=1}^{K} l_{kj}} \right)$$
$$= (l_i, m_i, u_i) \quad i = 1, \cdots, K. \tag{5.29}$$

The normalized row sums $\tilde{S}_i$ (i = 1, ..., K) are then compared using the degree of possibility values: [Calabrese et al (2013, p. 3749)]

$$V(\tilde{S}_i \geq \tilde{S}_j) = \begin{cases} 1 & \text{if } m_i \geq m_j \\ \frac{u_i - l_j}{(u_i - m_i) - (m_j - l_j)} & \text{if } l_j \leq u_i \;\; i, j = 1, \cdots, K, j \neq i \\ 0 & \text{otherwise} \end{cases} \tag{5.30}$$

and the relative crisp weight of each item i is calculated by normalizing the degree of possibility values:

$$w_i = \frac{V(\tilde{S}_i \geq \tilde{S}_j \mid j = 1, 2, ..., K, j \neq i)}{\sum_{k=1}^{K} V(\tilde{S}_k \geq \tilde{S}_j \mid j = 1, 2, ..., K, j \neq k)}, i = 1, 2, ..., K \tag{5.31}$$

where

$$V(\tilde{S}_i \geq \tilde{S}_j \mid j = 1, 2, ..., K, j \neq i) = \min_{j \in (1, \cdots, K)\, j \neq i} V(\tilde{S}_i \geq \tilde{S}_j) \;\; i = 1, 2, ..., K \tag{5.32}$$

The foregoing formula is used to compute the local crisp weights for the criteria and alternatives, and then the standard aggregation formula for the classical AHP is used to compute the global weights for the alternatives.

5.8.4.1 The output of the Chang model

The values for $w_k^{F|G}$, k=1, 2, 3, for the case at hand for the Chang model is 0.41, 0.28 and 0.25, respectively, which is comparable to the values in the crisp case.

5.8.4.2 Limitations of the Chang study

The limitations of the Chang (1996) study include:

- The normalization formula does not take into account constraints derived from the AHP method [Enea and Piazza (2004)]

- The method could lead to a wrong decision, because it may assign zero weights to some items (criteria, sub-criteria or alternatives), excluding them from the decision analysis. [Wang et al. (2008)]

5.9 Comment

The purpose of this chapter has been to discuss FL modifications of the AHP. To this end, we presented an overview of a hierarchical structure, the salient features of the AHP, the eigenvector method for determining the weights, alternate methods of computing vectors of priorities, the perceived limitations of the AHP, and the dominant features of the three most commonly used FAHP models, as well as some of their limitations. In addition, examples of FAHP applications in RA were mentioned.

5.10 Appendix 5A

The following tables and their values were adapted from Lootsma (1980), Figures 5, 6, 7 and 9.[44]

The pairwise comparison of the factors with respect to each of the criterion is shown in cells B2 to D4 of the following tables, and their vectors of priorities is shown in cells F2 to F4. Following Lootsma (1980), we use the normalized arithmetic mean of (5.13) to develop the latter. Thus, for example, the vectors of priorities of F_1, F_2 and F_3, with respect to C_1, are .25, .25 and .50, as shown in Table 5-5.

Table 5-5: Pairwise comparison of the factors with respect to C_1

	A	B	C	D	E	F
1	C_1	F_1	F_2	F_3	Total	Normalized
2	F_1	1	1	1/2	2.5	0.25
3	F_2	1	1	1/2	2.5	0.25
4	F_3	2	2	1	5	0.50
5						
6	Total				10	1.00

Adapted from Lootsma (1980) Figure 5.

The vectors of priorities of F_1, F_2 and F_3, with respect to C_2, are .43, .14 and .43, as shown in Table 5-6.

Table 5-6: Pairwise comparison of the factors with respect to C_2

	A	B	C	D	E	F
1	C_2	F_1	F_2	F_3	Total	Normalized
2	F_1	1	3	1	5.00	0.43
3	F_2	1/3	1	1/3	1.67	0.14
4	F_3	1	3	1	5.00	0.43
5						
6	Total				11.67	1.00

Adapted from Lootsma (1980) Figure 6.

[44] Lootsma (1980) actually used 7 criteria in his example, but this number was reduced to 4 when van Laarhoven and Pedrycz (1983) first produced the fuzzy version of his example. Subsequent fuzzy versions of his example followed suit.

The vectors of priorities of F_1, F_2 and F_3, with respect to C_3, are .60, .20 and .20, as shown in Table 5-7.

Table 5-7: Pairwise comparison of the factors with respect to C_3

	A	B	C	D	E	F
1	C_3	F_1	F_2	F_3	Total	Normalized
2	F_1	1	3	3	7.00	0.60
3	F_2	1/3	1	1	2.33	0.20
4	F_3	1/3	1	1	2.33	0.20
5						
6	Total				11.67	1.00

Adapted from Lootsma (1980) Figure 7.

The vectors of priorities of F_1, F_2 and F_3, with respect to C_4, are .40, .40 and .20, as shown in Table 5-8.

Table 5-8: Pairwise comparison of the factors with respect to C_4

	A	B	C	D	E	F
1	C_4	F_1	F_2	F_3	Total	Normalized
2	F_1	1	1	2	4.00	0.40
3	F_2	1	1	2	4.00	0.40
4	F_3	1/2	1/2	1	2.00	0.20
5						
6	Total				10.00	1.00

Adapted from Lootsma (1980) Figure 9.

6 Applying Fuzzy Optimization to Risk Assessment

6.1 Introduction

The residential housing market catastrophe largely contributes, among other events, to the global financial crisis of 2008 [Segal (2011, p. 330)]. This crisis has prompted policy makers and practitioners to question the efficacy of the existing standard for risk assessment and policies[45].

Risk is commonly associated with unexpected event. The terrorist attacks of September 11, 2001, the August 2005 Hurricane Katrina in New Orleans[46], and the April 2010 massive BP oil spill in the Gulf of Mexico are examples of such unexpected events. Knight (1921) started the discussion about risk and uncertainty in his now classic book “Risk, uncertainty, and profit”. Generally, risk refers to the possibility (or the possible consequences) of things going wrong [Panjer (2006, p. 3)]. Better et al. (2008)[47] and Ostrom and Wilhelmsen (2012, p 6) described risk as the probability of an unexpected event that results in negative consequences. In many cases huge amounts of money are involved. Risk also involves exposure. Holton (1997; 2004) defines risk as exposure to uncertainty, and hence views uncertainty and exposure to that uncertainty as the risk’s components. Often, a focus for policy makers and reinsurance companies is on low probability events with high consequences that lead to considerable damage, loss, death, and environmental impairment, for example (Brillinger, 2002). From a risk assessment perspective, risk may also involve unexpected events with positive consequences for the firm. Ignoring this side of risk leads to incomplete assessment, because downside and upside events may offset each other. Therefore, a complete definition is of the type given by Segal (2011, p. 19): risk is uncertainty, deviation from expected, and includes both positive and negative deviation.

Generally, risk events are grouped within the following risk categories: hazard risk, financial risk, strategic risk, and operational risk. A detailed definition and the composition of these groups are provided by Segal (2011, p. 116). Independent of the risk categories, a risk assessment (RA) is a systematic process for identifying and evaluating potential risks and opportunities that could positively or negatively affect the achievement of an enterprise's objectives [PricewaterhouseCoopers (2008)]. Generally, the RA process follows three steps: identification, analysis, and evaluation of risk. Detailed descriptions of these steps are provided in the literature [see for example Segal (2011, p. 113) and PricewaterhouseCoopers (2008)].

The flowchart in Figure 6-1 gives the components of the RA process.

[45] Francis (2011) challenged the regulators’ position towards a “fraud friendly environment”.
[46] US Army Engineers estimated that Hurricane Katrina was a 1-in-396-year event (Segal, 2011, p. 9)
[47] The term opportunity is used for an unexpected event that would have a positive impact (Better et al., 2008)

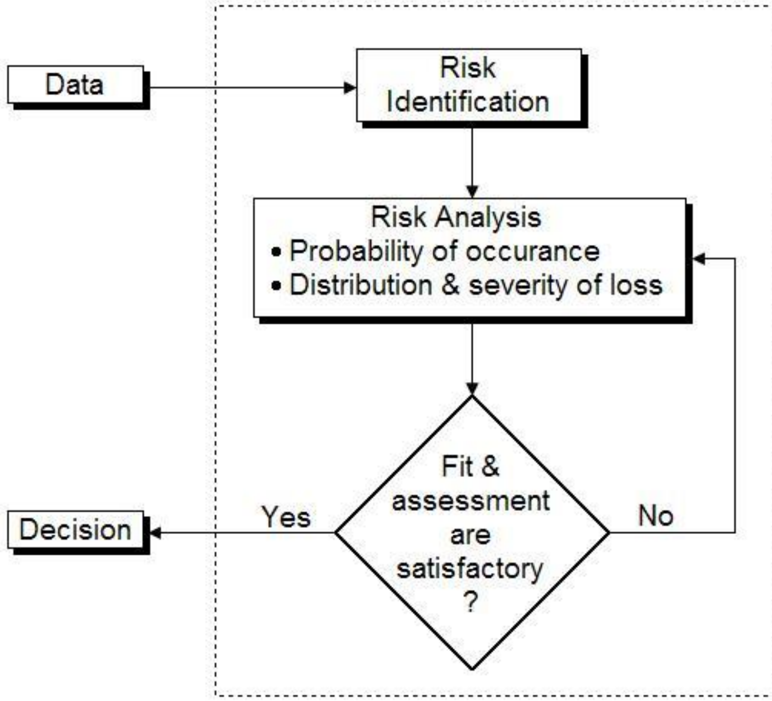

Figure 6-1: Risk Assessment Flowchart

A risk assessment process aims at providing a clear picture of what is known regarding the nature of a particular risk and the degree of uncertainty surrounding any estimates. Kunreuther (2002) suggested using a curve obtained by combining the set of events that could produce a given dollar loss. This is an exceedance probability (EP) curve to measure experts' knowledge (or lack of knowledge) and projection about a risk event. An EP curve shows the probabilities that certain level of losses will be exceeded. Figure 6-2 is an illustration of the Kunreuther EP curve for dollar losses to homes in Los Angeles from an earthquake.

The EP curve is obtained by first combining the set of events that could produce a given dollar loss. Then, the resulting probabilities of exceeding losses of different magnitudes are found. Based on these estimates, the mean EP is computed. The uncertainty associated with the probability of an event occurring and the magnitude of dollar losses is reflected in the 5% and 95% confidence interval curves.

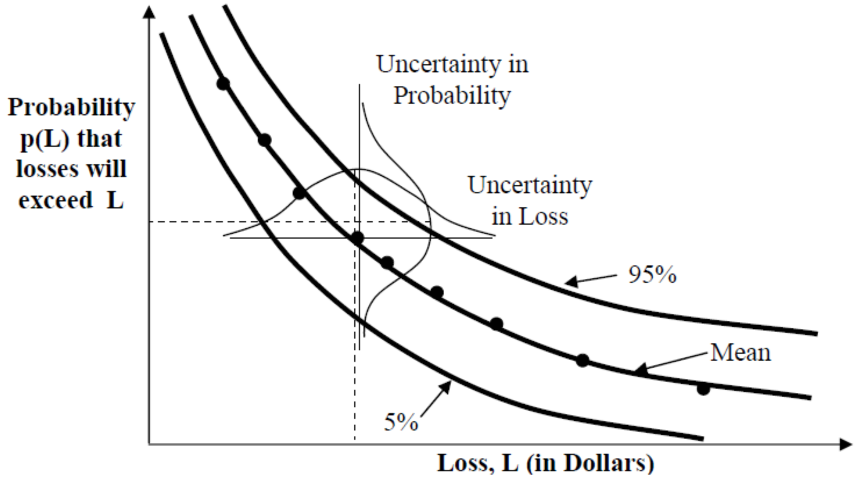

Figure 6-2: Exceedance probability (EP) curve[48]

6.2 Standard Optimization as a Risk Assessment Tool

Various qualitative and quantitative risk assessment techniques have been developed to help risk managers achieve their objective. Optimization is an example of such a quantitative technique, and includes many branches, such as linear / non-linear optimization, convex optimization, conic optimization, global optimization, discrete optimization, etc[49]. Optimization consists in finding an optimal solution to a given problem, under given circumstances. Progress in computational technique has made optimization more accessible and easier to use, hence more attractive to a larger number of practitioners. Optimization, as a powerful modeling and problem solving methodology, has a broad range of applications: in nature, for example, physical systems tend to a state of minimum energy, while molecules in an isolated chemical system react with each other until the total potential energy of their electrons is minimized, and rays of light follow paths that minimize their travel time [Nocedal and Wright (2000)]. Areas of application of optimization also include engineering[50] (transportation, production planning, design and data fitting), industry [51] (where, for example, airline companies schedule crews and aircraft to minimize cost aerospace), and management science. Optimization is also widely used in insurance and actuarial science where, for example, investors will seek to build up portfolios that minimize risks while achieving a high rate of return. Brockett and Xia (1995) and Shapiro (1986) give reviews of operational research in insurance.

[48] Kunreuther (2002, Figure 1). Used with the permission of the author.
[49] Nocedal and Wright (2000) provide a survey of the various branches in optimization.
[50] Rao (2009) present engineering applications of optimization.
[51] Ciriany and Leachman (1993) provide a survey of applications of optimization in industry.

In the next section we the review the mathematical formulation of an optimization problem, provide an example of optimization in insurance and actuarial science, and then present an application of optimization to risk assessment.

6.2.1 An Overview of Standard (Crisp) Optimization

The process of optimization aims at finding the optimal solution to a given problem. Generally, an optimization problem is of the form [Boyd and Vandenberghe (2009, p. 7)]:

$$\begin{aligned} &\text{Minimize } f_0(x) \\ &\text{Subject to } f_i(x) \le b_i \quad \text{for } i = 1, \dots, m \end{aligned} \tag{6.1}$$

where $x \equiv (x_1, \dots, x_n)$ is called the optimization variable, and is unknown. The functions $f_0 : R^n \to R$ and $f_i : R^n \to R$, $i = 1, \dots, m$ are, respectively, the objective function, and the set of inequality constraints. An optimal solution x* of the optimization (6.1) satisfies the constraints that, for any variable z with $f_1(z) \le b_1, \dots, f_m(z) \le b_m$, $f_0(z) \ge f_0(x^\star)$. Depending on the properties of the objective function and the constraints, optimization (6.1) can be a linear, quadratic, discrete, or a convex optimization problem.

(6.1) is a linear optimization if the objective and the constraint functions are linear:

$$f_i(\alpha x + \beta y) = \alpha f_i(x) + \beta f_i(y) \qquad \text{for } i = 0, 1, \dots, m \tag{6.2}$$

for all $x, y \in R^n$ and all $\alpha, \beta \in R$.

We obtain a convex optimization when the objective and the constraint functions are convex:

$$f_i(\alpha x + \beta y) \le \alpha f_i(x) + \beta f_i(y) \quad \text{for } i = 0, 1, \dots, m \tag{6.3}$$

for all $x, y \in R^n$ and all $\alpha, \beta \in R$. The parameters satisfy: $\alpha + \beta = 1$, $\alpha \ge 0$, $\beta \ge 0$

A quadratic problem is obtained when the objective function is quadratic. The constraints are usually linear functions (Rachev et al., 2008).

$$f_0(x) = c^T x + \tfrac{1}{2} x^T H x, \quad \text{for all } x \in R^n \tag{6.4}$$

where $c = (c_1, \dots, c_n)$ is a vector of coefficients for the linear part of the objective function and $H = \{c_{ij},\ i, j = 1, \dots, n\}$ is an n x n matrix defining the quadratic part of the objective. Examples of areas of application of quadratic optimization include portfolio analysis and international re-insurance.

When there is no constraint on the set of feasible solution, (6.1) becomes an unconstrained optimization. An unconstrained problem can be solved using the first and second-order conditions[52] on the function gradient ($\nabla f(x)$) and the Hessian matrix (H) of second order derivatives. For constrained optimization, the method of Lagrange multipliers is often applied [Details of which can be found in Boyd and Vandenberghe (2009) and Rachev (2008)].

6.2.2 Applications of Optimization in Insurance

Operations research in general and optimization in particular have a long history in insurance and actuarial sciences. In the applications to insurance, the objective function $f_0(x)$ in (6.1) can be such things as the expected value of the probability distribution of the throughput (the average quantity of non-defective parts produced per unit time) at a factory, or the fifth percentile of the distribution of the net present value (NPV) of a portfolio of investments [Better et al. (2008)].

Brockett and Xia (1995) give a detailed and technical review of applications of optimization in insurance. For instance, the authors present an example [adapted from Schleef (1989)] where linear optimization is used to set the cost of whole life insurance. In it, the main components of the whole life policy are the future premium from the insured, the amount of insurance protection expected, and the customer's budget for insurance. These components are optimized in order to find the product that best fits the customer needs.

The LP corresponding to this problem is of the form

Maximize [Discounted cash flow for policy]

Subject to [Customer budget at time t $\leq$ an amount B_t] and
[Level of insurance protection $\geq$ an amount I_t]

As another example, Brockett and Xia (1995) use optimization in a facility location problem, where an insurance company plans to engage in reorganization and geographical restructuration or expansion.

