THE TABLE OF ISOLATED MORTALITY

by

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Many will agree that the impact of the computer revolution on the actuarial profession has forced us to seek new ways to do our old things, quite often with results that surprise us, because it never occurred to us to look at the problem from any but the 'traditional' perspective. The classical methods of approximating an annuity value are predicated on the assumption that the interest element of the immediate annuity function is mathematically so inextricably entangled with the mortality element that it is not possible to separate the two, hence the necessity for approximation. We shall show that this premise, if not false, at least should be qualified. The mortality element can be partially isolated in a particular manner, allowing an exact annuity value to be readily obtained at any interest rate, as we shall presently show.

Analysis will show that the immediate annuity, a_x, may be expressed as a polynomial in the interest function d (or i) with the coefficients of each term being uniquely determined by the mortality element alone. Moreover, and this is the whole basis for this paper, these coefficients are readily available from a table generated by the successive summations of the l_x column of the pertinent mortality table. For want of a more appropriate nomenclature, we shall refer to this as the Table of Isolated Mortality, or TIM, for short. The derivation of the immediate annuity polynomial is shown in the Appendix.

As some amplification on the nature of the beast appears to be in order here, the polynomial is given below for convenience.

$$a_x = \frac{1}{l_x}\left[S^1_{x+1} - S^2_{x+1}.d + S^3_{x+2}.d^2 - S^4_{x+3}.d^3 + S^5_{x+4}.d^4 - S^6_{x+5}.d^5 + \ldots \right]$$

The coefficients S^i_y refers to the element in the Table of Isolated Mortality corresponding to age y in the i^{th} summation column, and $S^i_y = \sum_{t=y}^{\omega} S^{i-1}_t$

In theory, the polynomial should extend to the $(\omega - x)^{th}$ term. In practice, the polynomial may be truncated after a few terms depending on the interest rate for a value of a_x correct to, say, three decimal places. The practicality of this method depends on this property of the polynomial. The reason for this will become apparent when one considers the fact that while the coefficients increase in linear compound fashion with each successive term, the values of d^t decrease exponentially, so that convergence is quite rapid after the first few terms, the value of the truncated polynomial oscillating about the true value of a_x with each added term. If the interest function is i instead of d, convergence is assumptotic. The writer's choice of d instead of i is based on a belief that convergence is slightly better with the former. The precise number of terms required for a certain degree of accuracy depends on the interest rate and the mortality basis. The higher the interest rate, the flatter the mortality curve, the slower the rate of convergence.

As a vehicle for illustration, Table 1 has been prepared by generating the successive summations of the l_x column of the 1971 Group Annuity Male Mortality Table. The reader will note that beginning from the left, each successive column to the right is generated by summing the values of the one on its left, it being understood that summation is from age ω.

To illustrate how the table is to be used, let us write the immediate annuity polynomial for a male aged 60. As indicated in the formula, the coefficients will be along the solid line shown in the table. Note that the line starts at age 60+1, and <u>after the second term</u>, proceeds step-wise down the table. The polynomial, truncated after the 8^{th} term, is

$$\frac{1}{.8723878E+06}\left[\begin{array}{l} .1592913E+08 - (.1903783E+09)d + (.1566010E+10)d^2 - \\ (.1018604E+11)d^3 + (.5496143E+11)d^4 - (.2534858E+12)d^5 + \\ (.1020249E+13)d^6 - (.3637164E+13)d^7 + (.1161236E+14)d^8 - \ldots \end{array}\right]$$

for any interest rate. Values of d^t can be generated without too much effort by the relationship $d = \dfrac{i}{1+i}$.

Let us evaluate a_{60} for a particular value of i. For i = 4% , the value is 10.942.

II

It may have occurred to some readers that the increasing immediate annuity, Ia_x, may be susceptible to the same kind of analysis, and if so, the question comes to mind whether the coefficients of the resulting polynomial may be found from the Table if Isolated Mortality. The answer is, not surprisingly, yes. The derivation of the polynomial is shown in the Appendix. Note that the polynomial may be expressed in two equivalent forms. As a matter of choice the first form will be used in our illustrative example, but no significant advantage is seen in choosing one form over the other. Again, for convenience, we give below the required polynomial:

$$Ia_x = \frac{1}{l_x}\left[\begin{array}{l} S^2_{x+1} - (2S^3_{x+1} - S^2_{x+1})d + (3S^4_{x+2} - S^3_{x+2})d^2 - (4S^5_{x+3} - S^4_{x+3})d^3 + \\ (5S^6_{x+4} - S^5_{x+4})d^4 - (6S^7_{x+5} - S^6_{x+5})d^5 + \ldots\ldots \end{array}\right]$$

Note that each coefficient is a linear compound of two values from our table. As an example, let us write the increasing immediate annuity polynomial for a male aged 65. The coefficients are obtained along the broken lines shown in the table. Ia_{65} equals

$$\frac{1}{.8047687E+06}\left\{\begin{array}{l} .1192126E+09 - (2*.9552425E+09 - .1192126E+09)d + \\ (3*.5520945E+10 - .8360300E+09)d^2 - (4*.2660571E+11 - .4684910E+10)d^3 \\ + (5*.1099848E+12 - .2192079E+11)d^4 - (6*.3976378E+12 - .8806407E+11)d^5 \\ + (7*.1274764E+13 - .3095732E+12)d^6 - (8*.3660934E+13 - .9651911E+12)d^7 \\ + (9*.9491682E+13 - .2695743E+13)d^8 - \ldots \end{array}\right.$$

for any interest rate. For i = 5%, the value is 75.586

III

There are two other annuity polynomials which follow naturally from the derivation of the two just discussed, and these are the deferred life annuity, $n|a_x$, and the deferred increasing annuity, denoted here as $n|Ia_x$. No new concepts are involved in their derivation, and we give them here below:

$$n|a_x = \frac{v^n}{l_x}\left[S^1_{x+n+1} - S^2_{x+n+1}.d + S^3_{x+n+2}.d^2 - S^4_{x+n+3}.d^3 + S^5_{x+n+4}.d^4 - \ldots\ldots \right]$$

$$n|Ia_x = \frac{v^n}{l_x}\left(\begin{array}{l} S^2_{x+n+1} - (2S^3_{x+n+1} - S^2_{x+n+1})d + (3S^4_{x+n+2} - S^3_{x+n+2})d^2 - \\ 4S^5_{x+n+3} - S^4_{x+n+3})d^3 + (5S^6_{x+n+4} - S^5_{x+n+4})d^4 - \ldots\ldots\ldots \end{array} \right)$$

These two additional polynomials, while allowing us to obtain two additional annuities by means of TIM, also serve a much more important function. Taking advantage of the basic relationships between annuities and insurances, it is evident that most if not all single life standard annuity and insurance functions can be expressed as a linear combination of some or all of the four annuities for which we have derived polynomials. For example, take a complex function such as the decreasing insurance, $DA^{1}_{x:\overline{n|}}$. This can be shown to be equal to

$$v(n(a_x + 1) - Ia_x + n|Ia_x) - (n.a_x - 1|Ia_x + (n+1)|Ia_x)$$

Thus TIM proves to be a much more powerful tool than just another approximation method for simple annuities. If one has the facilities of a computer on a time-sharing basis, the successive summations of the l_x values may be generated simultaneously without much imaginative programming just once and stored. We could then obtain any single life actuarial function at a wide variety of interest rates without having to generate commutation functions each time.

