Increasing Insurances Under the
Uniform Distribution of Deaths Assumption
by
Mark D. J. Evans and Calvin D. Cherry

Frequently, the student of actuarial science is called upon to justify interest or life contingency formulas in his own mind by the use of so-called 'general reasoning'. This is all well and good, for it helps to better establish such formulas in the mind in preparation for the society examinations.

One may invite trouble, however, when using general reasoning to arrive at formulas rather than using mathematical proofs. A case in point is the derivation of the approximation for $(I^{(m)}\bar{A})_x$ in Life Contingencies by C. W. Jordan.

One can easily show that formula (3.27) for $(I^{(m)}A)_x$ is exact under the assumption of a uniform distribution of deaths throughout each year of age. Simply express $(I^{(m)}A)_x$ as

$$(IA)_x - A_x + \sum_{t=0}^{\infty} {}_{t|}(I^{(m)}A)^{1}_{x:\overline{1}|}$$

and since

$$(I^{(m)}A)^{1}_{x:\overline{1}|} = \sum_{t=1}^{m} \tfrac{t}{m} \cdot v \cdot {}_{\frac{t-1}{m}|\frac{1}{m}}q_x$$

$$= \frac{1}{m^2} v q_x \sum_{t=1}^{m} t = \frac{m+1}{2m} \cdot v q_x ,$$

$$\left(I^{(m)}A\right)_x = (IA)_x - A_x + \sum_{t=0}^{\infty} v^t \, {}_tp_x \cdot \frac{m+1}{2m} \cdot v\, q_{x+t}$$

$$= (IA)_x - A_x + \frac{m+1}{2m} A_x = (IA)_x - \frac{m-1}{2m} A_x$$

Next, Jordan puts bars over the A's in (3.27) to obtain formula (3.28) for $\left(I^{(m)}\bar{A}\right)_x$ and the implication is that this is also exact under the uniform distribution of deaths assumption. But this is not the case, as will be shown for m = 2.

Observe that when m = 2 in Jordan's formula (3.28):

$$\left(I^{(2)}\bar{A}\right)_x = (I\bar{A})_x - \tfrac{1}{4}\bar{A}_x = (I\bar{A})_x - \bar{A}_x + \tfrac{3}{4}\bar{A}_x$$

Consider the corresponding one year term benefit:

$$\left(I^{(2)}\bar{A}\right)^{1}_{x:\overline{1}|} = (I\bar{A})^{1}_{x:\overline{1}|} - \bar{A}^{1}_{x:\overline{1}|} + \tfrac{3}{4}\bar{A}^{1}_{x:\overline{1}|} = \frac{i}{\delta}\left(\tfrac{3}{4}\cdot v q_x\right)$$

Assuming uniform distribution of deaths:

$$\left(I^{(2)}\bar{A}\right)^{1}_{x:\overline{1}|} = v\left(\frac{(1+i)-(1+i)^{\frac{1}{2}}}{\delta}\cdot\frac{q_x}{2} + 2\cdot\frac{(1+i)^{\frac{1}{2}}-1}{\delta}\cdot\frac{q_x}{2}\right)$$

$$= \frac{v\,q_x}{2\delta}\left[(1+i) + (1+i)^{\frac{1}{2}} - 2\right]$$

$$= \frac{v\,q_x}{2\delta}\left[i + \tfrac{1}{2}i + \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\left(\tfrac{1}{2!}\right)i^2 + \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\left(\tfrac{1}{3!}\right)i^3 \right.$$

$$\left. + \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{2}\right)\left(\tfrac{1}{4!}\right)i^4 + \dots\right]$$

$$= \frac{v q_x}{2\delta}\left[\frac{3}{2} i - \frac{1}{8} i^2 + \frac{1}{16} i^3 - \frac{5}{128} i^4 + \dots\right]$$

$$= \frac{i}{\delta}\left(\frac{3}{4}\cdot v q_x\right) + \frac{v q_x}{2\delta}\left(-\frac{1}{8} i^2 + \frac{1}{16} i^3 - \frac{5}{128} i^4 + \dots\right)$$

$$\neq \frac{i}{\delta}\left(\frac{3}{4}\cdot v q_x\right)$$

Thus formula (3.28) does not reflect uniform distribution of deaths as one would believe.