In this case the problem is of the form

Minimize [sum of variable and fixed costs]

Subject to [Market potential increases]

Recently, Dhaene et al. (2012) use optimization to solve a capital allocation problem. The problem was formulated in the form of minimum distance, and a solution was obtained by minimizing the weighted sum for the deviations of the business unit’s losses from their respective capitals.

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[52] The first-order condition $\nabla f(x)=0$ is a necessary condition for finding a function extrema (minimum/maximum), and the second-order condition (sign of the Hessian) provides a sufficient additional condition for the extremum. [Boyd and Vandenberghe (2009); Rachev (2008)]

6.2.3 Optimization as a Risk Assessment Tool

In practice risk analysis often consists of (i) finding the probability of occurrence of an event, and (ii) computing the statistical distribution of the (potential) damages. As a result of this process, graphics or hazard risk maps can be obtained and used for prediction [Brillinger (2002)].

We present an example based on Better et al. (2008). An insurer has a number of potential projects for which the revenues for a horizon of approximately *n* periods (depending on the project) are given as probability distributions. For each project, there is an initial investment and a number of business development, engineering and earth sciences personnel. As a constraint, there is a budget limit for the investments, and a limited number of personnel of each skill category. A probability of success by project is also assigned. Following the authors, we assume without loss of generality that Project A has a probability of success 0.6. There is a window of opportunity for each project, which may start in different time periods. The insurer aims at selecting a set of optimal projects to invest in that will best further its corporate goals.

The authors compare three risk assessment methodologies, namely the mean-variance approach [Markowitz (1952)], the 5th percentile, and the value-at-risk (VaR), that were all implemented through optimization. The main optimization problem is of quadratic form:

$$\begin{aligned} &\text{Maximize} \quad r^T w - k w^T Q w, \\ &\text{subject to} \quad \sum_{i=1}^{n} c_i w_i = b \quad \text{and} \\ &\qquad\qquad\quad w_i \in \{0,1\} \end{aligned} \tag{6.5}$$

where r is a vector of portfolio returns, Q is a covariance matrix of returns, the coefficient k describes the insurer's risk aversion, the constant c_i represents the initial investment in project i, the term w_i is a binary variable representing the decision whether to invest in project i, and the constant b is the available budget.

With the mean-variance approach, the optimization problem is as follows (assuming that the objective is to maximize the expected NPV of the portfolio, while keeping the standard deviation of the NPV below a specified threshold of L1):

$$\begin{aligned} &\text{Maximize } \mu_{NPV} \quad (\text{objective function}) \\ &\text{Subject to } \sigma_{NPV} < L1 \ (\text{requirement}), \\ &\sum_i c_i x_i \le b \ (\text{budget constraint}), \\ &\sum_i p_{ij}\, x_i \le P_j \ \ \forall j \ \ (\text{personal constraints}), \\ &\text{All projects must start in year 1}, \\ &x_i \in \{0,1\}\ \forall i \quad (\text{binary decisions}). \end{aligned} \tag{6.6}$$

When the risk is controlled by the 5th percentile (assuming that the objective is to maximize the average return, as long as no more than 5% of the trial observations fall below a level of L2), the optimization problem is as follows:

$$
\begin{aligned}
&\text{Maximize } \mu_{NPV} \text{ (objective function)} \\
&\text{Subject to } P(5)_{NPV} < L2 \text{ (requirement)}, \\
&\sum_i c_i x_i \le b \text{ (budget constraint)}, \\
&\sum_i p_{ij}\, x_i \le P_j \ \forall j \ (\text{personal constraints}), \\
&\sum_{m_i \in M} x_i \le 1 \ \forall i \ (\text{mutually exclusivity}), \\
&x_i \in \{0,1\} \ \forall i \quad \text{(binary decisions)}.
\end{aligned} \tag{6.7}
$$

We often need to make decisions under uncertainty. In this case, we often cannot predict the exact outcome of a decision because the outcome depends on unknown factors. This leads to the concept of fuzzy optimization, which describes mathematical programming problems in which the functional relationship between the decision variables and the objective function is fuzzy (or known linguistically) [Carlsson et al. (1998)]

6.3 Fuzzy Optimization as a Risk Assessment Tool

Zadeh (1965) introduced the notion of fuzzy logic, which has since then gained recognition and was heavily applied in mathematics and computer sciences [Dubois and Prade (1980), Kandel (1986) and Zimmerman (1996)]. Various applications of fuzzy logic exist in the insurance literature, especially in insurance underwriting [DeWit (1982)], classification of insurance risk [Ebanks et al. (1992), Derrig and Ostaszewski (1995)], projected liabilities [Cummins and Derrig (1993); Sanchez and Gomez (2003)], future and present values [Buckley (1987)] and finance [Lemaire (1990)]. Ostaszewski (1993) and Shapiro (2004) provided extensive overviews of the possible applications of fuzzy logic in insurance.

6.3.1 Fuzzy Optimization

Bellman and Zadeh (1970) introduce the concept of fuzzy optimization in their seminal paper “Decision Making in a Fuzzy Environment”. Fuzzy optimization differs from classical optimization in the sense that the objective function and constraints are given the same importance: objective function and constraints are both (written in term of membership functions if appropriate and) optimized simultaneously. The membership functions are linked by a linguistic conjunction: “and” (for maximization) and “or” (for minimization).

For a maximization problem, where the optimal decision is the option with the highest degree of membership in the decision set, the optimal solution is by

$$x^* \equiv \arg[\max \min\{\mu_G(x), \mu_C(x)\}] \tag{6.8}$$

where G is the goal and C represent the constraints.
If the optimal decision is the option with the lowest degree of membership in the decision set, then the optimal solution is as follows

$$x^* \equiv \arg[\min \max\{\mu_G(x), \mu_C(x)\}] \tag{6.9}$$

A representation of the relationship between the MFs for the goal G, the constraint C and the decision D is given in Figure 3-6.

6.3.2 Basic steps in a fuzzy optimization problem

Generally, a fuzzy optimization problem can be solved through the following steps [Bellman and Zadeh (1970); DaSilva et al. (2002)]:

1. Fuzzification of the objective function: here, the membership associated with the objective function f(x) is computed through the following formula:

$$\mu_f(x) = \frac{f(x) - \min(f(x))}{\max(f(x)) - \min(f(x))} \tag{6.10}$$

   where max(f(x)) and min(f(x)) represent the maximum and minimum values in the feasible interval for the function f(x).

2. Fuzzification of the constraints is done the same way.

3. Membership of the optimal function: the membership functions for all the constraints and the objective function are combined, and the decision making formulas (6.8) and (6.9) are used depending on whether we have a maximization or minimization problem.

6.3.3 An example of a fuzzy optimization problem

The following example is adapted from DaSilva et al. (2002): Consider a (non-linear) quadratic function f(x) and assume that the aim is to maximize f under the given fuzzy constraints below.

$$\text{Max } f(x) = 10 - x - 25/x^2$$

Constraints:
C_1: x must be equal or greater than 2 and equal or less than 6.
C_2: a good value for x is equal or less than 3 and an acceptable value is not much greater.

Solution: we will follow the three steps described above.

1. Fuzzification of the objective function

   (2) =1.75 (min in [2,6]); $f(3.68) = 4.47$ (max of f in [2,6]);

   F(x) = $\mu_f(x) = \dfrac{f(x) - \min(f(x))}{\max(f(x)) - \min(f(x))}$ = 3.03 – (x/2.72) – (9.19/x^2)

2. One interpretation of the constraints is as follows

$$C_1(x) = \begin{cases} 0, & x < 2 \\ 1, & 2 \le x \le 6 \\ 0, & x > 6 \end{cases} \qquad C_2(x) = \begin{cases} 0, & x \le 3 \\ \frac{1}{3} + \frac{2}{x}, & x > 3 \end{cases}$$

   By merging both constraints C_1 and C_2, we obtain the following function

$$C(x) = \begin{cases} 0, & x < 2 \\ 1, & 2 \le x \le 3 \\ \frac{1}{3} + \frac{2}{x}, & 3 < x < 6 \end{cases}$$

3. A representation of the graphs of the objective function and the combined constraints is shown in Figure 6-3, where x_0 is the lowest value of x for which F(x) > 0, G(x) represents the decision function, $\mu_G(x) = \min\{C(x), F(x)\}$, and $x^* = \max\{\mu_G(x)\}$.

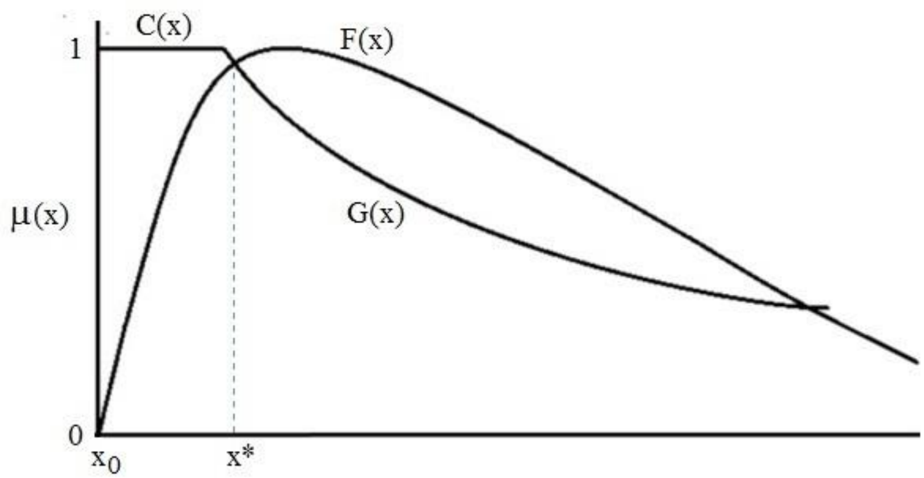

Figure 6-3 Optimal Solution[53]

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[53] Adapted from DaSilva et al. (2002) Figure 4.

6.3.4 Fuzzy Optimization as RA tool

This section is based on the paper by Liu and Zhang (2012) that deals with the risk assessment of natural disasters and sporadically occurring events in general, and on risk analysis and prediction for tropical cyclones in particular. The purpose of this risk analysis is to identify a functional relationship between the probability distributions of hazard causes (e.g., rainfall amounts or surface wind strengths) and hazard impacts (e.g., on human-beings, buildings, crops) using information matrices (i.e., inputs vs. outputs). The steps of their hazard risk assessment model can be summarized as follows:

1. Start with available data: the hazard-impact indicator matrix

   x_{ij} (i = 1, 2, . . . , n; j = 1, 2, . . . , k), from their k-year sample data with n indicators.

2. Standardize the hazard-impact indicator matrix by applying a scaling factor

$$r_{ij} = \frac{x_{ij}}{\max\limits_{1\le j\le k}\{x_{ij}\} + \min\limits_{1\le j\le k}\{x_{ij}\}}, i = 1, 2, \cdots, n$$

   This is done in order to scale down the raw data, for easier computation.

3. Obtain the fuzzy weighted hazard-impact indicator matrix.

   $\mathbf{R} = (r_{ij})_{nxk}$

   Note that the weights here do not add up to one, which is not an issue for the current case.

4. A second series of weights are applied to the hazard-impact indicators as follows:

   a. Linearization of r_{ij} using

$$z_j = \sum_{i=1}^{n} e_i r_{ij}, \; j = 1, 2, \cdots, k$$

   where e = (e_1, ..., e_n) is a unit vector.

   b. Compute S_z the standard deviation of z_j, which measures the spread of the data.

   c. Compute the local density, D_z, of the projected points,

$$D_z = \sum_i \sum_j (K-d_{ij})\, u(t)\, (K-d_{ij})$$

   where the window radius of the local density is defined as $K = 0.1S_z$, $d_{ij} = |z_i - z_j|$ denotes the distance, $t = K - d_{ij}$, and the unit step function u(t) is 0 for $t < 0$ and 1 for $t \geq 0$.

5. The associated objective function after projection is

$Q_e = S_z D_z$

6. The projection directions reflect the characteristics of hazard-impact and cause indicators. So the optimized projection direction of Q_e is obtained by maximizing the projected objective function

   $\max Q_e$

   s.t. $\sum_{i=1}^{n} e_i^2 = 1$ and $e_{i \geq 0}$

   and then computing the optimally projected direction of the index, e*, in order to capture important characteristics of the multi-dimensional information.

   Normalization of e* results in the categorical weights of the hazard-impact indicators:

   $$A_1 = (a_{i1};\ i = 1, 2, \ldots, n)$$

7. The ranking weights of the hazard-impact indicators, A2 = (a_{i2}; i = 1, 2, . . . , n), is obtained using a fuzzy variation of the AHP method.

8. Determine the combined weights between the categorical and ranking weights, A = (a_i; i = 1, 2, . . ., n).

9. Finally, solve the fuzzy optimization problem:

   $\min \sum \sum (c\ |a_{i1} - a_i|\ r_{ij} + (1\text{-}c)\ |a_{i2} - a_i|\ r_{ij}\ )$

   s.t. $\sum_{i=1}^{n} a_i^2 = 1$ and $a_i \geq 0$, $i = 1, 2, \ldots, n$

   to obtain the combined weights of the hazard-impact indicators. Here, c often is set to 0.5, which indicates that all categorical weights are assumed to make an equal contribution.

6.4 Conclusion

The aim of this chapter was to introduce the reader to fuzzy optimization and to show how it is related to risk assessment. To this end, we first discussed the use of standard optimization in risk assessment. Then we present an example dealing with fuzzy optimization in risk assessment. Fuzzy optimization differs from classical optimization in the sense that the objective function and constraints are given the same importance. In the example presented, fuzzy optimization was applied to hazard risk assessment in order to identify a functional relationship between the probability distributions of hazard causes and the hazard impacts.

7 Using Fuzzy Logic for Risk Assessment-- Cases Studies

7.1 Introduction

Risk is commonly associated with an unexpected event that results in large negative consequences.[54]

As mentioned in chapter four, the components of risk associated with an event generally consist of the likelihood of the occurrence (frequency) of the event and the severity of the consequence of this event.[55]

Also, as discussed in chapters one and two of this report, a realistic risk assessment should incorporate all probabilistic and possibilistic sources of uncertainty. Fuzzy logic appears to be the appropriate tool to help reach this goal. In this section, we present examples of RAA-FL in the areas of operational risks, hazard risks, financial risks, and strategic risks.

7.2 Using FL for Operational Risk Assessment: A Scenario Analysis Example

7.2.1 **Introduction to Operational risk**

Operational risk refers to unexpected changes associated with operations, and involves such things as human resources, technology, processes, and disasters [Segal (2011, p. 116)]. In particular, the operational risk is "the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events" [BCBS (2005, p. 140)]. For operational risk modeling, the Basel II accord [BSBC (2005, 2011)] recommends that loss data be organized according to seven proposed event types and eight suggested business lines. The Basel II accord also proposes three methods to measure the minimum capital charge required to cover the loss occurring from operational risk: the basic indicator approach, the standard approach, and the advanced measurement approach. These approaches are discussed in Bassell (2005, p.140-147).

Under the advanced measurement approach, the operational risk regulatory capital must combine the following four sources of data [Panjer (2006, p. 13); Gregoriou (2009, p. 4)]: (1) internal operational risk loss data, (2) external operational risk loss data, (3) scenario analysis of expert opinion, and (4) business environment and internal control factors. Scenario analysis finds its relevance in cases where the other sources of data do not provide robust estimates of the risk.

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[54] Examples include the hurricane Katrina in New Orleans of August 2005 [Segal, 2011, p.9] and the massive BP oil spill in the Gulf of Mexico of April 2010 [Upton (2011)]. Downside risk, notwithstanding, Segal (2011, p.19) counsels us that, from a risk management perspective, risk must also include unexpected events with positive consequences.

[55] Feng and Chung (2013) added the "detectability" as a third component of risk, while studying airport risks. Klugman, Panjer, and Wilmot (2004, p. 11) study the uncertainty in future payments with respect to any or all of occurrence (is there a payment), timing (when is the payment made?), and severity (how much is paid?).

One use of scenario analysis is to provide decision makers with experts' opinions on the company exposure in case of rare operational risk events that have high-severity losses. In particular, Basel II (2005, p. 150) states

> "A bank must use scenario analysis of expert opinion in conjunction with external data to evaluate its exposure to high-severity events. This approach draws on the knowledge of experienced business managers and risk management experts to derive reasoned assessments of plausible severe losses."

The common approach is to combine experts' opinions using bootstrapping or Bayesian inference, which require the empirical distribution of the severity and frequency, and the distribution for the priors [Shevchenko and Wuthrich (2006); Dutta and Babel (2009)]. However, several practical situations are characterized by insufficient or unreliable data, or data stated in linguistic form. Such instances provide a natural segue to the introduction of methodologies involving fuzzy logic.