IV

Other applications

There is another type of annuity to which TIM may be used with advantage. Suppose we have the situation in which the annual annuity payment escalates at some fixed rate j per year, as in a cost-of-living benefit. More precisely, suppose we have an annuity-due where the first annual payment is 1, the next (1+j), and the next $(1+j)^2$, etc. The present value of this annuity would be

$$1 + vp_x(1+j) + v^2{}_2p_x(1+j)^2 + v^3{}_3p_x(1+j)^3 + \ldots\ldots\ldots$$

$$= \sum_{t=0}^{\infty} \frac{(1+j)^t}{(1+i)^t} \cdot {}_tp_x = \sum_{t=0}^{\infty} (v')^t \cdot {}_tp_x = \ddot{a}'_x \quad \text{where } i' = \frac{(1+i)}{(1+j)} - 1$$

In the special case where j is greater than i, $\ddot{a}'_x$ will be of the form

$$\sum_{t=1}^{\infty} (1+i'')^t \cdot {}_tp_x \quad \text{where } i'' = \frac{(1+j)}{(1+i)} - 1$$

TIM takes it all in stride. The coefficients of the polynomial are obtained in exactly the same fashion. The polynomial undergoes a slight transformation however. d is replaced by i'', and instead of the alternating signs, all signs are now positive. To understand why this is so, one need only recall the binomial expansion of $(1+i'')$ and the associated sign rule. The transformed polynomial thus becomes

$$a'_x = \frac{1}{l_x}\left[S^1_{x+1} + S^2_{x+1}\cdot i + S^3_{x+2}\cdot i^{2} + S^4_{x+3}\cdot i^{3} + S^5_{x+4}\cdot i^{4} + S^6_{x+5}\cdot i^{5} + \ldots \right]$$

Joint-Life Annuities

If a particular mortality table follows Makeham's First Law, $u_x = A + Bc^x$, joint-life annuity values may also be obtained from TIM. Makehamized mortality tables are not so much in vogue to-day and Makeham's famous Law seems to be destined for the history books, and so the technique will not have the degree of utility that it might have had say a couple of decades ago, but it will be given here, if only to demonstrate the versatility of TIM.

In The Record, 1930 - Jones, it was demonstrated that where the mortality basis follows Makeham's Law, a joint-life annuity is identical to a single life annuity, with a change in the interest rate, and a change in the age x to z, as defined below. TIM provides a practical method for utilizing Jones' discovery.

Thus, $a_{x_1 x_2 x_3 \ldots x_n} = a_{yyy \ldots y_n}$ at interest rate $i = a'_z$ at interest rate i' where $i' = \frac{1 + i}{s^{n-1}} - 1$, and $z = y - \frac{\ln n}{\ln c}$

The joint age y, of course, comes from the Uniform Table of Seniority, s and c are the Makeham constants, and n the number of lives.

Since z is usually not an integral age, two values of a'_x on either side of a'_z will be needed. We may then interpolate linearly for a'_z .

V

Critique

It is not suggested here that TIM can replace that redoubtable cornerstone of actuarial science, the Commutation Function. Rather it is offerred here as an alternative in the special applications where only a few annuity values are required, but for a wide range of interest rates. Indeed, in this age of the high-speed electronic computers, there will be many who will regard the gain in computational efficiency as insignificant, preferring to grind out the values via the traditional commutation function route. Be that as it may; ingrained procedures will not be given up readily, the law of mathematical parsimony notwithstanding. Imaginative research in actuarial mathematics, however, as in the other mathematical disciplines, is not predicated on a prerequisite of utility; all that is required is that the frontiers of academic knowledge be pushed back a little bit. Moreover, a new look at an old problem can sometimes be quite refreshing. As to utility, the verdict will be left to the individual reader to decide.