Exact formulas for $(I^{(m)}A)_x^{(m)}$, $(I^{(m)}\bar{A})_x$ and $(\bar{I}\bar{A})_x$ can be derived assuming a uniform distribution of deaths:

$$(I^{(m)}A)_x^{(m)} = (IA)_x^{(m)} - A_x^{(m)} + \sum_{r=0}^{\infty} r E_x \sum_{t=1}^{m} \frac{t}{m}\cdot v^{\frac{t}{m}}\cdot {}_{\frac{t-1}{m}|\frac{1}{m}}q_{x+r}$$

$$= (IA)_x^{(m)} - \left(\frac{1}{d} - \frac{1}{d^{(m)}}\right) A_x^{(m)}$$

$$= \frac{i}{i^{(m)}}(IA)_x - \frac{i}{i^{(m)}}\left(\frac{1}{d} - \frac{1}{d^{(m)}}\right) A_x$$

$$(I^{(m)}\bar{A})_x = (I\bar{A})_x - \bar{A}_x + \sum_{r=0}^{\infty} r E_x \sum_{t=1}^{m} t\cdot v^{\frac{t-1}{m}}\cdot \bar{a}_{\overline{\frac{1}{m}|}}\cdot {}_{\frac{t-1}{m}|\frac{1}{m}}q_{x+r}$$

$$= (I\bar{A})_x - \left(\frac{1}{d} - \frac{1}{d^{(m)}}\right)\bar{A}_x$$

$$= \frac{i}{\delta}(IA)_x - \frac{i}{\delta}\left(\frac{1}{d} - \frac{1}{d^{(m)}}\right) A_x$$

$$(\bar{I}\bar{A})_x = (I\bar{A})_x - \bar{A}_x + \sum_{r=0}^{\infty} {}_rE_x \cdot (\bar{I}\bar{A})^{\,1}_{x+r:\overline{n|}}$$

$$= (I\bar{A})_x - \left(\frac{1}{d} - \frac{1}{\delta}\right)\bar{A}_x$$

$$= \frac{i}{\delta}(IA)_x - \frac{i}{\delta}\left(\frac{1}{d} - \frac{1}{\delta}\right)A_x$$

The following chart compares $\frac{m-1}{2m}$ to $\frac{1}{d} - \frac{1}{d^{(m)}}$ for various interest rates:

		$\frac{1}{d} - \frac{1}{d^{(m)}}$			
m	$\frac{m-1}{2m}$	3%	$4\frac{1}{2}$%	$5\frac{1}{2}$%	10%
1	0	0	0	0	0
2	.25	.251847	.252750	.253346	.255955
4	.375	.377309	.378438	.379182	.382444
12	$.458\overline{3}$	.460779	.461975	.462763	.466219
∞	.5	.502463	.503667	.504461	.507941

The difference between $\frac{1}{d} - \frac{1}{d^{(m)}}$ and $\frac{m-1}{2m}$ can be shown to be almost exactly equal to $\frac{m^2-1}{12m^2} \cdot \delta$. Thus, the error in using $\frac{m-1}{2m}$ increases in proportion to an increase in δ.

Derivation
of
$$(I^{(m)}A)_x^{(m)}$$

$$(I^{(m)}A)^{(m)\,1}_{x:\overline{1}|} = \sum_{t=1}^{m} \frac{t}{m}\cdot v^{\frac{t}{m}}\cdot {}_{\frac{t-1}{m}|\frac{1}{m}}q_x$$