7.2.2 **Overview of the Case Study**

The following is adapted from Durfee and Tselykh (2011). In this study, expert opinion is solicited to describe a potential large and plausible extreme operational risk loss event. Each team of experts is asked to produce optimistic, realistic, and pessimistic estimates for severity and frequency for each scenario storyline. This input from the experts will be referred to as scenario data. Following Durfee and Tselykh (2011), fuzzy logic is used to represent and model these scenarios. Figure 7-1 is a flowchart of the methodology used in this example, the details of which follow.

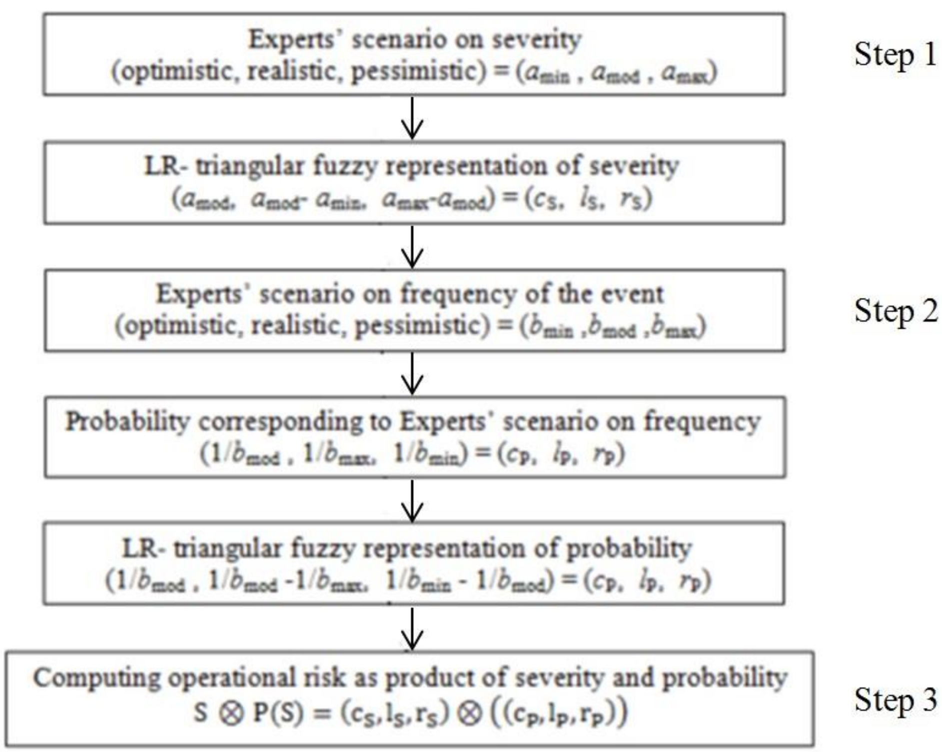

Figure 7-1 Flowchart of methodology

7.2.2.1 Step 1: Scenario data on financial impact of loss event (severity)

Pessimistic, realistic, and optimistic experts' predictions on the financial impact of the scenario are provided, and labeled $a_{\max}$, a_{mod}, $a_{\min}$ respectively. This can be represented by a LR-type triangular fuzzy number S as follows

$$\mathrm{S}=\left(\mathrm{a}_{\text{mod}},\ \mathrm{a}_{\text{mod}}-\ \mathrm{a}_{\min},\ \mathrm{a}_{\max}-\mathrm{a}_{\text{mod}}\right), \tag{7.1}$$

which can be rewritten as

$$\mathrm{S}=\left(\mathrm{c}_{\mathrm{S}},\ \mathrm{l}_{\mathrm{S}},\ \mathrm{r}_{\mathrm{S}}\right) \tag{7.2}$$

where $c_S= a_{mod}$, $l_S= a_{mod}- a_{min}$, $r_S= a_{max}-a_{mod}.$

The constants c_S, l_S, r_S are respectively the center value, the left spread, and the right spread of the triangular fuzzy number. Basel II recommends that the risk horizon for evaluating operational risk be set at **one year** [BSBC (2005)]. So **annual** frequencies are modeled.

7.2.2.2 Step 2: Scenario data on possible rate of occurrence of scenario event (frequency)

Similar to step 1, for each loss, experts' judgments about the frequency of the scenario will be given in the form [$b_{\min}$,b_{mod} ,$b_{\max}$]. The probability (annual frequency) of suffering a financial loss, S, will then be given by

$$\mathrm{P}\left(\mathrm{S}\right)=\left(1/\,\mathrm{b}_{\max},1/\,\mathrm{b}_{\text{mod}},1/\,\mathrm{b}_{\min}\right) \tag{7.3}$$

This can be written as a LR-type triangular fuzzy number

$$\mathrm{P}\left(\mathrm{S}\right)\ =\ \left(1/\,\mathrm{b}_{\text{mod}}\ ,\ 1/\,\mathrm{b}_{\text{mod}}-1/\,\mathrm{b}_{\max},1/\,\mathrm{b}_{\min}-\ 1/\,\mathrm{b}_{\text{mod}}\right)=\left(\mathrm{c}_{\mathrm{P}},\ \mathrm{l}_{\mathrm{P}},\ \mathrm{r}_{\mathrm{P}}\right) \tag{7.4}$$

where $c_{\mathrm{P}}= 1/b_{\text{mod}}$, $l_{\mathrm{P}}= 1/b_{\text{mod}} -1/b_{\max}$, $r_{\mathrm{P}}= 1/b_{\min} - 1/b_{\text{mod}}$

7.2.2.3 Step 3: Computing the operational risk

Fuzzy multiplication is used and the product of the fuzzy numbers in (7.2) and (7.4) gives the operational risk (OR) for each scenario

$$\begin{aligned} OR &= S\otimes P(S)=\left(c_{\mathrm{S}},\ l_{\mathrm{S}},\ r_{\mathrm{S}}\right)\otimes\left(c_{\mathrm{P}},\ l_{\mathrm{P}},\ r_{\mathrm{P}}\right)\\ &\approx (c_{\mathrm{S}}c_{\mathrm{P}}, c_{\mathrm{S}}r_{\mathrm{P}}+c_{\mathrm{P}}r_{\mathrm{S}}-r_{\mathrm{S}}r_{\mathrm{P}}, c_{\mathrm{S}}l_{\mathrm{P}}+c_{\mathrm{P}}l_{\mathrm{S}}+l_{\mathrm{S}}l_{\mathrm{P}})\\ &= (c_{OR}, l_{OR}, r_{OR}) \end{aligned} \tag{7.5}$$

where $c_{\mathrm{OR}} = c_{\mathrm{S}}\, c_{\mathrm{P}}$ $\quad l_{\mathrm{OR}} = c_{\mathrm{S}}\, r_{\mathrm{P}} + c_{\mathrm{P}}\, r_{\mathrm{S}} - r_{\mathrm{S}}\, r_{\mathrm{P}}$ $\quad r_{\mathrm{OR}} = c_{\mathrm{S}}\, l_{\mathrm{P}} + c_{\mathrm{P}}\, l_{\mathrm{S}} + l_{\mathrm{S}}\, l_{\mathrm{P}}$

Note that line 2 of (7.5) is an approximation. Figure 7-2[56] is a representation of the relationship between exact fuzzy multiplication and the approximation used.

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[56] Adapted from Hanss (2005, p. 59), Figure 2.7.

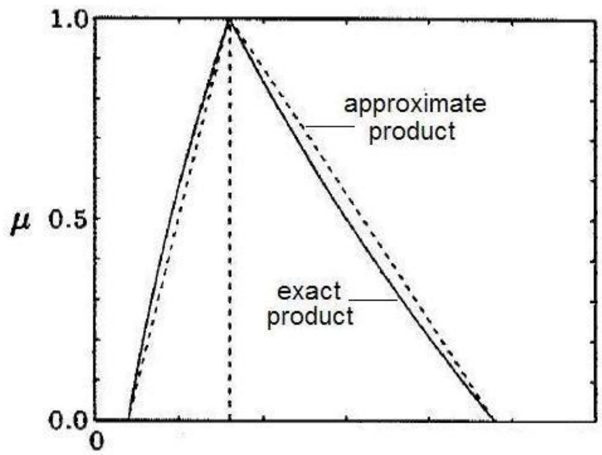

Figure 7-2 Exact vs. fuzzy multiplication

7.2.2.4 Step 4: The grade of membership

The grade of membership for a scenario is given by the α-cut[57] of the operational risk.

$$OR_{\alpha} = \left(c_{OR}, l_{OR}, r_{OR}\right)_{\alpha} = \left(c_{OR} - (1-\alpha)l_{OR},\ c_{OR} + (1-\alpha)\ r_{OR}\right) \quad (7.6)$$

7.2.3 Numerical Application

Table 7-1, which is based on Durfee and Tselykh (2011), shows scenario data that represents the financial impact of the loss event (severity)

Table 7-1 Severity for sample scenarios

Events	Type of Scenario	Severity (S) (000,000 omitted)			Fuzzy representation LR-TFN
		Optimistic a_{min}	Realistic a_{mod}	Pessimistic a_{max}	
1	External fraud (branch robberies)	\$ 1.2	\$ 1.7	\$ 2.2	(1.7, 0.5, 0.5)
2	Regulatory breaches[58] (can result in large fines and costly law suits)	\$ 1.5	\$ 3.5	\$ 5.0	(3.5, 2.0, 1.5)

[57] A representation of an α-cut of a fuzzy number was provided in Figure 3-4.
[58] A regulatory breach refers to violation or infraction of current regulations.

Based on Scenario 1, the LR-triangular fuzzy number is:

$A = (a_{mod}, \ a_{mod} - a_{min}, \ a_{max} - a_{mod}) = (1.7, \ 1.7 – 1.2, \ 2.2 – 1.7) = (1.7, 0.5, 0.5)$ for scenario 1
$= (3.5, \ 3.5 – 1.5, \ 5.0 – 3.5) = (3.5, 2.0, 1.5)$ for scenario 2

Table 7-2 shows scenario data that represents the frequency of the loss event

Table 7-2 Frequency for sample scenarios

Events	Type of Scenario	Frequency (P) (Exp. duration, in years, until occurrence)			Fuzzy representation LR-TFN
		Optimistic b_{min}	Realistic b_{mod}	Pessimistic b_{max}	
1	External fraud	5	10	15	(1/10 , 1/ 30, 1/10)
2	Regulatory breaches	4	8	12	(1/8 , 1/ 24, 1/8)

Based on scenario 1:

$[b_{min}, b_{mod}, b_{max}] = [5, 10, 15]$, and $P(S) = (1/b_{max}, 1/b_{mod}, 1/b_{min}) = (1/15, \ 1/10, \ 1/5)$

The corresponding LR-type triangular fuzzy number is

$P(S) = (1/b_{mod}, 1/b_{mod} - 1/b_{max}, \ 1/b_{min} - 1/b_{mod}) = (1/10, \ 1/10 – 1/15, \ 1/5 – 1/10)$
$= (1/10, \ 1/30, \ 1/10)$

Based on scenario 2,

$[b_{min}, b_{mod}, b_{max}] = [4, 8, 12]$, and $P(S) = (1/b_{max}, 1/b_{mod}, 1/b_{min}) = (1/12, \ 1/8, \ 1/4)$

The corresponding LR-type triangular fuzzy number is

$P(A) = (1/b_{mod}, 1/b_{mod} - 1/b_{max}, \ 1/b_{min} - 1/b_{mod}) = (1/8, \ 1/8 – 1/12, \ 1/4 – 1/8)$
$= (1/8, \ 1/24, \ 1/8)$

The Operational risk (OR)

The operational risk (OR) is computed using (7.5).

Under scenario 1, the operational risk is
$OR = (1.7, 0.5, 0.5) \otimes (1/10, 1/30, 1/10) \approx (.17, 2.7/30, 2.7/10) = (0.1700, 0.0900, 0.2700)$

While under scenario 2, the operational risk is
$OR = (3.5, 2.0, 1.5) \otimes (1/8, 1/24, 1/8) \approx (3.5/18, 7.5/24, 6.5/8) = (0.1944, 0.3125, 0.8125)$

α – cut of operational risk

The grade of membership for an operational risk is given by its α-cut as stated in (7.6).

Under scenario 1, for selected values of α (α = 0.85 and α = 0.95) the α – cut is

$$\begin{aligned} OR_{\alpha} = (0.1700, 0.0900, 0.2700)_{\alpha} &= (0.17 - (1-\alpha)\,0.09,\ 0.17 + (1-\alpha)\,0.27) \\ &= (0.1565, 0.2105) \quad \text{for } \alpha = 0.85 \\ &= (0.1655, 0.1835) \quad \text{for } \alpha = 0.95 \end{aligned}$$

Under scenario 2, for selected values of α (α = 0.85 and α = 0.95) the α – cut is

$$\begin{aligned} OR_{\alpha} = (0.1944, 0.3125, 0.8125)_{\alpha} &= (0.1944 - (1-\alpha)\,0.3125,\ 0.1944 + (1-\alpha)\,0.8125) \\ &= (0.1475, 0.3163) \quad \text{for } \alpha = 0.85 \\ &= (0.1788, 0.2350) \quad \text{for } \alpha = 0.95 \end{aligned}$$

Table 7-3 shows OR exposure and selected degrees of confidence (DOC) for sample scenarios.

Table 7-3 OR exposure and selected DOC for sample scenarios

Events	Type of Scenario	Fuzzy representation LR-TFN		Operational risk (OR) exposure LR-TFN	Degree of confidence	
		Frequency (c_P, l_P, r_P)	Severity (c_S, l_S, r_S) (000,000 omitted)		α = 0.85	α = 0.95
1	External fraud	(1/10, 1/30, 1/10)	(1.7, 0.5, 0.5)	(0.1700, 0.0900, 0.2700)	(0.1565, 0.2105)	(0.1655, 0.1835)
2	Regulatory breaches	(1/8, 1/ 24, 1/8)	(3.5, 2.0, 1.5)	(0.1944, 0.3125, 0.8125)	(0.1475, 0.3163)	(0.1788, 0.2350)

7.3 Using Fuzzy Logic for Hazard Risk Assessment

7.3.1 **Introduction:**

Hazard risk (also referred to as disaster risk) describes an unexpected loss associated with events such as property damage, natural perils, personal injury, and business interruption [Segal (2011, p. 374)]. The loss can be from natural or environmental sources, examples of which are tropical cyclones [Liu and Zhang (2012)] or earthquake [Chongfu (2003)]. The loss can also result from man-made sources, such as pollution [Kentel and Aral (2004)].

Health risk is a category of hazard risk, and results from both natural and man-made sources. It describes risk associated with events that can be of harm to human health from exposure to a substance, an activity, or an environmental hazard. Health risk assessment is an important challenge for insurance companies, especially life insurance companies and underwriting. Some studies have focused on fuzzy logic application in life insurance [Carreno and Jani (1993)] and in medical underwriting [Horgby et al. (1997)]. However, the literature on FL application in health risk assessment seems sparse.

In this case study section on health risk assessment we will present two methodologies: the first deals with assessment of health risk associated with obesity, and uses risk classification methodology. This example is adapted from Nawarycz et al. (2013). The second example deals with the risk assessment of cancer due to chemicals in drinking water, and is adapted from Kentel and Aral (2007) and Kentel (2004).

7.3.2 **RAA-FL for Health Risk Associated with Obesity**

In this application, the standard and fuzzy risk matrices are used to assess health risk associated with obesity. Obesity, especially abdominal obesity, which describes an excess of fat around abdominal organs, is a significant health risk factor. A common measure of obesity is the body mass index (BMI), which represents a person's weight (in kg) per square height in meters, that is, BMI = Weight / (Height ^{2}). [Carreno and Jani (1993)] An individual's waist circumference (WC) is also an indication of the presence (or not) of an abdominal obesity.

7.3.2.1 Using Standard (Crisp) Risk Classification

Classification by body mass index BMI

An individual is said to have a normal weight (NO) if his/her BMI is lower than 25 kg /m^2. An individual is said to be overweight (OW) if his/her BMI is between 25 and 30. The individual is obese (OB) if his/her BMI is 30 or greater. These relationships are shown in (7.7).

$$\text{weight} = \begin{cases} \text{Normal (NO)} & \text{BMI} < 25 \\ \text{Overweigth (OW)} & 25 \leq \text{BMI} < 30 \\ \text{Obese (OB)} & \text{BMI} \geq 30 \end{cases} \tag{7.7}$$

Classification by BMI and Waist Circumference (WC)

A combination of BMI and Waist Circumference (WC) provides an appropriate base for assessing cardio metabolic risk [Nawarycz et al. (2013)]. Nawarycz et al. also suggested three areas of WC based on recommendations from previous literature and by health policies agencies: without central (c) obesity (cNO) for WC less than 94cm (for males), central overweight (cOW) for WC between 94 and 102 cm (for males), and central obesity (cOB) for WC larger than 102 cm (for males). For women, cNO corresponds to WC less than 80 cm, cOW corresponds to values of WC between 80 cm and 88 cm, and cOB is for values of WC greater than 88 cm.

Health Risk Evaluation Using Standard (Crisp) Risk Classification

The health risk associated with obesity is assessed using the classification shown in Table 7-4.