TIM

x	LX	SUM1	SUM2	SUM3	SUM4	SUM5	SUM6	SUM7
15	0.9959165E+06	0.6037778E+08	0.1949927E+10	0.4396952E+11	0.7727655E+12	0.1123630E+14	0.1403195E+15	0.1543984E+16
16	0.9954852E+06	0.5938187E+08	0.1889549E+10	0.4201960E+11	0.7287960E+12	0.1046353E+14	0.1290832E+15	0.1403665E+16
17	0.9950432E+06	0.5838640E+08	0.1830168E+10	0.4013005E+11	0.6867764E+12	0.9734738E+13	0.1186197E+15	0.1274582E+16
18	0.9945885E+06	0.5739136E+08	0.1771781E+10	0.3829989E+11	0.6466463E+12	0.9047962E+13	0.1088850E+15	0.1155962E+16
19	0.9941200E+06	0.5639678E+08	0.1714390E+10	0.3652811E+11	0.6083465E+12	0.8401316E+13	0.9983700E+14	0.1047077E+16
20	0.9936369E+06	0.5540267E+08	0.1657993E+10	0.3481372E+11	0.5718184E+12	0.7792470E+13	0.9143564E+14	0.9472405E+15
21	0.9931371E+06	0.5440904E+08	0.1602591E+10	0.3315573E+11	0.5370047E+12	0.7221152E+13	0.8364273E+14	0.8558050E+15
22	0.9926187E+06	0.5341590E+08	0.1548182E+10	0.3155314E+11	0.5038490E+12	0.6684148E+13	0.7642158E+14	0.7721624E+15
23	0.9920787E+06	0.5242330E+08	0.1494766E+10	0.3000496E+11	0.4722959E+12	0.6180300E+13	0.6973743E+14	0.6957409E+15
24	0.9915172E+06	0.5143123E+08	0.1442343E+10	0.2851019E+11	0.4422910E+12	0.5708004E+13	0.6355714E+14	0.6260036E+15
25	0.9909312E+06	0.5043973E+08	0.1390412E+10	0.2706785E+11	0.4137808E+12	0.5265714E+13	0.5784915E+14	0.5624466E+15
26	0.9903178E+06	0.4944880E+08	0.1340472E+10	0.2567694E+11	0.3867130E+12	0.4851933E+13	0.5258344E+14	0.5045977E+15
27	0.9896741E+06	0.4845850E+08	0.1291024E+10	0.2433647E+11	0.3610361E+12	0.4465220E+13	0.4773152E+14	0.4520144E+15
28	0.9889971E+06	0.4746883E+08	0.1242565E+10	0.2304545E+11	0.3366996E+12	0.4104185E+13	0.4326631E+14	0.4042831E+15
29	0.9882831E+06	0.4647984E+08	0.1195096E+10	0.2180289E+11	0.3136542E+12	0.3767486E+13	0.3916213E+14	0.3610179E+15
30	0.9875290E+06	0.4549157E+08	0.1148617E+10	0.2060779E+11	0.2918513E+12	0.3453833E+13	0.3539466E+14	0.3218549E+15
31	0.9867301E+06	0.4450405E+08	0.1103125E+10	0.1945917E+11	0.2712436E+12	0.3161982E+13	0.3194083E+14	0.2864604E+15
32	0.9858815E+06	0.4351733E+08	0.1058621E+10	0.1835605E+11	0.2517845E+12	0.2890738E+13	0.2877886E+14	0.2545198E+15
33	0.9849784E+06	0.4253146E+08	0.1015104E+10	0.1729743E+11	0.2334284E+12	0.2638954E+13	0.2588813E+14	0.2257409E+15
34	0.9840151E+06	0.4154648E+08	0.9725729E+09	0.1628233E+11	0.2161310E+12	0.2405527E+13	0.2324919E+14	0.1998528E+15
35	0.9829859E+06	0.4056248E+08	0.9310267E+09	0.1530976E+11	0.1998488E+12	0.2189396E+13	0.2084368E+14	0.1766036E+15
36	0.9818829E+06	0.3957950E+08	0.8904643E+09	0.1437873E+11	0.1845390E+12	0.1989548E+13	0.1865428E+14	0.1557599E+15
37	0.9807007E+06	0.3859763E+08	0.8508849E+09	0.1348827E+11	0.1701603E+12	0.1805009E+13	0.1666475E+14	0.1371056E+15
38	0.9794307E+06	0.3761694E+08	0.8122872E+09	0.1263739E+11	0.1566721E+12	0.1634850E+13	0.1485974E+14	0.1204409E+15
39	0.9780625E+06	0.3663752E+08	0.7746703E+09	0.1182510E+11	0.1440347E+12	0.1478178E+13	0.1322489E+14	0.1055812E+15
40	0.9765866E+06	0.3565947E+08	0.7380329E+09	0.1105043E+11	0.1322096E+12	0.1334143E+13	0.1174671E+14	0.9235628E+14
41	0.9749918E+06	0.3468290E+08	0.7023736E+09	0.1031740E+11	0.1211592E+12	0.1201934E+13	0.1041257E+14	0.8060957E+14
42	0.9732476E+06	0.3370792E+08	0.6676908E+09	0.9610035E+10	0.1108469E+12	0.1080776E+13	0.9210635E+13	0.7019701E+14
43	0.9713011E+06	0.3273469E+08	0.6339830E+09	0.8947346E+10	0.1012369E+12	0.9699288E+12	0.8129859E+13	0.6098639E+14
44	0.9691059E+06	0.3176339E+08	0.6012485E+09	0.8308363E+10	0.9229455E+11	0.8686919E+12	0.7159931E+13	0.5285654E+14
45	0.9666163E+06	0.3079429E+08	0.5694853E+09	0.7707116E+10	0.8398622E+11	0.7763974E+12	0.6291239E+13	0.4569661E+14
46	0.9637918E+06	0.2982768E+08	0.5386911E+09	0.7137632E+10	0.7627912E+11	0.6924112E+12	0.5514842E+13	0.3940537E+14
47	0.9605939E+06	0.2886390E+08	0.5088635E+09	0.6598943E+10	0.6914153E+11	0.6161321E+12	0.4822431E+13	0.3389053E+14
48	0.9569879E+06	0.2790331E+08	0.4799997E+09	0.6090080E+10	0.6254261E+11	0.5469905E+12	0.4206300E+13	0.2906810E+14
49	0.9529417E+06	0.2694634E+08	0.4520965E+09	0.5610082E+10	0.5645253E+11	0.4844479E+12	0.3659310E+13	0.2486180E+14
50	0.9484247E+06	0.2599341E+08	0.4251502E+09	0.5157986E+10	0.5084244E+11	0.4279954E+12	0.3174863E+13	0.2120251E+14
51	0.9434122E+06	0.2504499E+08	0.3991570E+09	0.4732838E+10	0.4568844E+11	0.3771530E+12	0.2746866E+13	0.1802766E+14
52	0.9378772E+06	0.2410158E+08	0.3741120E+09	0.4333683E+10	0.4095162E+11	0.3314686E+12	0.2369715E+13	0.1528079E+14
53	0.9317998E+06	0.2316371E+08	0.3500106E+09	0.3959574E+10	0.3661794E+11	0.2905170E+12	0.2038246E+13	0.1291108E+14
54	0.9251589E+06	0.2223192E+08	0.3268470E+09	0.3609563E+10	0.3265837E+11	0.2538990E+12	0.1747730E+13	0.1087283E+14
55	0.9179371E+06	0.2130677E+08	0.3046152E+09	0.3282716E+10	0.2904880E+11	0.2212407E+12	0.1493831E+13	0.9125094E+13
56	0.9101172E+06	0.2038883E+08	0.2833085E+09	0.2978101E+10	0.2576609E+11	0.1921919E+12	0.1272591E+13	0.7631264E+13
57	0.9016877E+06	0.1947872E+08	0.2629199E+09	0.2694793E+10	0.2278799E+11	0.1664259E+12	0.1080399E+13	0.6358678E+13
58	0.8926356E+06	0.1857704E+08	0.2434412E+09	0.2431873E+10	0.2009320E+11	0.1436379E+12	0.9139734E+12	0.5278279E+13
59	0.8829157E+06	0.1768442E+08	0.2248642E+09	0.2188432E+10	0.1766133E+11	0.1235447E+12	0.7703356E+12	0.4364305E+13
60	0.8723878E+06	0.1680150E+08	0.2071798E+09	0.1963568E+10	0.1547290E+11	0.1058834E+12	0.6467908E+12	0.3593970E+13
61	0.8609429E+06	0.1592913E+08	0.1903783E+09	0.1756388E+10	0.1350933E+11	0.9041058E+11	0.5409074E+12	0.2947180E+13
62	0.8485109E+06	0.1506818E+08	0.1744491E+09	0.1566010E+10	0.1175295E+11	0.7690125E+11	0.4504968E+12	0.2406273E+13
63	0.8350509E+06	0.1421967E+08	0.1593810E+09	0.1391561E+10	0.1018694E+11	0.6514837E+11	0.3735956E+12	0.1955777E+13
64	0.8205102E+06	0.1338462E+08	0.1451613E+09	0.1232140E+10	0.8795382E+10	0.5496143E+11	0.3084472E+12	0.1582182E+13