Assuming a uniform distribution of deaths:

$${}_{\frac{t-1}{m}|\frac{1}{m}}q_x = \frac{l_{x+\frac{t-1}{m}} - l_{x+\frac{t}{m}}}{l_x} = \frac{l_x - \frac{t-1}{m}\cdot d_x - l_x + \frac{t}{m}\cdot d_x}{l_x}$$
$$= \frac{\frac{1}{m}\cdot d_x}{l_x} = \frac{1}{m}\cdot q_x$$

$$(I^{(m)}A)^{(m)\,1}_{x:\overline{1}|} = q_x\cdot\frac{1}{m^2}\cdot\left(v^{\frac{1}{m}} + 2v^{\frac{2}{m}} + 3v^{\frac{3}{m}} + \dots + mv^{\frac{m}{m}}\right)$$
$$= q_x\cdot(I^{(m)}a)^{(m)}_{\overline{1}|} = q_x\cdot\frac{\ddot{a}^{(m)}_{\overline{1}|} - v}{i^{(m)}} = vq_x\cdot\frac{\ddot{a}^{(m)}_{\overline{1}|} - v}{i^{(m)}v}$$

$$(I^{(m)}A)^{(m)}_x = (IA)^{(m)}_x - A^{(m)}_x + \sum_{t=0}^{\infty} {}_{t|}(I^{(m)}A)^{(m)\,1}_{x:\overline{1}|}$$

Simplifying the summation, we have

$$\sum_{t=0}^{\infty} {}_{t|}(I^{(m)}A)^{(m)\,1}_{x:\overline{1}|} = \sum_{t=0}^{\infty} {}_tE_x\cdot(I^{(m)}A)^{(m)\,1}_{x+t:\overline{1}|}$$

$$= \sum_{t=0}^{\infty} v^t {}_tp_x \cdot v q_{x+t} \cdot \frac{\ddot{a}_{\overline{1}|}^{(m)} - v}{i^{(m)} v} = \frac{\ddot{a}_{\overline{1}|}^{(m)} - v}{i^{(m)} v} \sum_{t=0}^{\infty} v^{t+1} {}_{t|}q_x$$

$$= \frac{\ddot{a}_{\overline{1}|}^{(m)} - v}{i^{(m)} v} \cdot A_x$$

Since $A_x^{(m)} = \frac{i}{i^{(m)}} A_x = \frac{\ddot{a}_{\overline{1}|}^{(m)}}{v} A_x$ under the uniform distribution of deaths assumption, we now have

$$(I^{(m)}A)_x^{(m)} = (IA)_x^{(m)} - \left(1 - \frac{(\ddot{a}_{\overline{1}|}^{(m)} - v)\, v}{i^{(m)} v\, \ddot{a}_{\overline{1}|}^{(m)}}\right) \cdot A_x^{(m)}$$

And simplifying the coefficient of $A_x^{(m)}$,

$$1 - \frac{\ddot{a}_{\overline{1}|}^{(m)} - v}{i^{(m)} \ddot{a}_{\overline{1}|}^{(m)}} = 1 - \frac{1}{i^{(m)} v^{\frac{1}{m}}} + \frac{v}{i^{(m)} \ddot{a}_{\overline{1}|}^{(m)}} = 1 - \frac{1}{d^{(m)}} + \frac{v}{d}$$

$$= 1 - \frac{1}{d^{(m)}} + \frac{1}{d} - 1 = \frac{1}{d} - \frac{1}{d^{(m)}}$$

Therefore:

$$(I^{(m)}A)_x^{(m)} = (IA)_x^{(m)} - \left(\frac{1}{d} - \frac{1}{d^{(m)}}\right) A_x^{(m)}$$

$$= \frac{i}{i^{(m)}} (IA)_x - \frac{i}{i^{(m)}} \left(\frac{1}{d} - \frac{1}{d^{(m)}}\right) A_x$$

$$\left(I^{(m)}\bar{A}\right)^{1}_{x:\overline{1|}} = \sum_{t=1}^{m} v^{\frac{t-1}{m}} \cdot \frac{t}{m} \cdot \frac{1-v^{\frac{1}{m}}}{\delta} \cdot q_x$$

$$= \frac{1-v^{\frac{1}{m}}}{\delta} \cdot q_x \cdot \frac{(1+i)^{\frac{1}{m}}}{m} \sum_{t=1}^{m} t\left(v^{\frac{1}{m}}\right)^t$$

$$\sum v_x \cdot \Delta u_x = u_x \cdot v_x - \sum u_{x+1} \cdot \Delta v_x$$

$$\Delta(u_x \cdot v_x) = u_{x+1} \cdot v_{x+1} + u_{x+1} \cdot v_x - u_{x+1} \cdot v_x - u_x \cdot v_x$$