Table 7-4 Risk classification for health risk evaluation with BMI and WC, women

	A	B	C	D
1	Waist Circumference (WC) in cm	Body Mass Index (BMI) in kg/m^2		
2		Normal (NO)	Overweight (OW)	Obese (OB)
3	cNO WC < 80 cm	Least Risk	Increased Risk	High Risk
4	cOW 80 cm < WC < 88 cm	Increased Risk	High Risk	Very High Risk
5	cOB WC > 88 cm	High Risk	Very High Risk	Very High Risk

Adapted from Nawarycz et al. (2013) Table II

7.3.2.2 Using Fuzzy Risk Classification

Rational for use of FL

The boundaries between subgroups for each factor within a risk category are not unique. As an example, the rule states that an individual has a normal weight (NO) if his/her BMI is lower than 25 kg /m^2. However, a BMI of 26 is, in some circumstances, considered as not overweight. Fuzzy logic allows for partial membership, and works in such a case.

Fuzzy membership function for BMI subcategories

For each subset (normal, overweight, and obese) a fuzzy formulation is done. The results are shown in Figure 7-3, which displays the membership function (MF) for the 3 subgroups.

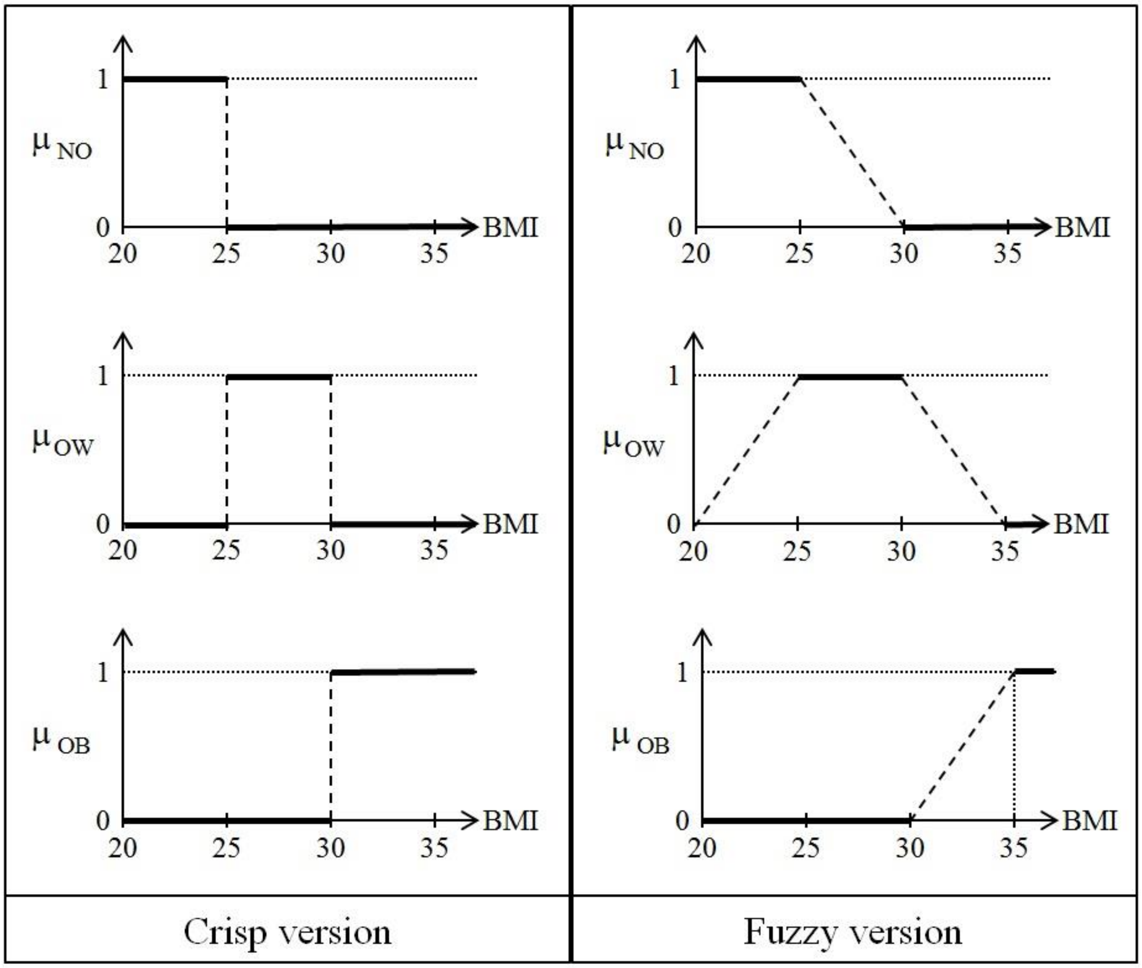

Figure 7-3 MF for BMI subsets

As an example, for the subset *NO-normal,* a crisp description of the membership would indicate: individuals with BMI less than 25 kg/m^2 have a "normal" level of BMI, and individuals with a BMI different from 25 kg/m^2 are either "overweight" or "obese". An extension to a fuzzy membership function implies that individuals with a BMI score around 25 (such as 25.5 or 26.2, etc.) belongs to some extent to the subset of normal BMI score.

The expressions for the membership functions of the BMI fuzzy subsets NO, OW, and OB are given by (7.8), (7.9) and (7.10), respectively [Nawarycz et al., 2013].

$$\mu_{BMI-NO}(\text{BMI}) = \begin{cases} 1 & \text{BMI} \leq 25 \\ 6 - 0.2 \times \text{BMI} & 25 < \text{BMI} \leq 30 \\ 0 & \text{BMI} > 30 \end{cases} \tag{7.8}$$

$$\mu_{BMI-OW}(\text{BMI}) = \begin{cases} 0 & \text{BMI} \leq 25 \\ 0.2 \times \text{BMI} - 5 & 25 < \text{BMI} \leq 30 \\ 7 - 0.2 \times \text{BMI} & 30 < \text{BMI} \leq 35 \\ 0 & \text{BMI} > 35 \end{cases} \tag{7.9}$$

$$\mu_{BMI-OB}(\text{BMI}) = \begin{cases} 0 & \text{BMI} \leq 30 \\ 0.2 \times \text{BMI -6} & 30 < \text{BMI} \leq 35 \\ 1 & \text{BMI} > 35 \end{cases} \tag{7.10}$$

In the current case study, we restrict the risk factors to the body mass index and the waist circumference. Since we have the membership function for the fuzzy BMI categories, we turn now to the computation of the membership function for the waist circumference (WC).

Calculating Fuzzy membership function for waist circumference WC subcategories

Again, the WC categories are: cNO without central obesity, cOW central overweight, and cOB central obesity. We computed the membership functions μ_{BMI-NO}, μ_{BMI-OW}, and μ_{BMI-OB} for the BMI categories. We use Zadeh extension principle (see p. 14 of this report) and a set of regression lines (obtained from epidemiology studies) to calculate the related memberships. We will use a linear regression equation f_2 from Nawarycz et al. (2013):

$$f_2 : \begin{cases} WC_{Males} = 2.66 \times BMI + 22.9 \\ WC_{Females} = 2.27 \times BMI + 23.1 \end{cases} \tag{7.11}$$

Let the following be a set of selected fuzzy numbers belonging to the NO set.

$$NO_{BMI} = \{0/25 + 0.8/26 + 0.6/27 + 0.4/28 + 0.2/29 + 0/30\} \tag{7.12}$$

In this equation, the summation sign stands for the union of (x, μ_A(x)) pairs, not summation, and "/" is only a marker, and does not imply division.

Using Zadeh's extension principle, we obtain the following membership function

$$
\begin{aligned}
cNO_{BMI,\,female} &= \left\{\mu(x) / f(x)\right\} \\
&= \left\{0 / f_2(25) + 0.8 / f_2(26) + 0.6 / f_2(27) + 0.4 / f_2(28) + 0.2 / f_2(29) \right. \\
&\quad \left. + 0 / f_2(30)\right\} \\
&= \left\{0 / 79.85 + 0.8 / 82.12 + 0.6 / 84.39 + 0.4 / 86.66 + 0.2 / 88.93 \right. \\
&\quad \left. + 0 / 91.2\right\}
\end{aligned}
\tag{7.13}
$$

In the same way, we obtain the MF for the categories OW and its correspondent cOW

$$
\begin{aligned}
OW_{BMI,\,female} &= \left\{0 / 25 + 0.2 / 26 + 0.6 / 28 + 1 / 30 + 0.6 / 32 + 0.2 / 34 + 0 / 35\right\} \\
cOW_{BMI,\,female} &= \left\{0 / f_2(25) + 0.2 / f_2(26) + 0.6 / f_2(28) + 1 / f_2(30) \right. \\
&\quad \left. + 0.6 / f_2(32) + 0.2 / f_2(34) + 0 / f_2(35)\right\} \\
&= \left\{0 / 79.85 + 0.2 / 82.12 + 0.6 / 84.39 + 1 / 91.2 \right. \\
&\quad \left. +0.6 / 95.74 + 0.2 / 100.28 + 0 / 102.55\right\}
\end{aligned}
\tag{7.14}
$$

Finally, we obtain the MF for the categories OB and its correspondent cOB

$$
\begin{aligned}
B_{BMI,\,female} &= \left\{0.0 / 29 + 0.0 / 30 + 0.4 / 32 + 0.8 / 34 + 1 / 35 + 1 / 36\right\} \\
cOB_{BMI,\,female} &= \left\{0 / f_2(29) + 0 / f_2(30) + 0.4 / f_2(32) + 0.8 / f_2(34) \right. \\
&\quad \left. +1 / f_2(35) + 1 / f_2(36)\right\} \\
&= \left\{0 / 79.85 + 0 / 84.39 + 0.4 / 91.2 + 0.8 / 95.74 \right. \\
&\quad \left. +1 / 102.55 + 1 / 104.82\right\}
\end{aligned}
\tag{7.15}
$$

The fuzzy risk is obtained using the fuzzy risk classification as done in Section 4.4 of this report.

The inputs are the fuzzy memberships of the BMI and WC categories.

Next, an example that uses the methodology of α-cut is discussed.

7.3.3 **RAA-FL for Health Risk Associated with Chemical Ingestion**

7.3.3.1 **Introduction**

This application is based on the paper by Kentel and Aral (2004) that uses FL together with probability theory to incorporate uncertainties into the health risk analysis associated with exposure to contaminated water. The use of FL is justified by the fact that the uncertainties in the data used do not come from randomness only. The data for this study was partly derived from expert judgment. In addition, information such as the cancer potency was lacking, and there was limited data for several variables of the model. A noteworthy feature of the paper was that it used α-cuts and interval arithmetic methodology.

7.3.3.2 A summary of the steps used for this case study

The methodology for hazard RA in general, and health RA in particular, relies on four main steps [Bardossy et al. (1991), Chongfu et al. (1995), Cohrssen and Covello (1989), Kentel and Aral (2004)]: (1) the release (or risk source) assessment, (2) the exposure assessment, (3) the dose-response (or consequence) assessment, and (4) the risk calculation (or loss assessment). Following the health RA discussed by Kentel and Aral (2004):

- The release assessment consists of identifying the system or activity at the source of the event. Here, the risk source, which is the contaminated tap water, is not studied because it is assumed to be well identified.

- The exposure assessment consists of quantifying the level of exposure to the risk agent. In this case, it involves quantifying the concentration of contaminated water (CW) that individuals were exposed to.

- The dose-response assessment involves determining the dose of the risk agent received by exposed populations and estimating the relationship between different doses and the magnitude of their adverse effects. This analysis of the consequence of the water contamination leads to the determination of the cancer potency factor (CPF), which is a fuzzy variable.

- The consequence and loss assessments address the damages resulting from a given event. In the case at hand, the health risk for cancer is given by the equation:

  $$\text{Risk} = \text{CW} \times f\left(x_1, \ldots, x_n\right) \times \text{CPF} \tag{7.16}$$

  where $f(x_1, \ldots, x_n)$ is a joint probability of random variables occurring in the risk formula.

  By re-sampling from the distributions of $P = f(x_1, \ldots, x_n) \times \text{CPF}$, the cumulative distribution function of P, P_{cdf}, can be generated and used in the analysis.

7.3.3.3 The parameters of the health risk equation

In this study, values and distributions for parameters are derived from previous literature and experiments. These previous studies suggested that CPF and CW be modeled as fuzzy variables. (TFNs, for simplicity).

Table 7-5 shows the value of selected parameters in the health risk assessment presented by Kentel and Aral (2004):

Table 7-5 Selected Parameters (types and values) of the health risk

Parameters	Variables Type	Values / Distributions
CPF	Fuzzy ((kg.day)/mg)	triangular: (0.08, 0.11, 0.14)
CW	Fuzzy (mg/L)	triangular: (0.012, 0.015, 0.018)

7.3.3.4 Using α-cuts and interval arithmetic to model the health risk

The essential feature of α-cuts[59] and interval arithmetic is that the α-cuts of fuzzy numbers are treated as intervals, subjected to interval arithmetic, and then the outcomes are merged to form a fuzzy number. See, for example, Buckley (2005, p. 16) and Hanss (2005, p. 20).

The methodology is implemented by first decomposing the fuzzy numbers into their α-cuts. The α-cuts for the fuzzy variables CW and CPF, for $\alpha \in [0,1]$, are:

$$\begin{aligned} \mathrm{CW}_{\alpha} &= \left(\mathrm{CW}_{\alpha\ \mathrm{lower}}, \mathrm{CW}_{\alpha\ \mathrm{upper}}\right) \\ \mathrm{CPF}_{\alpha} &= \left(\mathrm{CPF}_{\alpha\ \mathrm{lower}}, \mathrm{CPF}_{\alpha\ \mathrm{upper}}\right) \end{aligned} \tag{7.17}$$

where the "lower" and "upper" terms refer to the lower and upper extremes of the α-cut, which coincide with the smallest and largest x such that $\mu(x) \geq \alpha$, $0 < \alpha \leq 1$.

Then, for each value of α, $0 < \alpha \leq 1$, the α-cuts and interval arithmetic for the health risk takes the form:

$$\begin{aligned} \mathrm{Risk}_{\alpha\ \mathrm{lower}} &= \mathrm{CW}_{\alpha\ \mathrm{lower}} \otimes f\left(x_1, \ldots, x_n\right) \otimes \mathrm{CPF}_{\alpha\ \mathrm{lower}} \\ \mathrm{Risk}_{\alpha\ \mathrm{upper}} &= \mathrm{CW}_{\alpha\ \mathrm{upper}} \otimes f\left(x_1, \ldots, x_n\right) \otimes \mathrm{CPF}_{\alpha\ \mathrm{upper}} \end{aligned} \tag{7.18}$$

where ⊗ indicates interval multiplication.

[59] See page 13 of this report for a definition of an α-cut.

7.4 Using FL for Financial Risk Assessment: An example of portfolio cash RA

7.4.1 **Summary**

This section presents an overview of Kishore et al (2011), which calculated the portfolio cash availability risk from a portfolio variance matrix created by using the covariance values among project pairs. Expert opinions were used to create a fuzzy proportional derivative model, which was based on the net cash flow and the change in the cash flow. Then, by way of example, the model was used to predict portfolio risk for a probable future portfolio of projects for a hypothetical firm.

7.4.2 **A Methodology for Cash Risk Analysis**

7.4.2.1 **Methodology**

A flowchart of the methodology for cash risk assessment is shown in Figure 7-4.

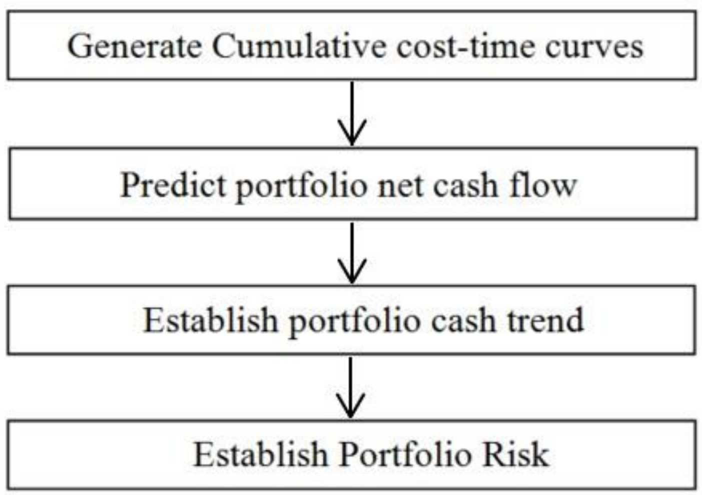

Figure 7-4 Methodology for cash risk assessment

7.4.2.2 **Fuzzy Rules used**

The general form of the *implication* relation is: [Kishore et al (2011)]

If (a set of conditions satisfied); *then* (a set of consequences inferred).