71GAM-male

T I M

X	SUM8	SUM9	SUM10	SUM11	SUM12	SUM13	SUM14	SUM15
15	0.1524987E+17	0.1371211E+18	0.1134874E+19	0.8722525E+19	0.6271099E+20	0.4243097E+21	0.2715726E+22	0.1651416E+23
16	0.1376589E+17	0.1216712E+18	0.9977529E+18	0.7587652E+19	0.5398847E+20	0.3615988E+21	0.2291416E+22	0.1379844E+23
17	0.1230223E+17	0.1081654E+18	0.8758817E+18	0.6589899E+19	0.4640082E+20	0.3076105E+21	0.1929817E+22	0.1150703E+23
18	0.1102765E+17	0.9586326E+17	0.7677163E+18	0.5714018E+19	0.3981042E+20	0.2612098E+21	0.1622207E+22	0.9577211E+22
19	0.9871686E+16	0.8483564E+17	0.6718530E+18	0.4946383E+19	0.3409691E+20	0.2213989E+21	0.1360997E+22	0.7955005E+22
20	0.8824612E+16	0.7496402E+17	0.5870174E+18	0.4274450E+19	0.2915062E+20	0.1873020E+21	0.1139599E+22	0.6594008E+22
21	0.7877374E+16	0.6613941E+17	0.5120534E+18	0.3687433E+19	0.2487618E+20	0.1581514E+21	0.9522908E+21	0.5454413E+22
22	0.7021571E+16	0.5826203E+17	0.4459140E+18	0.3175381E+19	0.2118875E+20	0.1332752E+21	0.7941456E+21	0.4502119E+22
23	0.6249409E+16	0.5124046E+17	0.3870519E+18	0.2729467E+19	0.1801337E+20	0.1120864E+21	0.6606706E+21	0.3707973E+22
24	0.5553672E+16	0.4499105E+17	0.3364115E+18	0.2341816E+19	0.1528391E+20	0.9407309E+20	0.5487842E+21	0.3047102E+22
25	0.4927672E+16	0.3943738E+17	0.2914205E+18	0.2005405E+19	0.1294209E+20	0.7878919E+20	0.4547113E+21	0.2498318E+22
26	0.4365228E+16	0.3450971E+17	0.2519831E+18	0.1713985E+19	0.1093669E+20	0.6584711E+20	0.3759222E+21	0.2043607E+22
27	0.3860630E+16	0.3014449E+17	0.2174734E+18	0.1462002E+19	0.9222701E+19	0.5491044E+20	0.3100754E+21	0.1667685E+22
28	0.3408615E+16	0.2628386E+17	0.1873289E+18	0.1244530E+19	0.7760699E+19	0.4568775E+20	0.2551650E+21	0.1357604E+22
29	0.3004332E+16	0.2287524E+17	0.1610451E+18	0.1057201E+19	0.6516170E+19	0.3792706E+20	0.2094772E+21	0.1102444E+22
30	0.2643315E+16	0.1987092E+17	0.1381699E+18	0.8961557E+18	0.5458970E+19	0.3141090E+20	0.1715502E+21	0.8929673E+21
31	0.2321460E+16	0.1722760E+17	0.1182991E+18	0.7579858E+18	0.4562815E+19	0.2595194E+20	0.1401393E+21	0.7214173E+21
32	0.2035000E+16	0.1490614E+17	0.1010715E+18	0.6396867E+18	0.3804829E+19	0.2138913E+20	0.1141873E+21	0.5812782E+21
33	0.1780480E+16	0.1287115E+17	0.8616543E+17	0.5386152E+18	0.3165143E+19	0.1758431E+20	0.9279820E+20	0.4670911E+21
34	0.1554740E+16	0.1109067E+17	0.7329434E+17	0.4524498E+18	0.2626529E+19	0.1441917E+20	0.7521390E+20	0.3742930E+21
35	0.1354887E+16	0.9535927E+16	0.6220369E+17	0.3791555E+18	0.2174080E+19	0.1179264E+20	0.6079474E+20	0.2990793E+21
36	0.1178283E+16	0.8181041E+16	0.5266776E+17	0.3169518E+18	0.1794925E+19	0.9618561E+19	0.4900210E+20	0.2387846E+21
37	0.1022524E+16	0.7002759E+16	0.4448672E+17	0.2642841E+18	0.1477973E+19	0.7823635E+19	0.3938356E+20	0.1892825E+21
38	0.8854182E+15	0.5980239E+16	0.3748396E+17	0.2197974E+18	0.1213689E+19	0.6345062E+19	0.3155992E+20	0.1498989E+21
39	0.7649774E+15	0.5094823E+16	0.3150372E+17	0.1823135E+18	0.9938923E+18	0.5131973E+19	0.2521426E+20	0.1183390E+21
40	0.6593964E+15	0.4329848E+16	0.2640890E+17	0.1508098E+18	0.8115789E+18	0.4138081E+19	0.2006231E+20	0.9312475E+20
41	0.5670404E+15	0.3670451E+16	0.2207905E+17	0.1244009E+18	0.6607691E+18	0.3326503E+19	0.1594424E+20	0.7304244E+20
42	0.4864308E+15	0.3103411E+16	0.1840860E+17	0.1023219E+18	0.5363582E+18	0.2665734E+19	0.1261773E+20	0.5709822E+20
43	0.4162339E+15	0.2616980E+16	0.1530519E+17	0.8391328E+17	0.4340464E+18	0.2129366E+19	0.9951999E+19	0.4448049E+20
44	0.3552475E+15	0.2200746E+16	0.1268821E+17	0.6860810E+17	0.3501331E+18	0.1695320E+19	0.7822633E+19	0.3452850E+20
45	0.3023912E+15	0.1845499E+16	0.1048747E+17	0.5591988E+17	0.2815250E+18	0.1345188E+19	0.6127312E+19	0.2670587E+20
46	0.2566948E+15	0.1543107E+16	0.8641972E+16	0.4543241E+17	0.2256051E+18	0.1063663E+19	0.4782125E+19	0.2057857E+20
47	0.2172894E+15	0.1266413E+16	0.7098868E+16	0.3679044E+17	0.1801728E+18	0.8380579E+18	0.3718463E+19	0.1579645E+20
48	0.1833989E+15	0.1069123E+16	0.5812457E+16	0.2969157E+17	0.1433624E+18	0.6578851E+18	0.2880405E+19	0.1207799E+20
49	0.1543308E+15	0.8857245E+15	0.4743336E+16	0.2387912E+17	0.1136908E+18	0.5145027E+18	0.2222520E+19	0.9197583E+19
50	0.1294690E+15	0.7313940E+15	0.3857613E+16	0.1913578E+17	0.8981175E+17	0.4008119E+18	0.1708018E+19	0.6975063E+19
51	0.1082664E+15	0.6019252E+15	0.3126219E+16	0.1527817E+17	0.7067600E+17	0.3110002E+18	0.1307206E+19	0.5267046E+19
52	0.9023879E+14	0.4936590E+15	0.2524294E+16	0.1215195E+17	0.5539783E+17	0.2403242E+18	0.9962062E+18	0.3959839E+19
53	0.7495800E+14	0.4034204E+15	0.2010635E+16	0.9627663E+16	0.4324588E+17	0.1649264E+18	0.7558820E+18	0.2963634E+19
54	0.6204693E+14	0.3284625E+15	0.1627214E+16	0.7597028E+16	0.3361821E+17	0.1416805E+18	0.5709556E+18	0.2207752E+19
55	0.5117410E+14	0.2664156E+15	0.1298752E+16	0.5969816E+16	0.2602118E+17	0.1080623E+18	0.4292751E+18	0.1636797E+19
56	0.4204900E+14	0.2152415E+15	0.1032336E+16	0.4671065E+16	0.2005137E+17	0.8204116E+17	0.3212128E+18	0.1207522E+19
57	0.3441774E+14	0.1731425E+15	0.8170950E+15	0.3638732E+16	0.1538030E+17	0.6198979E+17	0.2391716E+18	0.8863103E+18
58	0.2805907E+14	0.1387748E+15	0.6439026E+15	0.2821637E+16	0.1174157E+17	0.4660949E+17	0.1771818E+18	0.6471387E+18
59	0.2278081E+14	0.1107157E+15	0.5051279E+15	0.2177735E+16	0.8919938E+16	0.3486791E+17	0.1305724E+18	0.4699569E+18
60	0.1841652E+14	0.8793491E+14	0.3944127E+15	0.1672607E+16	0.6742205E+16	0.2594798E+17	0.9570452E+17	0.3393845E+18
61	0.1482256E+14	0.6951839E+14	0.3064773E+15	0.1278195E+16	0.5069599E+16	0.1920577E+17	0.6975655E+17	0.2436800E+18
62	0.1187538E+14	0.5469584E+14	0.2369590E+15	0.9717173E+15	0.3791408E+16	0.1413617E+17	0.5055076E+17	0.1739235E+18
63	0.9469106E+13	0.4282047E+14	0.1822632E+15	0.7347583E+15	0.2819690E+16	0.1034477E+17	0.3641461E+17	0.1233728E+18
64	0.7513329E+13	0.3335138E+14	0.1394427E+15	0.5524952E+15	0.2084932E+16	0.7525079E+16	0.2606984E+17	0.8695818E+17