The specific set of rules for a firm with a fixed nonzero amount of cash reserves:

a) *If* {net cash flow at time t is negative and change in cash flow since time (t - 1) is negative}, *then* {cash availability at time (t+1) is *small*;}

b) *If* {net cash flow at time t is zero and change in cash flow since time (t - 1) is zero}, *then* {cash availability at time (t+1) is *moderate*;}

c) *If* {net cash flow at time t is positive and change in cash flow since time (t - 1) is positive}, *then* {cash availability at time (t+1) is large;}

7.4.2.3 Portfolio cash-flow risk

The total portfolio cash-flow risk for a project is represented by the standard deviation of the project cash flow. It assesses the potential variation or uncertainty in the company cash position related to the portfolio of projects. The portfolio cash-flow risk (PR) is computed as follows:

$$PR = \sqrt{\sum_{ij}^{n} x_i x_j \sigma_{ij}} \tag{7.19}$$

where, for a portfolio composed of n projects, x_i represents the proportion of the total costs of all projects in the portfolio incurred by project i, x_j is the proportion of the total costs of all projects in the portfolio incurred by project j, and σ_{ij} denotes the covariance between projects i and j. The covariance term (σ_{ij}) measures the effect of project i on the cash flow of project j, and can be calculated as follows

$$\sigma_{ij} = \rho_{ij} \times \sigma_i \times \sigma_j \tag{7.20}$$

where ρ_{ij} denotes the correlation coefficient between projects i and j, and σ_i and σ_j are the standard deviations of the cash-flow values for project i and j, respectively.

Table 7-6 displays the components in the computation of a portfolio cash-flow risk for n projects (P_i, *i =1,…, n*).

Table 7-6: Portfolio cash-flow risk computation for n projects

Project	P_1	P_2	$\cdots$	P_i	$\cdots$	P_n
P_1	x_1^2	$x_1 x_2 \sigma_{12}$	$\cdots$	$x_1 x_i \sigma_{1i}$	$\cdots$	$x_1 x_n \sigma_{1n}$
P_2	$x_2 x_1 \sigma_{21}$	x_2^2	$\cdots$	$x_2 x_i \sigma_{2i}$	$\cdots$	$x_2 x_n \sigma_{2n}$
$\cdots$	$\cdots$	$\cdots$	$\ddots$	$\cdots$	$\cdots$	$\cdots$
P_i	$x_i x_1 \sigma_{i1}$	$x_i x_1 \sigma_{i1}$	$\cdots$	x_i^2	$\cdots$	$x_i x_n \sigma_{in}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\ddots$	$\cdots$
P_n	$x_n x_1 \sigma_{n1}$	$x_n x_1 \sigma_{n1}$	$\cdots$	$x_n x_i \sigma_{ni}$	$\cdots$	x_n^2
Total portfolio cash-flow variance = $\sum_{i,j}^{n} x_i x_j \sigma_{ij}$						
Total portfolio cash-flow risk = $\sqrt{\sum_{i,j}^{n} x_i x_j \sigma_{ij}}$						

Adapted from Kishore et al. (2011), Table 2.

7.4.2.4 Net Cash Flow

Series of simulations (based on variable scenarios) are done to measure the change in a portfolio net cash flow. Denote by a_1 the lower limit, a_2 the model value, and a_3 the upper limit for the cash values. Then the net cash flow can be represented by the following triangular form

$$\mathrm{NCF} = (\mathrm{a}_1,\ \mathrm{a}_2,\ \mathrm{a}_3). \tag{7.21}$$

It follows [Shaheen et al (2007, p. 327)] that this fuzzy version of the NCF has a mean of

$$EV_{TFN} = \frac{a_1 + a_2 + a_3}{3} \tag{7.22}$$

and a variance and standard deviation, respectively, of

$$Var_{TFN} = \frac{a_1^2 + a_2^2 + a_3^2 + a_1 a_2 + a_1 a_3 + a_2 a_3}{18} \text{ and } \sqrt{Var_{TFN}}\ . \tag{7.23}$$

7.4.3 **Financial Risk Case Study**

Three different portfolio compositions were assessed in the Kishore et al (2011) study, each incorporating three types of projects (for a total of 9 projects):

- Type 1: projects priced under $100,000.
- Type 2: projects with a price range of $100,000 to $1,000,000.
- Type 3: projects priced above $1,000,000.

Three project portfolios were then created with a mix of these project types: (a) 50:40:10, (b) 40:50:10, and (c) 30:60:10, where the portfolio designation represents % Type 1: % Type 2: % Type 3, respectively.

Table 7-7 summarizes a portfolio cash-flow risk for a hypothetical firm.

Table 7-7 Portfolio cash-flow risk

Project	P1	P2	P3	P4	P5	P6	P7	P8	P9
P1	40	2	2	2348	65	65	3,502	3,502	0
P2	2	0	0	145	4	4	217	217	0
P3	2	0	0	145	4	4	217	217	0
P4	2348	145	145	136,615	3,788	3,788	-203,700	-203,700	0
P5	65	4	4	3,788	105	105	5,648	5,648	0
P6	65	4	4	3,788	105	105	5,648	5,648	0
P7	3502	217	217	-203,700	5,648	5,648	-303,725	-303,725	0
P8	3502	217	217	-203,700	5,648	5,648	-303,725	-303,725	0
P9	0	0	0	0	0	0	0	0	955,826,058
Portfolio cash-flow risk ($) = $ $\sqrt{955,229,989}$ = (±) $30,907									

7.5 Using Fuzzy Logic for Strategic Risk Assessment

7.5.1 Summary and Steps

This subsection deals with an example from Zhang et al. (2013) that uses similarity measures between fuzzy numbers to resolve a strategic RA problem based on multiple criteria group decision making.

The problem: Investment Case Selection.
An investment company wants to invest a sum of money in the best option. There is a panel of three experts and five possible alternatives to invest the money:

A_1 is a car company;
A_2 is a food company;
A_3 is a computer company;
A_4 is an arms company;
A_5 is a TV company.

The three experts must make a decision based on the following four criteria:

$C1$ is the risk analysis;
$C2$ is the growth analysis;
$C3$ is the social-political impact analysis;
$C4$ is the environmental impact analysis

The Steps
The decision procedure for the Zhang et al. method can be summarized as follows:

- Step 1. Characterize the linguistic values of each alternative and criterion using TFNs.
- Step 2. Compute the group preference vector for each alternative A_i ($i = 1, 2, \dots, m$).
- Step 3. Compute the expected weight value w_j^C for criterion C_j ($j = 1, 2, \dots, n$).
- Step 4. Compute the three weighted similarity measures for alternative A_i ($i = 1, 2, \dots, m$).
- Step 5. Rank the alternatives, and select the best one(s) in accordance with the weighted similarity measure.

The details of each step follows.

Step 1 involves developing a table, the cells of which are TFNs of the form

$$(a_{ijk1}, a_{ijk2}, a_{ijk3}) \tag{7.24}$$

where i=1,..., 5 is the index for the alternatives, j=1,..., 4 is the index for the criteria, k=1,..., 3 is the index for the experts, and the 1, 2, and 3 denote the lower, model and upper values of the TFN.

Step 2 involves computing the group preference vector for each alternative A_i ($i = 1, 2, \dots, m$), which Zhang et al. represent as:

$$V_i = \left\{ \begin{array}{c} \left\langle C_1, \left(\sum_{k=1}^{p} \lambda_k a_{i1k1}, \sum_{k=1}^{p} \lambda_k a_{i1k2}, \sum_{k=1}^{p} \lambda_k a_{i1k3} \right) \right\rangle, \\ \left\langle C_2, \left(\sum_{k=1}^{p} \lambda_k a_{i2k1}, \sum_{k=1}^{p} \lambda_k a_{i2k2}, \sum_{k=1}^{p} \lambda_k a_{i2k3} \right) \right\rangle, \\ \cdots \\ \left\langle C_n, \left(\sum_{k=1}^{p} \lambda_k a_{ink1}, \sum_{k=1}^{p} \lambda_k a_{ink2}, \sum_{k=1}^{p} \lambda_k a_{ink3} \right) \right\rangle \end{array} \right\} \tag{7.25}$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_p)$ is a weight vector that gives the weight associated with each of the experts.

Step 3 involves computing the expected weight value w_j^C for criterion C_j, $j = 1, 2, \dots, n$. It takes the form $w_j^C = (a_{j1}, a_{j2}, a_{j3})$, $j = 1, 2, \dots, n$, and is obtained by the decision maker using a methodology such as the AHP of Chapter 5.

Step 4 involves comparing the alternatives A_i ($i = 1, 2, \dots, m$), with respect to the criteria, with an ideal alternative A_P.

This concept of an ideal alternative is an abstraction, but it does provide a useful theoretical construct to evaluate the alternatives. Essentially, it involves characterizing each criterion as either positive or negative, accordingly as it has a positive or negative impact with respect to the alternatives, and defining the positive ideal solution as the one involving the best values attainable with respect to the criteria, and the negative-ideal solution as the one involving the worst values attainable with respect to the criteria. Then, the vector representing A_P is constructed by choosing its cells to be the shortest distance from the ideal solution and the farthest distance from the negative-ideal solution.

The details of the construction of A_P is given in Appendix 7-A.

Zhang et al. use the following the three weighted similarity measures for comparing the alternatives A_i ($i = 1, 2, \dots, m$) with the ideal alternative A_P:

$$S^J(A_i, A_P) = \sum_{j=1}^{n} w_j^C \frac{\sum_{l=1}^{3} a_{ijl}\, b_{Pjl}}{\sum_{l=1}^{3} a_{ijl}^2 + \sum_{l=1}^{3} b_{Pjl}^2 - \sum_{l=1}^{3} a_{ijl}\, b_{Pjl}}$$

$$S^E(A_i, A_P) = \sum_{j=1}^{n} w_j^C \frac{2\sum_{l=1}^{3} a_{ijl}\, b_{Pjl}}{\sum_{l=1}^{3} a_{ijl}^2 + \sum_{l=1}^{3} b_{Pjl}^2} \tag{7.26}$$

$$S^C(A_i, A_P) = \sum_{j=1}^{n} w_j^C \frac{\sum_{l=1}^{3} a_{ijl}\, b_{Pjl}}{\sqrt{\sum_{l=1}^{3} a_{ijl}^2}\sqrt{\sum_{l=1}^{3} b_{Pjl}^2}}$$

Step 5 involves the ranking of the alternatives, and selection of the best one(s) in accordance with the weighted similarity measures.

Table 7-8, which is based on Zhang et al. (2013), Table 6, is an example of the type of output this methodology produces.

Table 7-8: Decision results of different similarity measures

	S^J	S^E	S^C
A_1	0.6486	0.7141	0.7997
A_2	0.6465	0.7136	0.7988
A_3	0.7329	0.7630	0.7998
A_4	0.7730	0.7859	0.7999
A_5	0.7674	0.7803	0.7998
Ranking	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$	$A_4 \succ A_5 \succ A_3 \succ A_1 \succ A_2$
Best	A_4	A_4	A_4

7.6 Appendix 7-A

Following the methodology of Yoon (1987), the construction of A_P begins with a performance matrix based on m alternatives and n criteria, $(x_{ij})_{mxn}$, where x_{ij} denotes a performance rating.

The first step is to normalize the cells

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}} \tag{7.27}$$

which renders them non-dimensional and results in the normalized matrix $(r_{ij})_{mxn}$.

Since the weights associated with the criteria, w_j^C, j = 1,..., n, are relevant to the analysis, $(r_{ij})_{mxn}$ is adjusted to accommodate these weights in the form of $v_{ij} = w_j^C r_{ij}$, i = 1,..., m, j = 1,..., n, which results in the weighted normalized matrix $(v_{ij})_{mxn}$.

Let

J^+ denote the set {j = 1,..., n | j is associated with the criteria having a positive impact}

J^- denote the set {j = 1,..., n | j is associated with the criteria having a negative impact}

Then, the ideal alternative A_P is computed as

$$\mathrm{A_P} = \{\max_i(\mathrm{v_{ij}} \mid j \in J^+),\ \min_i(\mathrm{v_{ij}} \mid j \in J^-) \mid \mathrm{i} = 1,2,\cdots,m\} \tag{7.28}$$

8 Closing Comments and Observations

This study investigated risk assessment applications of fuzzy logic (RAA-FL). It began with a review of the literature in this area, one product of which was an annotated bibliography of representative articles. Next, we presented a review of the FL methodology that would be used during the course of the study. Given this background, we turned to the issue of how crisp models, which have fuzzy components that are inadequately accommodated by the model, can be reformulated as fuzzy models. The topics addressed, in this regard, were the use of FL to model the risk matrix, the Analytic Hierarchy Process, and optimization. The final portion of the study presented RAA-FL case studies that involved operational risk, hazard risk, financial risk and strategic risk.

Since most of the current RAA-FL articles do not have an actuarial perspective, an important task of this study was to identify and describe RAA-FL methodologies that can be repurposed for actuarial use. Here, the goal was to provide sufficient detail so that actuaries who have read our study will have a sense of the methodologies involved and will be able to recognize potential opportunities for implementation. To the extent we have met this goal, the study will have served its purpose.

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10 Annotated bibliography

Adamcsek, E. (2008) "The Analytic Hierarchy Process and its Generalizations" Thesis, Eötvös Loránd University.

This thesis begins with a discussion of the principles and axioms of the analytical process, including hierarchical decomposition of the decision, pairwise comparisons, pairwise matrix evaluation, and additive weighted aggregation of priorities. It then turns to a discussion of group decision making, including aggregating individual judgments and priorities, computing weights, the consistency of the weighted geometric mean complex judgment matrix, the logarithmic least squares approach, and the generalized approach. It closes with a discussion of the analytic network process.

An, M., Chen, Y., Baker, C. J. (2011) "A fuzzy reasoning and fuzzy-analytical hierarchy process based approach to the process of railway risk information: A railway risk management system," Information Sciences 181(18), 3946-3966.

This article presents the development of a risk management system for railway risk analysis using a fuzzy reasoning approach and the fuzzy analytical hierarchy process (FAHP). The former is employed to estimate the risk level of each hazardous event in terms of failure frequency, consequence severity, and consequence probability. The FAHP is then used to determine the relative importance of the risk contributions so that the risk assessment can progress from the hazardous event level to the hazard group level and finally to railway system level. A case study is conducted based on the risk assessment of a shunting facility.

Babad, Y. M., Berliner, B. (1995) "Reduction of uncertainty through actualization of intervals of possibilities and its applications to reinsurance." Proceedings of the 25th International Congress of Actuaries 2, 23–36.

One drawback of interval and fuzzy arithmetic is the possible increase in imprecision, as expressed by the width of the resulting intervals and fuzzy numbers. This paper demonstrates how to overcome this drawback, and improve the resulting precision, by using actualization, which requires that once a value from an interval is selected, it must consistently be used wherever the interval is used. Actualization is then illustrated in matrix operations and stop-loss reinsurance.

Bellman, R., Zadeh, L. A. (1970) "Decision-making in a fuzzy environment," Management Science 17, 141-164.

This paper presents problems involving multistage decision processes in a fuzzy environment and suggest tentative ways of attacking them. In particular, the paper shows application to decision-making of the concept of a fuzzy algorithm -a concept which may be of use in problems which are less susceptible to quantitative analysis than those commonly considered.

Better, M., Glover, F., Kochenberger, G., Wang, H. (2008) "Simulation Optimization Applications in

Risk Management," International Journal of Information Technology and Decision Making, 7(4), 571-587

This article explores the use of a simulation optimization approach to analyze risky decisions. The methodology is exemplified using applications from finance and business process design.

Bezdek, J. C. (1981) "Pattern Recognition with Fuzzy Objective Function Algoritms", Plenum Press, New York

While fuzzy systems allow us to convert and embed empirical qualitative knowledge into reasoning systems capable of performing approximate pattern matching and interpolation, these systems generally cannot adapt or learn because they are unable to extract knowledge from existing data. This book discusses a fuzzy clustering methodology for overcoming this limitation, called the (fuzzy) c-means algorithm. The essence of the c-means algorithm is that it produces reasonable centers for clusters of data, in the sense that the centers capture the essential feature of the cluster, and then groups data vectors around cluster centers that are reasonably close to them.

Bilgiç, T., Türkšen, I. B. (1995) "Measurement of membership functions: theoretical and empirical work," in: Dubois, D., Prade, H. (Eds.), Fundamentals of Fuzzy Sets, Kluwer Academic, Boston, 195-230.

This chapter presents a review of various interpretations of the membership function and of methods for obtaining it. Techniques from measurement theory are used to show that different interpretations of the membership function call for different elicitation methods.

Bodoff, N. (2008) “Risk Management: The Current Financial Crisis, Lessons Learned and Future Implications.” SOA, www.soa.org/library/essays/rm-essay-2008-bodoff.pdf

Bodoff believes that the central issue that underlies the current crisis and many others is how one measures profit. Specifically, he postulates that the current crisis derives from the lack of “risk adjustment” when reporting profit, and that a key reform crucial to mitigating future crises is to ensure that we always measure profit on a “risk-adjusted basis.”

Boender, C. G. E., de Grann, J. G., Lootsma, F. A. (1989) "Multi-criteria decision analysis with fuzzy pairwise comparison," Fuzzy Sets and Systems, 29, 133–143.

The authors present a modification of the fuzzy multi-criteria method proposed by Van Laarhoven and Pedrycz (1983). First the weights of the decision criteria are calculated by the minimization of a logarithmic regression function. Next the weights of the decision alternatives are calculated for each criterion separately. Then, the fuzzy final scores of the alternatives are determined by an aggregation of the calculated weights. They also show that their method yields weights that are optimal with respect to the logarithmic regression function.

Bouyssou, D., Marchant, T., Pirlot, M., Perny, P., Tsoukias, A., Vincke, P. (2000) Evaluation and Decision Models: A Critical Perspective, Kluwer Academic Publishers, Boston.

The purpose of this book is to provide a conceptual framework for decision and evaluation models. The authors postulate that there is no best model, and decision makers must understand the principles of formal versions of these models and apply them critically.

Boyd, S., Vandenberghe, L. (2009) Convex Optimization, Cambridge University Press, 7th Ed.

Convex optimization is a special class of mathematical optimization problems, which includes least-squares and linear programming problems. The main goal of this book is to help potential users develop the skills and background needed to recognize, formulate, and solve convex optimization problems.

Brillinger, D. R. (2002) Some Examples of Risk Analysis in Environmental Problems, University of California, Berkeley.

This article views risk analysis as the problem of estimating the probabilities of rare and damaging events, such as floods, earthquakes, forest fires, and space debris collisions. Various ways to collect and extrapolate data are described and examples involving the foregoing events are presented.

Brockett, P. L., Xia, X. (1995) Operations research in insurance: a review. Transactions of Society of Actuaries 47: 7–87

This paper provides a set of examples of operations research techniques pertinent to actuaries and shows how the expanding field of general quantitative reasoning in risk management can have a positive impact on the insurance industry. Of particular relevance to this report, is the discussion of how a fuzzy LP problem can be resolved by reformulating it as a crisp LP problem.

Buckley, J. J. (1987) The fuzzy mathematics of finance. Fuzzy Sets and Systems 21:257–273

This paper was the first to demonstrate the applicability of fuzzy difference equations in the field of finance. Two methods were considered: one that dealt with fuzzy analogues of the elementary compound interest problem; and a fuzzy difference equations method. It turned out that both methods produce the same general formula. The main contribution of the paper was the simplicity of the method presented, which accommodates formulas for cases that are specialized and more realistic than the very basic one of compound interest.

Buckley, J. J. (2005) Fuzzy Probabilities. Physica-Verlag, Berlin Heidelberg.

This book advocates using fuzzy numbers, which are constructed from a set of confidence intervals, to estimate the parameters of probability distributions, instead of using point estimates calculated from random samples. Then, for example, a fuzzy normal random variable can have a normal distribution with a fuzzy mean and/or variance. Applications discussed queuing theory, Markov chains, inventory control, decision theory and reliability theory.

Buckley, J. J. (1985a) "Ranking alternatives using fuzzy numbers," Fuzzy Sets and Systems 15(1), 21-31.

This paper investigated the problem of employing expert opinion to rank alternatives across a set of criteria. Given that experts use fuzzy numbers to express their preferences, fuzzy arithmetic is used to compute a fuzzy ranking. This leads to a partition of the alternatives into ranked sets. The total ranking process is shown to possess a number of important properties. An example was presented to illustrate the method.

Buckley, J. J. (1985b) "Fuzzy hierarchical analysis," Fuzzy Sets and Systems 17(3), 233-247.

This paper extends hierarchical analysis to the case where the participants are allowed to employ fuzzy ratios in place of exact ratios. The pairwise comparison of the alternatives and the criteria in the hierarchy produce fuzzy positive reciprocal matrices, and the geometric mean method is employed to calculate the fuzzy weights for each fuzzy matrix. The latter are combined in the usual manner to determine the final fuzzy weights for the alternatives. The procedure easily extends to multiple expert situations and to the case of missing data. Examples are presented that show the final fuzzy weights and the final ranking.

Calabrese, A., Costa, R., Menichini, T. (2013) "Using Fuzzy AHP to manage Intellectual Capital assets: An application to the ICT service industry." Expert Systems with Applications 40(9): 3747.

This paper uses the FAHP to assess the relative importance of intellectual capital (IC) components, with respect to their contribution to the company value creation, in order to obtain guidelines for IC management and investments. Specifically, a modified Chang (1996) version of the FAHP is used to accommodate the linguistic variables that experts and managers use in the evaluation process of the company intangible assets. As an example, the FAHP methodology is applied to a group of Information and Communication Technology (ICT) service companies.

Carlsson, C., Fullér, R. (1998) "Optimization under fuzzy rule constraints," The Belgian Journal of Operational Research, Statistics and Computer Science, 38, 17-24.

This paper deals with a mathematical programming problem in which the functional relationship between the decision variables and the objective function is not completely known. It is assumed that the knowledge-base consists of a block of fuzzy if-then rules, where the antecedent part of the rules contains some linguistic values of the decision variables, and the consequence part is a linear combination of the crisp values of the decision variables. They determine the crisp functional relationship between the objective function and the decision variables, and show how to solve the resulting (usually nonlinear) programming problem to find a fair optimal solution to the original fuzzy problem.

CAS (2003) "Overview of Enterprise Risk Management," Casualty Actuarial Society, Enterprise Risk Management Committee, http://www.casact.org/pubs/forum/03sforum/

This document presents an overview of Enterprise Risk Management (ERM). It begins with an explanation of the evolution to and rationale for ERM and then lays out its conceptual framework and ERM vocabulary, including a description of the measures, models and tools supporting the discipline. This is followed by a discussion of relevant case studies from various industries and some practical considerations in implementing ERM.

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Chan, F. T. S., Kumar, N. (2007) "Global supplier development considering risk factors using fuzzy extended AHP-based approach," Omega 35(4), 417-431.

This paper used the fuzzy extended AHP of Chang (1992) to identify and discuss some of the important and critical decision criteria, including risk factors, for the development of an efficient system for global supplier selection. Decision criteria included cost, quality, service performance and supplier's profile. The fuzzy pairwise comparisons of the customers and experts were formulated as triangular fuzzy numbers, and the implementation of the system was demonstrated by a problem having four stages of hierarchy that contains different criteria and attributes.

Chang, D.-Y. (1996). "Applications of the extent analysis method on fuzzy AHP." European Journal of Operational Research 95(3): 649-655.

This paper used triangular fuzzy numbers (TFNs) to extend the AHP to a FAHP, where arithmetic means were used to determine the priority vector, and the final ranking was done using crisp numbers. The term "extent analysis" refers to an analysis of the extent to which one fuzzy number dominates another when doing a pairwise comparison.

Chen, J. E., Otto, K. N. (1995) "Constructing membership functions using interpolation and measurement theory," Fuzzy Sets and Systems 73(3), 313-327.

This article discusses the constructing of membership functions based on first determining the best and worst values, and then using a constrained interpolation scheme to assign the remaining values

Csutora, R., Buckley, J. J. (2001) "Fuzzy hierarchical analysis: The Lamda-Max method," Fuzzy Sets and Systems 120(2), 181-195.

This article presents a method of finding the fuzzy weights in the fuzzy AHP, which is the direct fuzzification of Saaty's λmax method (eigenvector method) of computing the weights, and gives the same weights. An example is presented where there are five criteria and three alternatives.

Cummins D., Derrig, R. A. (1997) "Fuzzy Financial Pricing of Property-liability Insurance," North American Actuarial Journal 1(4), 21-40.

This article discusses the use of fuzzy set theory (FST) in actuarial science. In particular, it deals with the potential use of FST in insurance pricing and propose rules for project decision-making using FST. The results indicate that FST can lead to significantly different decisions than the conventional approach.

DaSilva, L. E. B, Torres, G. L., Dos Reis, L. O. M., Haddad, J. (2002) "Application of Fuzzy Optimization in Energy Saving" Rev. Ciênc. Exatas, Taubaté, v. 5-8, p. 21- 35.

This paper discusses the development of a fuzzy optimization process. It begins with an overview of the classical optimization process and its potential drawbacks, including constraints that are not well defined. Then, the theoretical aspects and implementation issues of the fuzzy optimization process is discussed. The paper concludes with an example based on power system energy saving.

de Figueiredo, R. J. P. (2007) "A nonlinear functional analytic framework for modeling and processing fuzzy sets," in Nikravesh, M., Kacprzyk, J., Zadeh, L. A. Forging new frontiers, fuzzy pioneers I, Berlin: Springer-Verlag, 171-191.

This paper extends the standard membership function, $\mu_A(x)$, where {x} is the universe of discourse and A is a fuzzy set of elements in {x}, to a membership functional of the form $\mu_A(\tilde{A}, J, x)$, where there is an attribute, or an event characterized by the respective attribute, $\tilde{A}$, and a judgment criterion, J, on the basis of which the membership of x in A is judged.

Department of Defense (1993) Military Standard MIL-STD-882c, System Safety Program Requirements, AMSC Number F6861.

This standard provides uniform requirements for developing and implementing a system safety program of sufficient comprehensiveness to identify the hazards of a system and to impose design requirements and management controls to prevent mishaps. The aim is to eliminate hazards or reduce the associated risk to an acceptable level.

Dombi, J. (1990) "Membership Function as an Evaluation," Fuzzy Sets and Systems 35(1), 1-21.

This article discusses a theoretical basis for membership functions that is easily implemented in practice. Specifically, it presents a class of membership functions that can be described with four parameters: the interval [a, b], the sharpness, λ, which is an indicator of increasing membership, and the decision level, ν, which is the intersection value of $y = \mu(x)$ and $y = x$, the characteristic value of the shape. The model is validated with empirical data, and by showing that the linear form is a special case.

Down, A., Coleman, M., Absolon, P. (1994) Risk Management for Software Projects, McGraw-Hill, London.

This book provides a practitioner's guide to risk management for software development, featuring proven techniques used at IBM. It shows how to set up an optimum software project risk environment and manage it through to a successful project conclusion. A number of examples and case studies are provided.

Dubois, D. (2011) "The role of fuzzy sets in decision sciences: Old techniques and new directions," Fuzzy Sets and Systems 184, 3-28.

This article provides a tentative assessment of the role of fuzzy sets in decision analysis. It discusses membership functions, aggregation operations, linguistic variables, fuzzy intervals and the valued preference relations they induce. The importance of the notion of bipolarity and the potential of qualitative evaluation methods are also pointed out. Dubois takes a critical standpoint on the state-of-the-art, in order to highlight the actual achievements and question what is often considered debatable by decision scientists observing the fuzzy decision analysis literature.

Dubois, D. J., Prade, H. M. (1980) Fuzzy sets and systems: theory and applications. Academic Press, San Diego, CA

This book was a rather exhaustive research monograph on the first 15 years of fuzzy set theory and its applications, and, as such, it covers almost all of the important developments in the theory of fuzzy sets during that period. The applications span a wide variety of topics ranging from industrial process control to medical diagnosis and group decision processes.

Dubois, D., Kerre, E., Mesiar, R., Prade, H. (2000) "Fuzzy interval analysis," in Dubois, D., Prade, H. (Eds.) Fundamentals of Fuzzy Sets, The Handbooks of Fuzzy Sets Series, Boston, MA: Kluwer, 483–581.

This chapter provides an overview of the literature dealing with fuzzy intervals and their operations, where fuzzy intervals are fuzzy sets of the real line that generalize intervals. It also provides a synthesis of the literature on the comparison of fuzzy numbers, and a brief survey of the major trends with respect to fuzzy interval applications.

Dubois, D., Prade, H. (1997) Fuzzy Sets and Systems: Theory and Applications. Academic Press, San Diego, CA.

See: Dubois, D. J., Prade, H. M. (1980) Fuzzy sets and systems: theory and applications. Academic Press, San Diego, CA

Durfee, A., Tselykh, A. (2011) "Evaluating Operational Risk Exposure Using Fuzzy Number Approach to Scenario Analysis", EUSFLAT-LFA, Atlantis Press, France.

This article describes the use of trapezoidal fuzzy numbers of various heights to implement the operational risk levels derived from expert opinions as to probabilities and severities, which were collected from scenarios. The focus of the study is the level of operational risk and the adequate capital required to cover it.

Dutta, K., Babel, D. (2009) "Scenario Analysis in the Measurement of Operational Risk Capital: A Change of Measure Approach", Wharton School of Business, University of Pennsylvania, Working paper #101.

This article proposes a method for the measurement of operational risk exposure of an institution using scenario analysis and internal loss data. The authors provide a revised interpretation of scenario data, which they view as consistent with the loss experience of an institution, with regard to both the frequency and severity of the loss. Then, using this interpretation, they show how the scenario data together with historical data could be used to effectively measure operational risk exposure and, using the Change of Measure (= implied probability measure / historical probability measure) concept, evaluate each scenario in terms of its effect on the operational risk measure.

Elsayed, T. (2009) "Fuzzy inference system for the risk assessment of liquefied natural gas carriers," Applied Ocean Research 31, 179-185

This article explores a risk matrix where the input probabilities and consequences used in risk assessment are represented by fuzzy sets, to take into account uncertainties associated with the assignment of their values. Both the Mamdani FIS, in which output risk values are fuzzy sets, and the Sugeno FIS, in which output risk values are constant or linear, are investigated. A case study of a liquefied natural gas ship loading/offloading at a terminal is presented.

Enea, M., Piazza, T. (2004) "Project selection by constrained fuzzy AHP," Fuzzy Optim. Decis. Making 3, 39-62.

This paper is based upon a constrained fuzzy extension of the AHP. The constraint is meant to avoid situations where the traditional mathematical operators give meaningless results when applied to fuzzy numbers. Using this methodology, the authors are able to reduce the level of uncertainty below that obtained by earlier researchers. Among their findings, the authors conclude that the geometric mean is preferable to the arithmetic one when accommodating the opinions of a group of experts.

Feng C-M, Chung, C-C (2013) “Assessing the Risks of Airport Airside through the Fuzzy Logic-Based Failure Modes, Effect, and Criticality Analysis” Mathematical Problems in Engineering, 1-11.

This article implements a fuzzy risk matrix, where the risk of an event is defined as an expected value that depends on its frequency of occurrence, the severity of its failure effects, and the likelihood that subsequent testing of the design will detect that the potential failure mode actually occurs. This latter is referred to as detectability. The focus is on airport airside risk, that is, risks associated with the apron-gate area, taxiway system, holding pad, runway, and terminal airspace.

Fera, M., Macchiaroli, R. (2010) "Appraisal of a new risk assessment model for SME," Safety Science 48(10), 1361-1368.

This paper concerns risk assessment (RA) in small and medium enterprises (SMEs). To accommodate the situation, the authors propose a crisp RA model that merges a geometric mean version of the AHP with risk matrix methodology.

Hanss, M. (2005) Applied Fuzzy Arithmetic: An Introduction with Engineering Applications, Berlin: Springer-Verlag.

This book is divided into two parts. The first part is an introduction to the theory of fuzzy arithmetic. It presents the derivation of fuzzy arithmetic from the original fuzzy set theory and expounds on its implementation with existing mathematical formulations. The second part presents a diversified exposition of engineering application of fuzzy arithmetic.

Holton, G. (1997) "Subjective Value at Risk", Financial Engineering News, 1(Aug). Reprinted by SOA, 1998, "Risks and Rewards Newsletter" October 1998 – Issue No. 31.

Holton makes the case that markets are too complex and ever-changing for any model to fully describe and that selecting a model is a subjective process. He goes on to note that risk limits enable an organization to manage risk by limiting traders to taking positions within a specified range, and that the role of the VaR model is to objectively define what that range is.

Holton, G. (2004) "Defining Risk", Financial Analysts Journal 60(6): 19-25.

This article reviews the development of the definition of risk and explores the nature of risk, as the term is commonly used. While Holton defines risk as exposure to a proposition of which one is uncertain, he acknowledges that, while his definition clarifies common usage, it is flawed. He goes on to conclude that because operational definitions apply only to that which can be perceived, we can never operationally define risk. At best, we can operationally define only our perception of risk.

Hopkin, P. (2010) Fundamentals of Risk Management: Understanding, evaluating and implementing effective risk management, Philadelphia: Kogan Page Limited

This book provides a comprehensive introduction to risk management, from the perspective of commercial and business risk. This includes a discussion of international standards and models and the influence of ISO 31000. With a focus on operational, change and strategic risk, it provides a broad range of case studies and well-known examples.

Hu, S., Fang, Q., Xia, H., Xi, Y. (2007) "Formal safety assessment based on relative risks model in ship navigation," Reliability Engineering and System Safety 92, 369-377

In this study of safety assessment in ship navigation, the parameters of the standard fuzzy risk matrix (RM) model are extended to frequency, relative severity, and obligated severity, where the latter involve contingent consequences. The parameter weights are established using the AHP.

Hussey, D. E. (1978) "Portfolio Analysis: Practical Experience with the Directional Policy Matrix," Long Range Planning 11(4), 2-8

Apparently, this was the first article to discuss the RM. The context was strategic portfolio analysis. Hussey, in describing the RM, asserted that "[t]his three dimensional approach results in what is one of the most sophisticated, the most valuable, and the most understandable approaches to portfolio analysis so far available."

IEC (2009) Risk management -- Risk assessment techniques, IEC/FDIC 31010: 2009(E), Final Draft International Standard, International Electrotechnical Commission

This international standard is a supporting standard for ISO 31000 and provides guidance on selection and application of systematic techniques for risk assessment. The application of a range of techniques is introduced, with specific references to other international standards where the concept and application of techniques are described in greater detail.