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TIM

X	LX	SUM1	SUM2	SUM3	SUM4	SUM5	SUM6	SUM7
65	0.8047687E+06	0.1256411E+08	0.1317767E+09	0.1067015E+10	0.7563203E+10	0.4616604E+11	0.2534858E+12	0.1273735E+13
66	0.7876593E+06	0.1175935E+08	0.1192126E+09	0.9552425E+09	0.6476166E+10	0.3860284E+11	0.2073199E+12	0.1020249E+13
67	0.7640367E+06	0.1097169E+08	0.1074532E+09	0.8360300E+09	0.5520445E+10	0.3212666E+11	0.1687171E+12	0.8129296E+12
68	0.7487987E+06	0.1020265E+08	0.9648155E+08	0.7285706E+09	0.4684919E+10	0.2660571E+11	0.1365905E+12	0.6442125E+12
69	0.7269427E+06	0.9453852E+07	0.8627891E+08	0.6320952E+09	0.3956345E+10	0.2192079E+11	0.1099684E+12	0.5076221E+12
70	0.7033643E+06	0.8726910E+07	0.7682507E+08	0.5458163E+09	0.3324249E+10	0.1796445E+11	0.8806467E+11	0.3976373E+12
71	0.6779686E+06	0.8023546E+07	0.6809618E+08	0.4689915E+09	0.2778433E+10	0.1464020E+11	0.7006966E+11	0.3095732E+12
72	0.6508444E+06	0.7345578E+07	0.6007464E+08	0.4008934E+09	0.2309442E+10	0.1186177E+11	0.5545947E+11	0.2394735E+12
73	0.6223149E+06	0.6694734E+07	0.5272907E+08	0.3408189E+09	0.1908546E+10	0.9552331E+10	0.4359770E+11	0.1840141E+12
74	0.5927665E+06	0.6072415E+07	0.4603435E+08	0.2880901E+09	0.1567729E+10	0.7643783E+10	0.3404537E+11	0.1404164E+12
75	0.5624044E+06	0.5479649E+07	0.3996195E+08	0.2420560E+09	0.1279639E+10	0.6076056E+10	0.2640159E+11	0.1063710E+12
76	0.5313073E+06	0.4917245E+07	0.3448230E+08	0.2020940E+09	0.1037583E+10	0.4796420E+10	0.2032553E+11	0.7996945E+11
77	0.4993927E+06	0.4385938E+07	0.2956507E+08	0.1676117E+09	0.8354893E+09	0.3758840E+10	0.1552911E+11	0.5964393E+11
78	0.4664707E+06	0.3886546E+07	0.2517914E+08	0.1380466E+09	0.6676776E+09	0.2923351E+10	0.1177027E+11	0.4411482E+11
79	0.4326072E+06	0.3420076E+07	0.2129259E+08	0.1128675E+09	0.5298312E+09	0.2255473E+10	0.8846926E+10	0.3234454E+11
80	0.3961319E+06	0.2987469E+07	0.1787253E+08	0.9157491E+08	0.4169638E+09	0.1725642E+10	0.6591455E+10	0.2349762E+11
81	0.3633227E+06	0.2589338E+07	0.1488507E+08	0.7370238E+08	0.3253891E+09	0.1306678E+10	0.4865815E+10	0.1690616E+11
82	0.3286454E+06	0.2226016E+07	0.1229573E+08	0.5881733E+08	0.2516869E+09	0.9832891E+09	0.3557138E+10	0.1204035E+11
83	0.2945678E+06	0.1897371E+07	0.1006971E+08	0.4652162E+08	0.1928696E+09	0.7316024E+09	0.2573849E+10	0.8483213E+10
84	0.2614869E+06	0.1602804E+07	0.8172340E+07	0.3645192E+08	0.1463479E+09	0.5387331E+09	0.1842247E+10	0.5909365E+10
85	0.2298167E+06	0.1341318E+07	0.6569536E+07	0.2827958E+08	0.1098960E+09	0.3923853E+09	0.1303514E+10	0.4067120E+10
86	0.1999171E+06	0.1111502E+07	0.5228219E+07	0.2171005E+08	0.8161643E+08	0.2824893E+09	0.9111283E+09	0.2763606E+10
87	0.1720856E+06	0.9115856E+06	0.4116716E+07	0.1648184E+08	0.5990638E+08	0.2008731E+09	0.6286390E+09	0.1852476E+10
88	0.1464770E+06	0.7395200E+06	0.3205131E+07	0.1236512E+08	0.4342456E+08	0.1409667E+09	0.4277660E+09	0.1223839E+10
89	0.1232624E+06	0.5930430E+06	0.2465611E+07	0.9159991E+07	0.3105944E+08	0.9754216E+08	0.2867944E+09	0.7960730E+09
90	0.1024669E+06	0.4697806E+06	0.1872568E+07	0.6694380E+07	0.2189946E+08	0.6648272E+08	0.1892574E+09	0.5092736E+09
91	0.8407900E+05	0.3673136E+06	0.1402788E+07	0.4821812E+07	0.1520508E+08	0.4458326E+08	0.1227747E+09	0.3200164E+09
92	0.6806287E+05	0.2832346E+06	0.1035475E+07	0.3419024E+07	0.1038326E+08	0.2937819E+08	0.7819142E+08	0.1972418E+09
93	0.5433588E+05	0.2151717E+06	0.7522406E+06	0.2383549E+07	0.6964240E+07	0.1899493E+08	0.4881323E+08	0.1190504E+09
94	0.4276310E+05	0.1608359E+06	0.5370688E+06	0.1631309E+07	0.4580691E+07	0.1203070E+08	0.2981830E+08	0.7023714E+08
95	0.3307576E+05	0.1180728E+06	0.3762329E+06	0.1094241E+07	0.2949382E+07	0.7450004E+07	0.1778762E+08	0.4041883E+08
96	0.2509907E+05	0.8499706E+05	0.2581601E+06	0.7180086E+06	0.1855141E+07	0.4500622E+07	0.1033762E+08	0.2263122E+08
97	0.1866859E+05	0.5969799E+05	0.1731631E+06	0.4598484E+06	0.1137133E+07	0.2645481E+07	0.5836994E+07	0.1229361E+08
98	0.1358177E+05	0.4122940E+05	0.1132651E+06	0.2866854E+06	0.6772854E+06	0.1508348E+07	0.3191513E+07	0.6456613E+07
99	0.9640840E+04	0.2764763E+05	0.7203575E+05	0.1734202E+06	0.3906001E+06	0.8310629E+06	0.1663165E+07	0.3265100E+07
100	0.6660613E+04	0.1800679E+05	0.4438815E+05	0.1013845E+06	0.2171798E+06	0.4404628E+06	0.8521029E+06	0.1581935E+07
101	0.4463773E+04	0.1134618E+05	0.2638136E+05	0.5699637E+05	0.1157953E+06	0.2232830E+06	0.4116401E+06	0.7298326E+06
102	0.2890494E+04	0.6882406E+04	0.1503518E+05	0.3061502E+05	0.5879898E+05	0.1074677E+06	0.1883571E+06	0.3181925E+06
103	0.1800142E+04	0.3991913E+04	0.8152773E+04	0.1557984E+05	0.2818396E+05	0.4866877E+05	0.8086944E+05	0.1298354E+06
104	0.1068915E+04	0.2191771E+04	0.4160863E+04	0.7427062E+04	0.1260412E+05	0.2050480E+05	0.3218073E+05	0.4890598E+05
105	0.5969922E+03	0.1122856E+04	0.1969094E+04	0.3266202E+04	0.5177062E+04	0.7500680E+04	0.1167592E+05	0.1678526E+05
106	0.3073423E+03	0.5258643E+03	0.8462373E+03	0.1297108E+04	0.1910861E+04	0.2723620E+04	0.3775245E+04	0.5109336E+04
107	0.1415794E+03	0.2185221E+03	0.3203730E+03	0.4508706E+03	0.6137532E+03	0.8127588E+03	0.1051625E+04	0.1334092E+04
108	0.5577249E+02	0.7694272E+02	0.1018512E+03	0.1304978E+03	0.1628826E+03	0.1990057E+03	0.2388669E+03	0.2824663E+03
109	0.1743202E+02	0.2117023E+02	0.2490843E+02	0.2864664E+02	0.3230464E+02	0.3612305E+02	0.3986125E+02	0.4359946E+02
110	0.3738210E+01	0.3738210E+01	0.3738210E+01	0.3738210E+01	0.3738210E+01	0.3738210E+01	0.3738210E+01	0.3738210E+01