IMA (2007) Enterprise Risk Management: Tools and Techniques for Effective Implementation, Montvale, NJ: Institute of Management Accountants.

In this Statement on Management Accounting (SMA), several techniques for identifying risks are discussed and illustrated with examples from company experiences. A suggested approach for determining the root causes (drives) of the risks is described and followed by a discussion of several qualitative and quantitative procedures for assessing risks. Practical ERM implementation considerations are also explored.

Jang, J.-S. R., Sun, C.-T., Mizutani, E. (1997) Neuro-Fuzzy and Soft Computing: A Computational Approach to Learning and Machine Intelligence. Englewood Cliffs, NJ: Prentice-Hall

This book provides an overview of soft computing methodologies, including fuzzy sets, neural networks, genetic algorithms and their composite use for developing high performance intelligent systems. It places particular emphasis on the theoretical aspects of covered methodologies, as well as empirical observations and verifications of various applications in practice. The principles are explained with many examples and illustrations and Matlab is used for visualization and simulations.

Jia, L., Zhang, Y., Tao, L., Jing, H., Bao, S. (2013) "A methodology for assessing cleaner production in the vanadium extraction industry." Journal of Cleaner Production, in press, http://dx.doi.org/10.1016/j.jclepro.2013.05.016.

This study uses a fuzzy AHP model to assess the extent of cleaner production in the vanadium production process. The criteria of assessment indicators included both quantitative and qualitative indicators.

Kaplan, S., Garrick, B. J. (1981) "On the Quantitative Definition of Risk", Journal of Risk Analysis 1(1), 11-27.

This article suggested a quantitative definition of risk in terms of the idea of a “set of triplets,” composed of a scenario identification or description, the probability of that scenario, and the consequence or evaluation measure of that scenario. The definition is extended to include uncertainty and completeness, and the use of Bayes’ theorem is described in this connection. The definition is used to discuss the notions of “relative risk,” “relativity of risk,” and “acceptability of risk.”

Karwowski, W., Mital, A. (1986) "Potential applications of fuzzy sets in industrial safety engineering", Fuzzy Sets and Systems 19(2), 105-120.

This paper discusses potential applications of fuzzy set theory to risk analysis in the area of industrial safety engineering, where the risk analysis utilizes fuzzy conditional statements and compositional rules of inference. Using Mamdani's conjunctive logic, risk is evaluated in terms of a linguistic representation of the likelihood of the occurrence of a hazardous event, exposure, and possible consequences of that event.

Kentel, E., Aral, M. (2004) “Probabilistic-fuzzy health risk modeling,” Stochastic Environment Research Risk Assessment 18: 324-338.

This paper deals with health risk analysis in the case of human exposure to contaminated water. The main goal is to quantify the risk of developing cancer from exposure to this risk agent. The authors make an exposure assessment and a dose-response assessment in order to calculate the health risk, using fuzzy operations and alpha-cut methodology.

Kishore, V., Abraham, D. M., Sinfield, J. V. (2011) “Portfolio Cash Assessment Using Fuzzy Systems Theory,” Journal of Construction Engineering and Management 137(5), 333-343.

This paper proposes a methodology on the basis of fuzzy systems theory (FST) to forecast cash requirements of a portfolio, taking into account the effect of changing portfolio composition on portfolio cash-flow risk. Portfolio cash-flow risk is calculated from a variance matrix created by using covariance among cash flows of pairs of projects. Expert opinions regarding project selection, project risk assessment and cash control were collected to create a fuzzy proportional derivative (PD) model that predicts portfolio risk.

Klir, G. J., Yuan, B. (1996) Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems: Selected Papers by Lotfi A. Zadeh. World Scientific, New Jersey.

This book consists of selected papers written by Lotfi A. Zadeh and published during the period 1949-1995. Since Zadeh has been the principal contributor to the development in the field of fuzzy set theory over the last 30 year of this period, the papers contain virtually all the major ideas in fuzzy set theory, fuzzy logic, and fuzzy systems in their historical context.

Knight, F. H. (1921) Risk, Uncertainty, and Profit. Boston: Houghton Mifflin.

This book represents an attempt to state the essential principles of the conventional economic doctrine more accurately, and to show their implications more clearly, than had previously been done. The particular technical contribution to the theory of free enterprise which this book makes is a fuller and more careful examination of the role of the entrepreneur or enterpriser, the recognized "central figure" of the system, and of the forces which fix the remuneration of his special function. Of particular relevance in the context of this study is the book's discussion of the difference between risk and uncertainty.

Koissi, M.-C., Shapiro, A. F. (2006) “Fuzzy Formulation of the Lee-Carter Model for Mortality Forecasting”, Insurance: Mathematics and Economics, 39, 287-309.

In this paper, a fuzzy formulation of the Lee-Carter (LC) model is proposed. The standard LC model, which uses singular value decomposition, assumes that the errors have a constant variance over all ages. This statement, however, does not often hold. The advantage of the fuzzy approach is that the errors are viewed as fuzziness of the model structure; hence the homoscedasticity is not an issue.

Krätschmer, V. (2001) "A unified approach to fuzzy random variables," Fuzzy Sets and Systems 123(1), 1-9.

The concept of fuzzy random variable was introduced as an analogous notion to random variables in order to extend statistical analysis to situations when the outcomes of some random experiment are fuzzy sets. In this paper, a set-theoretical concept of fuzzy random variable was presented that was shown to be a unification of the previous concepts, in certain situations.

Kumar, G., Maiti, J. (2012) "Modeling risk based maintenance using fuzzy analytic network process," Expert Systems with Applications 39, 9946-9954.

This article uses the analytic network process (ANP) of Saaty (2001) and the extent analysis of Chang (1996) to model maintenance policy selection. Essentially, Chang’s extended analysis is used to develop the fuzzy priority vector and then convert it into a crisp priority vector, and the preferred maintenance policy alternative is determined using the ANP analysis.

Kunreuther, H. (2002) “Risk Analysis and Risk Management in an Uncertain World”, Risk Analysis, 22 (4): 655-664

This paper discusses how one can link the tools of risk assessment and our knowledge of risk perception to develop risk management options for dealing with extreme events. In particular, it suggests ways that risk analysis experts can apply their expertise and talent to the risks associated with terrorism and discusses the changing roles of the public and private sectors in dealing with extreme events.

Kutlu, A. C., Ekmekçioğlu, M. (2012) "Fuzzy failure modes and effects analysis by using fuzzy TOPSIS-based fuzzy AHP," Expert Systems with Applications 39(1): 61.

Failure mode and effects analysis (FMEA) is a risk matrix-type technique whereby, for each failure modes, three risk factors (severity, occurrence, and detectability) are evaluated and a risk priority number is obtained by their multiplication. This study uses an approach to fuzzy FMEA based on integrating the fuzzy AHP of Chang (1996) and the fuzzy TOPSIS of Chen (2000), where the former is used to determine the weight vector of the three risk factors and the latter is used to obtain the most important failure modes.

Lee, S. K., Mogi, G., Hui, K. S. (2013) "A fuzzy analytic hierarchy process (AHP)/data envelopment analysis (DEA) hybrid model for efficiently allocating energy R&D resources: In the case of energy technologies against high oil prices," Renewable and Sustainable Energy Reviews 21, 347-355.

In this paper, the five criteria, economic impact, commercial potential, inner capacity, technical spin-off, and development cost, were used to assess strategic energy technologies against high oil prices. A two-stage multi-criteria decision making (MCDM) approach was used to evaluate the relative weights of criteria. In the first stage, a fuzzy AHP was used to allocate the relative weights of criteria; in the second stage, the data envelopment analysis approach was used to measure the relative efficiency of the energy technologies against high oil prices from an economic perspective.

Liu, B. (2012) Uncertainty Theory, 4th Ed., Uncertainty Theory Laboratory, Tsinghua University

The purpose of this book is to provide an axiomatic approach to deal with human uncertainty, based on four axioms of normality, duality, subadditivity, and product. In it, Liu advocates the use of uncertainty theory, in lieu of fuzzy logic, because the latter violates the law of excluded middle and the law of contradiction, among other reasons.

Liu, H., and Zhang, D.-L. (2012) “Analysis and prediction of hazard risks caused by tropical cyclones in Southern China with fuzzy mathematical and grey models”, Applied mathematics Modelling, 36: 626-637.

In this study, a hazard-risk assessment model and a hazard-year prediction model are constructed by integrating recent advances in fuzzy mathematics, grey theory and information spread technique. First, a genetic fuzzy mathematical algorithm is developed to calculate the categorical and ranking weights of hazard impact and cause indicators. Then, the hazard impact and cause index series are found and a primitive information matrix and a fuzzy relation matrix are created in order to make a fuzzy rough inference of the hazard risks.

Liu, J., Martínez, L., Wang, H., Mª Rodríguez, R., Novozhilov, V. (2010) "Computing with Words in Risk Assessment," International Journal of Computational Intelligence Systems 3(4), 396-419

This paper presents an overview of risk assessment applications of computing with words. The primary fuzzy logic topics that were used for the risk assessment applications included fuzzy numbers, fuzzy rule-based processes, fuzzy extensions of typical probabilistic methodology, and the ordinal linguistic approach.

Lootsma, F. A. (1981) "Performance evaluation of nonlinear optimization methods via multi-criteria decision analysis and via linear model analysis," in: M. J. D. Powell, Ed., Nonlinear Optimization 1981 (Academic Press, London, 1982) 419-453.

This article showed that logarithmic regression could extend Saaty's hierarchical analysis to the case of multiple estimates for the comparison ratios and situations where there were no estimates for certain ratios (missing data).

Markowski, A. S., Mannan, M. S. (2008) "Fuzzy risk matrix," Journal of Hazardous Materials 159, 152-157.

This paper describes a procedure for developing a fuzzy risk matrix based on a Mamdani fuzzy inference system. A unique aspect of the study is that it develops three types of fuzzy risk matrixes (low-cost, standard-cost, and high-cost) by the rezoning of cells within the matrix.

Mathworks (2006) Fuzzy logic toolbox. User's Guide. Natick, MA.

This toolbox is a collection of fuzzy logic functions built on the MATLAB numeric computing environment. Of particular relevance to this project is that it provides tools for creating and editing fuzzy inference systems, and displaying the results in a 3-D surface view.

Miri Lavasani, S. M., Yang, Z., Finlay, J., Wang, J. (2011) "Fuzzy risk assessment of oil and gas offshore wells," Process Safety and Environmental Protection 89(5), 277-294.

This article used fuzzy logic in the RA of oil and gas offshore wells. Essentially, basic risk items in a hierarchical framework are expressed as fuzzy numbers, which are combinations of the likelihood of a failure event and the associated failure consequence, and the AHP is used to estimate the weights required for grouping non-commensurate risk sources.

Moore, R. E. (1966) Interval Analysis, Englewood Hills, N.J: Prentice-Hall.

This book is primarily concerned with the analysis of closed intervals, and is the forerunner of much of the modern literature on interval arithmetic.

Musilek, P., Gupta, M. M. (2000) Neural networks and fuzzy systems. In: Sinha, N. K., Gupta, M. M. (Eds) Soft computing and intelligent systems. Academic Press, 137-160

This chapter gives an overview of the two constituent fields of fuzzy neural networks, neural networks and fuzzy systems. It begins with an introduction to the theory of neural networks and a detailed treatment of single-neuron structures. This is followed by an introduction to fuzzy sets and fuzzy numbers and the mathematical tools used to manipulate them. The remainder of the chapter is concerned with the synergy between the neural networks and fuzzy systems.

Nieto-Morote, A., Ruz-Vila, F. (2011) "A fuzzy approach to construction project risk assessment," International Journal of Project Management 29(2), 220-231.

This article presents a risk assessment methodology based on fuzzy logic, to deal with subjective judgment, and on the AHP, which is used here to structure a large number of risks. Trapezoidal fuzzy numbers are used to capture the vagueness in the linguistic variables and an algorithm is used to handle the inconsistencies in the fuzzy preference relation when pair-wise comparison judgments are necessary.

Ostaszewski, K. (1993) Fuzzy Set Methods in Actuarial Science. Society of Actuaries, Schaumburg, IL.

This book presents the basic concepts of fuzzy set theory, including fuzzy sets, fuzzy measures, and approximate reasoning. In all cases, references for further studies, and for investigations of more advanced concepts, are provided. Then, a review of some fundamental concepts of actuarial science is given from the perspective of possible applications of fuzzy set-theoretic methods.

Ostrom, L. T., Wilhelmsen, C. A. (2012) Risk Assessment: Tools, Techniques, and Their Applications, John Wiley & Sons.

Designed as a practical, in-the-field toolkit, this book details how a risk assessment is performed, showing the proper tool to be used at various steps in the process, as well as locating the tool that best fits the risk assessment task at hand. It progresses from simple to more complex risk assessment techniques, all of which have been used by the authors in their daily work.

Panjer, H. (2006) "Operational Risk: Modeling Analytics", Wiley Series in Probability and Statistics.

This book is designed to provide a framework of the mathematical models and methods used in the measurement and modeling of operational risks in both the banking and insurance sectors. Beginning with a foundation for operational risk modeling and a focus on the modeling process, it progresses to a discussion of probabilistic tools for operational risk modeling and statistical methods for calibrating models of operational risk.

Peña-Reyes, C. A., Sipper, M. (1999), A fuzzy-genetic approach to breast cancer diagnosis. Artificial Intelligence in Medicine 17, 131–155.

This article used genetic algorithm-constructed FISs to automatically produce diagnostic systems for breast cancer diagnosis. The Pittsburgh-style of genetic algorithms (every individual is encoded as a string with variable length) was used to generate the database and rulebase for the FISs, based on data furnished by specialists. They claimed to have obtained the best classification performance to date for breast cancer diagnosis and, because their final systems involve just a few simple rules, high human-interpretability.

PricewaterhouseCoopers (2008) "A Practical Guide to Risk Assessment" http://www.pwc.com/en_US/us/issues/enterprise-risk-management/assets/risk_assessment_guide.pdf

This document provides practical guidance on RA by examining the issues and detailing the benefits and opportunities available to organizations that systematically embed RAs into their existing business processes. The topics discussed include the role of RA as the foundation of an effective enterprise risk management program, key principles for effective and efficient RA, and essential steps for performing a RA.

PricewaterhouseCoopers (2009) “Extending Enterprise Risk Management (ERM) to address emerging risks”. http://www.pwc.com/gx/en/research-publications/pdf/pwcglobalriskserm.pdf

This document makes the case that organizations need to take a new look at their risk management processes and allocation of resources to ensure that emerging risks are effectively identified, assessed, and managed from strategic planning to day-to-day processes at all levels of the organization. Moreover, to address risks that may seem unknown or unknowable, organizations must adopt a systematic approach to emerging risk identification, assessment, and management. The article goes on to discuss the nature of the steps of this systematic approach.

Rachev, S., Stoyanov, S., Fabozzi, F. (2008) Advanced Stochastic Models, Risk Assessment, and Portfolio Optimization: The Ideal Risk, Uncertainty, and Performance Measures, Wiley.

This book extends traditional approaches of risk measurement and portfolio optimization by combining distributional models with risk or performance measures into one framework. Topics addressed include the fundamentals of probability metrics, new approaches to portfolio optimization, and a variety of essential risk measures. Numerous examples are provided.

Ruan, D., Huang, C. (2000) Fuzzy Sets and Fuzzy Information - Granulation Theory, Beijing Normal University Press

This is a book of key selected papers by Lotfi A. Zadeh. Briefly, it includes his papers on fuzzy sets, fuzzy systems, linguistic variables, fuzzy if-then rules, fuzzy graphs, information granularity, and computing with words. The book is intended as a quick reference for those working in fuzzy mathematics and engineering.

Saaty, T. L. (1978) "Exploring the interface between hierarchies, multiple objectives and fuzzy sets", Fuzzy Sets and Systems 1(1), January.

This paper gives a method for measuring the relativity of fuzziness by structuring the functions of a system hierarchically in a multiple objective framework. It is here that we find an early assertion by Saaty that his 9-point scale provides a measure of the grade of membership of elements in a fuzzy set. In later articles he would maintain that this is the reason why it is redundant to fuzzify the AHP.

Saaty, T. L. (1980) The Analytic Hierarchy Process, New York: McGraw-Hill.

This is the Saaty's classical book on the AHP, which is an effective decision-making tool for planning, structuring priorities, weighing alternatives, allocating resources, analyzing policy impacts and resolving conflicts. The book presents the theory underlying the AHP, and provides examples and applications of its use.

Saaty, T. L. (1999) "The Seven Pillars of the Analytic Hierarchy Process," http://www.ergonomia.ioz.pwr.wroc.pl/download/AhpSaatyTheSevenPillars.pdf.

This article discusses the highlights of the seven pillars of the AHP, which are: ratio scales, paired comparisons, conditions for eigenvector sensitivity, homogeneity and clustering to extend the scale, additive synthesis of priorities, allowing rank preservation or rank reversal, and group decision making that synthesizes individual judgments.