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TIM

X	SUM8	SUM9	SUM10	SUM11	SUM12	SUM13	SUM14	SUM15
65	0.5931147E+13	0.2583805E+14	0.1060913E+15	0.4130526E+15	0.1532437E+16	0.5440147E+16	0.1854476E+17	0.6088839E+17
66	0.4657413E+13	0.1990692E+14	0.8025329E+14	0.3069613E+15	0.1119384E+16	0.3907712E+16	0.1310462E+17	0.4234363E+17
67	0.3637164E+13	0.1524952E+14	0.6034637E+14	0.2267081E+15	0.8124229E+15	0.2788328E+16	0.9196904E+16	0.2923901E+17
68	0.2824235E+13	0.1161236E+14	0.4509685E+14	0.1663617E+15	0.5857149E+15	0.1975905E+16	0.6408577E+16	0.2004211E+17
69	0.2180023E+13	0.8788123E+13	0.3348450E+14	0.1217649E+15	0.4193534E+15	0.1390190E+16	0.4432675E+16	0.1363353E+17
70	0.1672401E+13	0.6608100E+13	0.2469640E+14	0.8778039E+14	0.2980887E+15	0.9708371E+15	0.3042485E+16	0.9200859E+16
71	0.1274764E+13	0.4935699E+13	0.1808831E+14	0.6308399E+14	0.2103084E+15	0.6727484E+15	0.2071647E+16	0.6158378E+16
72	0.9651911E+12	0.3660934E+13	0.1315262E+14	0.4499569E+14	0.1472244E+15	0.4624402E+15	0.1398899E+16	0.4086734E+16
73	0.7257176E+12	0.2695743E+13	0.9491682E+13	0.3184307E+14	0.1022287E+15	0.3152160E+15	0.9364588E+15	0.2687835E+16
74	0.5417035E+12	0.1970026E+13	0.6795938E+13	0.2235140E+14	0.7038562E+14	0.2129873E+15	0.6212429E+15	0.1751376E+16
75	0.4012872E+12	0.1428423E+13	0.4825913E+13	0.1555547E+14	0.4803427E+14	0.1426017E+15	0.4082557E+15	0.1130133E+16
76	0.2949161E+12	0.1027036E+13	0.3397590E+13	0.1072955E+14	0.3247876E+14	0.9456748E+14	0.2656542E+15	0.7218777E+15
77	0.2149467E+12	0.7321198E+12	0.2370555E+13	0.7331963E+13	0.2174921E+14	0.6208872E+14	0.1710867E+15	0.4562234E+15
78	0.1553028E+12	0.5171731E+12	0.1638436E+13	0.4961409E+13	0.1441725E+14	0.4031751E+14	0.1089980E+15	0.2851372E+15
79	0.1111880E+12	0.3618704E+12	0.1121263E+13	0.3327973E+13	0.9455846E+13	0.2592226E+14	0.6865846E+14	0.1761945E+15
80	0.7884348E+11	0.2506824E+12	0.7593937E+12	0.2201710E+13	0.6132873E+13	0.1646641E+14	0.4273620E+14	0.1074811E+15
81	0.5534591E+11	0.1718389E+12	0.5087107E+12	0.1442317E+13	0.3931163E+13	0.1033354E+14	0.2626980E+14	0.6474487E+14
82	0.3843975E+11	0.1164931E+12	0.3366718E+12	0.9336066E+12	0.2488845E+13	0.6402379E+13	0.1593627E+14	0.3847507E+14
83	0.2639940E+11	0.7805331E+11	0.2203788E+12	0.5967348E+12	0.1555240E+13	0.3913533E+13	0.9533893E+13	0.2253881E+14
84	0.1791619E+11	0.5165393E+11	0.1423254E+12	0.3763560E+12	0.9585057E+12	0.2358294E+13	0.5620360E+13	0.1300492E+14
85	0.1200683E+11	0.3373774E+11	0.9067155E+11	0.2340306E+12	0.5821447E+12	0.1399748E+13	0.3262066E+13	0.7304564E+13
86	0.7934707E+10	0.2173091E+11	0.5693386E+11	0.1433590E+12	0.3481191E+12	0.8176389E+12	0.1862278E+13	0.4122498E+13
87	0.5176103E+10	0.1379171E+11	0.3520294E+11	0.8642521E+11	0.2047601E+12	0.4695198E+12	0.1044639E+13	0.2260219E+13
88	0.3323629E+10	0.8615105E+10	0.2141173E+11	0.5122228E+11	0.1183349E+12	0.2647547E+12	0.5751196E+12	0.1215581E+13
89	0.2099790E+10	0.5291479E+10	0.1279663E+11	0.2981054E+11	0.6711263E+11	0.1464249E+12	0.3103598E+12	0.6404614E+12
90	0.1303717E+10	0.3191692E+10	0.7505150E+10	0.1701392E+11	0.3730208E+11	0.7931226E+11	0.1639350E+12	0.3301016E+12
91	0.7944430E+09	0.1887975E+10	0.4313461E+10	0.9508766E+10	0.2028817E+11	0.4201022E+11	0.8462270E+11	0.1661666E+12
92	0.4744266E+09	0.1093532E+10	0.2425485E+10	0.5195305E+10	0.1077940E+11	0.2172205E+11	0.4261251E+11	0.8154389E+11
93	0.2771850E+09	0.6191058E+09	0.1331953E+10	0.2769821E+10	0.5584097E+10	0.1094265E+11	0.2089046E+11	0.3893141E+11
94	0.1581348E+09	0.3419208E+09	0.7128471E+09	0.1437868E+10	0.2814280E+10	0.5358551E+10	0.9947808E+10	0.1804095E+11
95	0.8749768E+08	0.1837861E+09	0.3709263E+09	0.7250209E+09	0.1376412E+10	0.2544273E+10	0.4589257E+10	0.8093147E+10
96	0.4747885E+08	0.9588843E+08	0.1871403E+09	0.3540946E+09	0.6513910E+09	0.1167861E+10	0.2044986E+10	0.3503894E+10
97	0.2484763E+08	0.4840358E+08	0.9125182E+08	0.1669546E+09	0.2972964E+09	0.5164698E+09	0.8771256E+09	0.1458907E+10
98	0.1255403E+08	0.2356195E+08	0.4284224E+08	0.7570274E+08	0.1303419E+09	0.2191736E+09	0.3606559E+09	0.5817818E+09
99	0.6097419E+07	0.1100794E+08	0.1928029E+08	0.3286050E+08	0.5463920E+08	0.8883166E+08	0.1414823E+09	0.2211260E+09
100	0.2832319E+07	0.4910516E+07	0.8272362E+07	0.1358027E+08	0.2177870E+08	0.3419246E+08	0.5265052E+08	0.7964373E+08
101	0.1250384E+07	0.2078197E+07	0.3361846E+07	0.5307855E+07	0.8198500E+07	0.1241377E+08	0.1845816E+08	0.2699310E+08
102	0.5205523E+06	0.8278130E+06	0.1283664E+07	0.1946009E+07	0.2890645E+07	0.4215269E+07	0.6044395E+07	0.8534955E+07
103	0.2023598E+06	0.3072607E+06	0.4558365E+06	0.6623604E+06	0.9444369E+06	0.1324624E+07	0.1829126E+07	0.2490560E+07
104	0.7252444E+05	0.1049009E+06	0.1485758E+06	0.2065239E+06	0.2822765E+06	0.3799873E+06	0.5045025E+06	0.6614340E+06
105	0.2355849E+05	0.3237649E+05	0.4367496E+05	0.5794815E+05	0.7575256E+05	0.9771081E+05	0.1245152E+06	0.1569315E+06
106	0.6773230E+04	0.8818004E+04	0.1129847E+05	0.1427319E+05	0.1780444E+05	0.2195825E+05	0.2680439E+05	0.3241637E+05
107	0.1663896E+04	0.2044775E+04	0.2480469E+04	0.2974715E+04	0.3531251E+04	0.4153812E+04	0.4846141E+04	0.5611977E+04
108	0.3298040E+03	0.3808796E+03	0.4356935E+03	0.4942454E+03	0.5565361E+03	0.6225647E+03	0.6923315E+03	0.7658364E+03
109	0.4733766E+02	0.5107587E+02	0.5481407E+02	0.5855228E+02	0.6229048E+02	0.6602869E+02	0.6976689E+02	0.7350510E+02
110	0.3738210E+01	0.3738210E+01	0.3738210E+01	0.3738210E+01	0.3738210E+01	0.3738210E+01	0.3738210E+01	0.3738210E+01