Saaty, T. L. (2001) Decision making with dependence and feedback: The analytic network process. Pittsburgh, PA: RWS Publications.

The ANP provides a general framework to deal with decisions without making assumptions about the independence of higher level elements from lower level elements and about the independence of the elements within a level. This book uses the ANP to show how to make decisions when alternatives and criteria are interdependent, and how to cope with dependence between different groups of people, goals and criteria.

Saaty, T. L., Tran, L. T. (2007). "On the invalidity of fuzzifying numerical judgments in the Analytic Hierarchy Process." Mathematical and Computer Modelling 46, 962-975.

This article makes the case that it is redundant to fuzzify the AHP because when judgments are allowed to vary in choice over the values of a fundamental scale, as in the AHP, these judgments are themselves already fuzzy. Moreover, making them fuzzier can make the validity of the outcome worse.

Saaty, T. L. (2008) "Decision making with the analytic hierarchy process," Int. J. Services Sciences 1(1), 83-98.

This paper describes the AHP and expounds on related topics, such as priority scales and consistency. It also presents an illustration.

Saaty, T. L., Vargas, L. G. (2012) Models, Methods, Concepts & Applications of the Analytic Hierarchy Process, Second Edition.

This book is a collection of selected applications of the AHP in economic, social, political and technological areas. Its focus is the three themes: economics, the social sciences, and the linking of measurement with human values.

Segal, S. (2011) Corporate Value of Enterprise Risk Management: The Next Step in Business Management, Wiley Corporate F&A.

This book presents an ERM approach that is centrally focused on measuring, protecting, and increasing company value. Features of the book include a presentation of 10 key ERM criteria for evaluating the robustness of any ERM program, a discussion of techniques to avoid the five common mistakes in risk identification, and case studies that illustrate key elements of the value-based ERM approach.

Shah, S. (2003) “Measuring Operational Risk Using Fuzzy Logic Modeling.” http://www.irmi.com/expert/articles/2003/shah09.aspx

This article provides a nontechnical description of how fuzzy logic modeling techniques can be used to assess operational risks. After giving examples of membership functions based on key risk indicators, the mechanics of deriving the expected loss is hinted at, and a risk matrix is shown that relates market conduct risk to product complexity and agent years of experience.

Shang, K., Hossen, Z. (2013) Applying Fuzzy Logic to Risk Assessment and Decision-Making, CAS/CIA/SOA Joint Risk Management Section

This study explores areas where fuzzy logic models may be applied to improve risk assessment and risk decision-making. It discusses the methodology, framework and process of using fuzzy logic systems for risk management, and presents practical examples.

Shapella, A., Stein, O. (2012) "Understand ORSA Before Implementing It," in SOA (2012) “Risk Metrics for Decision Making and ORSA” 3rd essay e-book, 34-37. http://www.soa.org/Library/Essays/orsa-essay-2012-toc.aspx

This article concerns the regulatory requirement of the NAIC that U.S. insurers perform an Own Risk and Solvency Assessment (ORSA). Its purpose is to provide an overview of the evolution and rationale for ORSA, as well as practical implications for insurers with respect to the design of an ORSA process.

Shapiro, A. F. (1986) “Applications of Operations Research Techniques in Insurance”, in Insurance and Risk Theory, ed. M. Goovaerts, E de Vylder, and J. Haezendonck. Norwell, Mass.: Kluwer Academic Publishers, 129-143

A literature review was conducted as part of a study of applications of operations-research techniques in insurance. This paper provides a cursory overview of some of the literature reviewed.

Shapiro, A. F. (2002) "The Merging of Neural Networks, Fuzzy Logic, and Genetic Algorithms," Insurance: Mathematics and Economics 31, 2002, 115–131

This article presents an overview of the merging of NNs, FL and GAs. The topics addressed include the advantages and disadvantages of each technology, the potential merging options, and the explicit nature of the merging.

Shapiro, A. F. (2003) “Capital Market Applications of Neural Networks, Fuzzy Logic and Genetic Algorithms”, 13th International AFIR Colloquium, 1, 493-514.

This paper presents an overview of studies that have focused on capital market applications of neural networks, fuzzy logic and genetic algorithms. The specific purposes of the paper are twofold: first, to review the capital market applications of these technologies so as to document the unique characteristics of capital markets as an application area; and second, to document the extent to which these technologies, and hybrids thereof, have been employed.

Shapiro, A. F. (2004) "Fuzzy logic in insurance," Insurance: Mathematics and Economics 35, 399-424.

This article presents an overview of studies that have focused on insurance applications of fuzzy logic (FL). Application areas addressed include classification, underwriting, projected liabilities, fuzzy future and present values, pricing, asset allocations and cash flows, and investment. The specific purposes of the article are two-fold: first, to review FL applications in insurance so as to document the unique characteristics of insurance as an application area; and second, to document the extent to which FL technologies have been employed.

Shapiro, A. F. (2013) "Modeling Future Lifetime as a Fuzzy Random Variable," Insurance: Mathematics and Economics 53(3), 864-870

A recent article modeled life annuities as fuzzy random variables (FRVs). However, the authors assumed that the uncertainty insofar as mortality is entirely due to randomness and that the uncertainty with respect to interest rates is entirely due to fuzziness. The concern is that such a dichotomy may be problematic since, in actuality, the uncertainty of both the mortality parameter and the interest rate parameter can have both random and fuzzy features. The purpose of this article is to address the mortality portion of this dichotomy and, to this end, we model future lifetime as a FRV.

Shevchenko, P. V., Wüthrich, M. V. (2006) “The Structural Modelling of Operational Risk via Bayesian Inference: Combining Loss Data with Expert Opinion”, Journal of Operational Risk, 1(3), 3-26.

This paper presents examples of the Bayesian inference methods for operational risk quantification, in light of the Basel II regulatory requirements that the internal model of a bank must include the use of internal data, relevant external data, scenario analysis and factors reflecting the business environment and internal control systems. In this context, the role of scenario analysis is to estimate the frequency and severity of risk events based on expert opinions with respect to the aforementioned factors.

Shi, S., Jiang, M., Liu, Y., Li, R. (2012) "Risk assessment on Falling from Height based on AHP-fuzzy," Procedia Engineering 45, 112-118.

This paper used AHP and the fuzzy comprehensive evaluation method to assess the risk of falling from a height on a construction site.

Sinha, N. K., Gupta, M. M. (Eds.) (2000) Soft Computing and Intelligent Systems: Theory and Applications. Academic Press, San Diego, CA.

This book is intended as a comprehensive view of the general field of soft computing and intelligent control systems. To this end, it has sections on the foundations of soft computing and intelligent control systems, the theory of soft computing and intelligent control systems, the implications and applications of intelligent control, and future perspectives. Moreover, the book is intended to be pedagogically sound.

Sivanandam, S. N., Sumathi, S., Deepa, S. N. (2007) Introduction to Fuzzy Logic using MATLAB, Springer-Verlag Berlin Heidelberg

This book is designed to give a broad, yet in-depth, overview of the field of fuzzy systems, with a MATLAB orientation. The topic covered include: an introduction to fuzzy logic and Matlab, the definition, properties, and operations of fuzzy sets, membership functions, the process of fuzzification, the process and methods of defuzzification, fuzzy rule-based system, and the Matlab fuzzy logic tool box.

Smithson, M., Verkuilen, J. (2006) Fuzzy Set Theory: Applications in the Social Sciences. Sage Publications, Thousand Oaks, CA

This book, which was written for sociologists, seeks to combine the intuitive appeal of fuzzy sets with the rigor of standard quantitative analysis. Because of its mathematical comprehensiveness and detail, however, its readers need to already possess some familiarity with fuzzy logic.

SOA (2008) Risk Management: The Current Financial Crisis, Lessons Learned and Future Implications. www.soa.org/library/essays/rm-essay-2008.pdf

This book of 35 short essays highlights key lessons learned with respect to the financial crisis of 2007-08. Topics covered included the impact of combining risks, the need for a risk culture that balances incentive compensation with desired performance, and the need to align the authority to make decisions with bottom line accountability.

SOA (2011) “Risk Management: Part Two - Systemic Risk, Financial Reform, and Moving Forward from the Financial Crisis.” www.soa.org/library/essays/fin-crisis-essay-2011.pdf

The intent of this publication is to offer thought leadership on the ERM discipline and the essential elements needed to maintain risk transfer systems in times of unusual stresses and unlikely events. It contains the opinions of authors written in response to a call for essays from the actuarial societies. The topics covered included modeling, role of government vs. role of market, emerging systemic risks, treating systemic risk, company management/board governance, and regulation.

Subramanian, N., Ramanathan, R. (2012) "A review of applications of Analytic Hierarchy Process in operations management," Int. J. Production Economics 138, 215-241.

This paper presents a review of the AHP articles that appeared in peer reviewed journals during 1990 to 2009, which focused on applications in operations management. They found few papers that contributing to the theory of AHP modeling, and that there was a significant research gap in the application of the AHP in the areas of forecasting, layout of facilities and stock management.

Sweeting, P. (2011) Financial enterprise risk management, Cambridge University Press, New York

This book presents the procedures needed to build and maintain an ERM framework for a financial institution. As well as outlining the construction of such frameworks and an assortment of risk mitigation strategies, it discusses the internal and external contexts within which risk management must be carried out and the range of qualitative and quantitative techniques that can be used to identify, model and measure risks.

Vaidya, O., Kumar, S. (2006) "Analytic hierarchy process: An overview of applications," European Journal of Operational Research 169(1), 1-29.

This article presents a literature review of applications of the AHP. Using various tabular formats and charts, papers are categorized according to themes, areas of applications, region and year. A total of 150 application papers are cited, of which 27 are critically analyzed.

van Laarhoven, P. J. M., Pedrycz, W. (1983). A fuzzy extension of Saaty's priority theory. Fuzzy Sets and Systems, 11, 229–241.

This article was the first to present a fuzzy version of the AHP. The main features of the approach were the following: logarithmic least squares were used to derive fuzzy weights and fuzzy performance scores, the membership functions took the form of triangular fuzzy numbers, approximate fuzzy multiplication was used, and multiple decision-makers were accommodated.

Vargas, L. G. (1990). "An overview of the analytic hierarchy process and its applications." European Journal of Operational Research 48(1), 2-8.

This article begins with an overview of principles and axioms associated with the AHP. Topics discussed include the design of a hierarchy, hierarchy evaluation, and the four axioms of the AHP, the axioms of reciprocal comparison, homogeneity, independence, and expectations. Thereafter, areas of research and applications are discussed.

Wall, K. D. (2011) "The trouble with risk matrices," Working paper, Naval Postgraduate School (DRMI)

In this article, Wall contends that the theoretical basis of the risk matrix is superficial and the validity of the qualitative information it employs is highly suspect.

Wang, Y.-M., Elhag, T. M. S., Hua, Z. (2006) "A modified fuzzy logarithmic least squares method for fuzzy analytic hierarchy process," Fuzzy Sets and Systems 157, 3055-3071.

This paper critiques the van Laarhoven and Pedrycz (1983) method for fuzzifying the AHP. The problems they identify are related to the normalization of local fuzzy weights, inconsistent lower and upper bound values, incomplete fuzzy comparison matrices, and global fuzzy weights.

Wang, Y.-M., Luo, Y., Hua, Z. (2008) "On the extent analysis method for fuzzy AHP and its applications," Eur. J. Oper. Res. 186, 735-747.

This paper critiques the Chang (1996) extent analysis method for fuzzifying the AHP. In it, examples are used to show that the priority vectors determined by the method do not represent the relative importance of decision criteria or alternatives, and that misapplication of the method may cause the exclusion of some useful decision information.

Wu, W., Cheng, G., Hu, H., Zhou, Q. (2013) "Risk analysis of corrosion failures of equipment in refining and petrochemical plants based on fuzzy set theory," Engineering Failure Analysis 32, 23-34.

This paper presents an explicit description of a model for analyzing risk of corrosion failures of refining and petrochemical equipment based on the combination of fuzzy synthetic evaluation and fuzzy logic. The four parts of the model are discussed, which includes the identification of the source of failure, the estimation of failure likelihood, the estimation of severity of failure consequence, and the determination of a risk index.

Yager, R. R., Ovchinnikov, S., Tong, R. M., Ngugen, H. T. (1987) Fuzzy Sets and Applications: Collected Papers of Lotfi A. Zadeh. John Wiley & Sons, New York.

This book is a collection of reprints of almost all the basic papers of Lotfi A. Zadeh during the first 20 years of fuzzy logic. Topics include linguistic modelling of complex systems, approximate reasoning, and the use of fuzzy logical and natural language tools in expert systems. The book is relatively easy reading, due to Zadeh's mainly intuitive and heuristic approach to fuzzy logic during this period.

Zadeh, L. A. (1965). “Fuzzy sets,” Information and Control 8(3), 338-353.

This is the seminal paper in which Zadeh introduced the concept of fuzzy sets. The topics addressed included membership functions and the fuzzy set context of notions such as inclusion, union, intersection, complement, relation, and convexity.

Zadeh, L. A. (1968) "Probability measures of fuzzy events," J. Math. Ann. Appl. 23(2), 421-427.

This article uses the concept of a fuzzy set to extend the notions of an event and its probability to fuzzy events. It has the limited objective of showing how the notion of a fuzzy event can be given a precise meaning in the context of fuzzy sets.

Zadeh, L. A. (1975, 1976) The concept of linguistic variable and its application to approximate reasoning (Parts 1-3). Information Sciences 8:199–249, 301–357, and 9:43–80

These articles define the concept of a linguistic variable, that is, a variable whose values are words or sentences in a natural or artificial language. Related concepts, such as term-set, compatibility function, and hedges, are also defined. The point is made that by providing a basis for approximate reasoning, fuzzy logic may offer a more realistic framework for human reasoning than the traditional two-valued logic.

Zadeh, L. A. (1981) "Fuzzy systems theory: a framework for the analysis of humanistic systems" In: Cavallo, R.E. (Ed.), Recent Developments in Systems Methodology in Social Science Research. Kluwer, Boston, 25-41.

This article seeks to convey the perception of some of the basic ideas that underlie fuzzy systems theory. In it, Zadeh promoted the term fuzzy systems theory because he believed that its distinguishing characteristic will be a conceptual framework for dealing with the pervasive fuzziness of almost all phenomena that are associated with human behavior.

Zadeh, L. A. (1994) “The Role of Fuzzy Logic in Modeling, Identification and Control,” Modeling Identification and Control, 15(3), 191-203.

In this article, the basic ideas underlying fuzzy logic and its applications to modeling, identification and control are described and illustrated by examples. The case is made that in order to design systems having a high Machine Intelligence Quotient, soft computing, rather than hard computing, may be needed, where at that juncture, the principal constituents of soft computing were fuzzy logic, neurocomputing and probabilistic reasoning.

Zadeh, L. A. (2008) “Is there a need for fuzzy logic”, Information Sciences 178(13), 2751-2779.

This paper makes the case that the progression from bivalent logic to fuzzy logic is a significant positive step in the evolution of science because, in large measure, the real-world is a fuzzy world. To deal with fuzzy reality what is needed is fuzzy logic. The paper goes on to discuss the distinguishing features and important contributions of fuzzy logic.

Zadeh, L. A. (2012) Fuzzy Logic, Department of EECS, University of California, Berkeley, 1177-1200

This article presents fuzzy logic in a nontraditional perspective based on graduation, granulation, precisiation and the concept of a generalized constraint. In addition to defining and elaborating on these and related terms, the conceptual structure and principal contributions of fuzzy logic are discussed.

Zeng, J., An, M., Smith, N. J. (2007) "Application of a fuzzy based decision making methodology to construction project risk assessment." International Journal of Project Management 25(6): 589.

This article presents a RA methodology, based on a modified AHP, to cope with risks in complicated construction situations. To begin, fuzzy aggregation, based on trapezoidal membership functions, is used to create group decisions. Then defuzzification is employed to transform the fuzzy number scales into crisp scales for the computation of priority weights.

Zhang, H., Gao, D., Liu, W. (2012) "Risk assessment for Liwan relief well in South China Sea," Engineering Failure Analysis 23, 63-68.

This paper proposes a multi-factor risk assessment model for a relief well based on the AHP and the fuzzy comprehensive evaluation method.

Zhang, L., Xu, X., Tao, L. (2013), "Some Similarity Measures for Triangular Fuzzy Number and Their Applications in Multiple Criteria Group Decision-Making," Journal of Applied Mathematics, Article ID 538261, 1-7.

The authors propose some similarity measures between two TFNs. A methodology for multiple criteria group decision-making (MCGDM) problems with triangular fuzzy information is proposed; the criteria values take the form of linguistic values, which can be converts to TFNs.

Zimmermann, H. J. (1996) Fuzzy Set Theory and its Applications, 3 ed. Kluwer Academic Publishers, Boston, MA.

The primary goal of this book is to provide a textbook for courses in fuzzy set theory, as well as a resource that can be used as an introduction to the subject. The chapter topics include possibility theory, fuzzy logic and approximate reasoning, expert systems, fuzzy control, fuzzy data analysis, decision making and fuzzy set models in operations research.