71GAM-male

APPENDIX

Analysis of a_x

$$a_x = \sum_{t=1}^{\infty} v^t \, {}_tp_x = \sum_{t=1}^{\infty} (1 - d)^t \, {}_tp_x = \frac{1}{l_x} \sum_{t=1}^{\infty} (1 - d)^t l_{x+t}$$

Expanding $(1 - d)^t$ binomially and collecting terms, we arrive at the following expression for a_x:

$$\frac{1}{l_x}\Big[(l_{x+1} + l_{x+2} + l_{x+3} + l_{x+4} + l_{x+5} + l_{x+6} + \ldots\ldots.)d^0 -$$

$$(l_{x+1} + 2\, l_{x+2} + 3\, l_{x+3} + 4\, l_{x+4} + 5\, l_{x+5} + 6\, l_{x+8} + \ldots)d^1 +$$

$$(l_{x+2} + 3\, l_{x+3} + 6\, l_{x+4} + 10\, l_{x+5} + 15\, l_{x+6} + 21\, l_{x+7} + \ldots)d^2 -$$

$$(l_{x+3} + 4\, l_{x+4} + 10\, l_{x+5} + 20\, l_{x+6} + 35\, l_{x+7} + 56\, l_{x+8} + \ldots)d^3 +$$

$$(l_{x+4} + 5\, l_{x+5} + 15\, l_{x+6} + 35\, l_{x+7} + 70\, l_{x+8} + 126\, l_{x+9} + \ldots)d^4 -$$

$$(l_{x+5} + 6\, l_{x+6} + 21\, l_{x+7} + 56\, l_{x+8} + 126\, l_{x+9} + 252\, l_{x+10} + \ldots)d^5 +$$

$$(l_{x+6} + 7\, l_{x+7} + 28\, l_{x+8} + 84\, l_{x+9} + 210\, l_{x+10} + 462\, l_{x+11} + \ldots)d^6 - \ldots\text{etc}\Big]$$

The coefficient of d^o is the summation of the l_x column to l_{x+1}, which the reader will recognize as the familiar actuarial function T_{x+1}. Let us define this as S^1_{x+1}, however. The coefficient of d^1 is the double summation of the l_x column to l_{x+1}. We will denote this as S^2_{x+1}. We will denote S^i_y as the i^{th} summation to age y. It may be seen by inspection that

$$a_x = \frac{1}{l_x}\Big[S^1_{x+1} - (S^2_{x+1})d + (S^3_{x+1} - S^2_{x+1})d^2 - (S^4_{x+1} - 2S^3_{x+1} + S^2_{x+1})d^3 +$$

$$(S^5_{x+1} - 3S^4_{x+1} + 3S^3_{x+1} - S^2_{x+1})d^4 -$$

$$(S^6_{x+1} - 4S^5_{x+1} + 6S^4_{x+1} - 4S^3_{x+1} + S^2_{x+1})d^5 +$$

$$(S^7_{x+1} - 5S^6_{x+1} + 10S^5_{x+1} - 10S^4_{x+1} + 5S^3_{x+1} - S^2_{x+1})d^6 - \ldots\ldots \Big]$$

The coefficients of S^i_{x+1} should be readily recognized as the binomial coefficients. A further simplification is possible. Notice that the coefficient of d^2 can be reduced to S^3_{x+2}, the coefficient of d^3 can be reduced to $(S^4_{x+2} - S^3_{x+2}) = S^4_{x+3}$, etc., finally yielding the following result:

$$a_x = \frac{1}{l_x}\left[S^1_{x+1} - (S^2_{x+1})d + (S^3_{x+2})d^2 - (S^4_{x+3})d^3 + (S^5_{x+4})d^4 - (S^6_{x+5})d^5 + (S^7_{x+6})d^6 - \ldots\ldots \right]$$

which is our desired formula.

II <u>Analysis of Ia_x</u>

$$Ia_x = \sum_{t=1}^{\infty} tv^t\,{}_tp_x = \sum_{t=1}^{\infty} t(1-d)^t\,{}_tp_x = \frac{1}{l_x}\sum_{t=1}^{\infty} t(1-d)^t l_{x+t}$$

Expanding $(1 - d)^t$ binomially, and collecting coefficients, we arrive at tho following:

$$Ia_x =$$

$$\frac{1}{l_x}\Big[(l_{x+1} + 2\,l_{x+2} + 3\,l_{x+3} + 4\,l_{x+4} + 5\,l_{x+5} + 6\,l_{x+6} + \ldots)d^0 -$$

$$(l_{x+1} + 4\,l_{x+2} + 9\,l_{x+3} + 16\,l_{x+4} + 25\,l_{x+5} + 36\,l_{x+6} + \ldots)d^1 +$$

$$(2\,l_{x+2} + 9\,l_{x+3} + 24\,l_{x+4} + 50\,l_{x+5} + 90\,l_{x+6} + 147\,l_{x+7} + \ldots)d^2 -$$

$$(3\,l_{x+3} + 16\,l_{x+4} + 50\,l_{x+5} + 120\,l_{x+6} + 245\,l_{x+7} + 448\,l_{x+8} + \ldots)d^3 +$$

$$(4\,l_{x+4} + 25\,l_{x+5} + 90\,l_{x+6} + 245\,l_{x+7} + 560\,l_{x+8} + 1134\,l_{x+9} + \ldots)d^4 -$$

$$(5\,l_{x+5} + 36\,l_{x+6} + 147\,l_{x+7} + 448\,l_{x+8} + 1134\,l_{x+9} + 2520\,l_{x+10} + \ldots)d^5 +$$

$$(6\,l_{x+6} + 49\,l_{x+7} + 224\,l_{x+8} + 756\,l_{x+9} + 2100\,l_{x+10} + 5082\,l_{x+11} + \ldots)d^6$$

$$- \ldots\ldots \text{ etc}\Big]$$

Using the notation of the previous section, the above may be written as

$$Ia_x = \frac{1}{l_x}\left[S^2_{x+1} - (2S^3_{x+1} - S^2_{x+1})d + (3S^4_{x+2} - S^3_{x+2})d^2 - (4S^5_{x+3} - S^4_{x+3})d^3 + (5S^6_{x+4} - S^5_{x+4})d^4 - (6S^7_{x+5} - S^6_{x+5})d^5 + (7S^8_{x+6} - S^7_{x+6})d^6 - \ldots\right]$$

Alternatively, we may write this formula as

$$Ia_x = \frac{1}{l_x}\left[S^2_{x+1} - (S^3_{x+1} + S^3_{x+2})d + (2S^4_{x+2} + S^4_{x+3})d^2 - (3S^5_{x+3} + S^5_{x+4})d^3 + (4S^6_{x+4} + S^6_{x+5})d^4 - (5S^7_{x+5} + S^7_{x+6})d^5 + (6S^8_{x+6} + S^8_{x+7})d^6 - \ldots \right]$$

The above two formulas may be written more concisely as

$$Ia_x = \frac{1}{l_x}\left[S^2_{x+1} + \sum_{t=1}^{\infty} (-1)^t((1+t)S^{2+t}_{x+t} - S^{1+t}_{x+t})d^t \right]$$

$$Ia_x = \frac{1}{l_x}\left[S^2_{x+1} + \sum_{t=1}^{\infty} (-1)^t(tS^{2+t}_{x+t} + S^{2+t}_{x+t+1})d^t \right]$$

In order to make the quantum jump from the binomial expansion to the formula in our summation form, the reader is well advised to refer to Pascal's Triangle where among its many remarkable properties one will find the successive summations of the binomial coefficients.

END

ADDENDUM

The Fortran program "TIM1" was used to generate the Table of Isolated Mortality. The 1971 Group Annuity Mortality q_x values were accessed via a subroutine.

The other Fortran program was used to obtain three continuous annuity values using the TIM technique. The age, sex and interest rate for the three annuity values were supplied to the program via a Data File "TIM DATA". If a whole table of annuity values were required, the program could easily be adapted by introduction of a DO LOOP. Male and Female q_x values were available from the subroutine.

```
      INTEGER X,Z
      REAL QX(2,110),LX(2,110),SUM(15,110),AX,D,VINT,VINTX
      IMSETB = 0
      IFSETB = 0
      IWED = 1
      ISEX = 1
      CALL GA71(QX,IWED,IMSETB,IFSETB)
      LX(1,5)=1000000.
      LX(2,5)=1000000.
      DO 160 J=1,2
      DO 160 I=1,105
160   LX(J,5+I)=LX(J,4+I)*(1-QX(J,4+I))
      DO 170 I=1,15
170   SUM(I,110)=LX(ISEX,110)
      DO 180 I=1,105
      SUM(1,(110-I))=SUM(1,(111-I))+LX(ISEX,110-I)
      DO 180 J=2,15
180   SUM(J,(110-I))=SUM(J,(111-I))+SUM((J-1),(110-I))
190   CONTINUE
      WRITE(4,302)
      DO 300 I=15,64
      WRITE(4,500)I,LX(1,I),SUM(1,I),SUM(2,I),SUM(3,I),SUM(4,I),SUM(5,I)
     1,SUM(6,I),SUM(7,I)
300   CONTINUE
      WRITE(4,302)
302   FORMAT('1'//2X,'X',8X,'LX',12X,'SUM1',11X,'SUM2',11X,'SUM3',11X,'S
     1UM4',11X,'SUM5',11X,'SUM6',11X,'SUM7'//)
303   FORMAT('1'//2X,'X',7X,'SUM8',11X,'SUM9',11X,'SUM10',10X,'SUM11',10
     1X,'SUM12',10X,'SUM13',10X,'SUM14',10X,'SUM15'//)
      DO 600 I=65,110
      WRITE(4,500)I,LX(1,I),SUM(1,I),SUM(2,I),SUM(3,I),SUM(4,I),SUM(5,I)
     1,SUM(6,I),SUM(7,I)
500   FORMAT(2X,I3,8(2X,E13.7))
600   CONTINUE
      WRITE(4,303)
      DO 501 I=15,64
      WRITE(4,500)I,SUM(8,I),SUM(9,I),SUM(10,I),SUM(11,I),SUM(12,I),SUM(
     113,I),SUM(14,I),SUM(15,I)
501   CONTINUE
      WRITE(4,303)
      DO 700 I=65,110
      WRITE(4,500)I,SUM(8,I),SUM(9,I),SUM(10,I),SUM(11,I),SUM(12,I),SUM(
     113,I),SUM(14,I),SUM(15,I)
700   CONTINUE
230   CALL EXIT
      END
```

```
      INTEGER X,Z
      REAL QX(2,110),LX(2,110),SUM(15,110),AX,D,VINT,VINTX
      READ(1,*)IMSETB,IFSETB
      IWFD = 1
      ILTSEX = 0
      CALL GA51EX(QX,IWED,IMSETB,IFSETB)
      LX(1,5)=1000000.
      LX(2,5)=1000000.
      DO 160 J=I,2
      DO 160 1=1,105
160   LX(J,5+I)=LX(J,4+I)*(1-QX(J,4+I))
      WRITE(4,200)
200   FORMAT('1'//'INT RATE    AGE   SEX    ABARX'//)
5     READ(1,*,END=230)Z,ISEX,VINT
      IF(Z.EQ.0)GO TO 230
      IF(ISEX.EQ.ILTSEX)GO TO 190
      DO 170 1=1,15
170   SUM(I,110)=LX(ISEX,110)
      DO 180 1=1,105
      SUM(1,(110-I))=SUM(1,(111-I))+LX(ISEX,110-I)
      DO 180 J=2,15
180   SUM(J,(110-I))=SUM(J,(111-I))+SUM((J-1),(110-I))
190   CONTINUE
      VINT = VINT*.01
      D = VINT/(1+VINT)
      X = Z
      AX=(((((((((((((SUM(15,(X+14)))*D-SUM(14,(X+13)))*D+SUM(13,(X+12)
     1))*D-SUM(12,(X+11)))*D+SUM(11,(X+10)))*D-SUM(10,(X+9)))*D+SUM(9,(X
     2+8)))*D-SUM(8,(X+7)))*D+SUM(7,(X+6)))*D-SUM(6,(X+5)))*D+SUM(5,(X+4
     3)))*D-SUM(4,(X+3)))*D+SUM(3,(X+2)))*D-SUM(2,(X+1)))*D+SUM(1,(X+1))
      AX=AX/LX(ISEX,X) + .5
      VINTX = VINT*100
      WRITE(4,210,END=230)VINTX,Z,ISEX,AX
210   FORMAT(F6.4,1X,'%',4X,I2,4X,I1,4X,F7.4)
      ILTSEX = ISEX
      GO TO 5
230   CALL EXIT
      END
```

```
0,0
65,1,6.0,
65,1,5.0,
50,1,4.75,
38,2,5.25,
20,2,3.75,
0/
```