$$= u_{x+1} \cdot \Delta v_x + v_x \cdot \Delta u_x$$

$$\sum_{t=1}^{m} t\left(v^{\frac{1}{m}}\right)^t = \frac{t\left(v^{\frac{1}{m}}\right)^t}{v^{\frac{1}{m}}-1}\Bigg|_1^{m+1} - \sum_{t=1}^{m} \frac{\left(v^{\frac{1}{m}}\right)^{t+1}}{v^{\frac{1}{m}}-1}$$

$$= \frac{t\left(v^{\frac{1}{m}}\right)^t}{v^{\frac{1}{m}}-1} - \frac{\left(v^{\frac{1}{m}}\right)^{t+1}}{\left(v^{\frac{1}{m}}-1\right)^2}\Bigg|_1^{m+1}$$

$$= \frac{(m+1)v^{\frac{m+1}{m}} - v^{\frac{1}{m}}}{v^{\frac{1}{m}}-1} - \frac{v^{\frac{m+2}{m}} - v^{\frac{2}{m}}}{\left(v^{\frac{1}{m}}-1\right)^2}$$

$$\left(I^{(m)}\bar{A}\right)^{1}_{x:\overline{1|}} = \frac{1-v^{\frac{1}{m}}}{\delta} \cdot q_x \cdot \frac{(1+i)^{\frac{1}{m}}}{m}\left[\frac{(m+1)v^{\frac{m+1}{m}} - v^{\frac{1}{m}}}{v^{\frac{1}{m}}-1} - \frac{v^{\frac{m+2}{m}} - v^{\frac{2}{m}}}{\left(v^{\frac{1}{m}}-1\right)^2}\right]$$

$$= \frac{1}{\delta m} \cdot q_x \cdot \left[\frac{(m+1)v-1}{-1} - \frac{v^{\frac{m+1}{m}} - v^{\frac{1}{m}}}{1-v^{\frac{1}{m}}}\right]$$

$$= \frac{1}{\delta m} \cdot q_x \cdot \left[1 - (m+1)v + v^{\frac{1}{m}} \cdot \frac{m \cdot d}{d^{(m)}} \right]$$

$$= \frac{1}{\delta m} \cdot q_x \cdot \left[1 - (m+1)(1-d) + \frac{m \cdot d}{d^{(m)}} \left(1 - \frac{d^{(m)}}{m} \right) \right]$$

$$= \frac{1}{\delta m} \cdot q_x \cdot \left[1 - m - 1 + d \cdot m + d + \frac{m \cdot d}{d^{(m)}} - d \right]$$

$$= \frac{1}{\delta m} \cdot q_x \cdot m \cdot \left(-1 + d + \frac{d}{d^{(m)}} \right)$$

$$= \frac{1}{\delta} \cdot q_x \cdot \left(-1 + d + \frac{d}{d^{(m)}} \right) = \frac{d}{\delta} \cdot q_x \cdot \left(\frac{-1}{d} + 1 + \frac{1}{d^{(m)}} \right)$$

$$= \bar{A}^{\,1}_{x:\overline{1|}} \cdot \left(\frac{-1}{d} + 1 + \frac{1}{d^{(m)}} \right)$$

$$\left(I^{(m)} \bar{A} \right)_x = \left(I \bar{A} \right)_x - \bar{A}_x + \sum_{t=0}^{\infty} {}_tE_x \cdot \bar{A}^{\,1}_{x+t:\overline{1|}} \cdot \left(\frac{-1}{d} + 1 + \frac{1}{d^{(m)}} \right)$$

$$= \left(I \bar{A} \right)_x - \bar{A}_x + \bar{A}_x \cdot \left(\frac{-1}{d} + 1 + \frac{1}{d^{(m)}} \right)$$

$$= \left(I \bar{A} \right)_x - \left(\frac{1}{d} - \frac{1}{d^{(m)}} \right) \cdot \bar{A}_x$$

$$= \frac{i}{\delta} \left(IA \right)_x - \frac{i}{\delta} \left(\frac{1}{d} - \frac{1}{d^{(m)}} \right) A_x$$

$$
\begin{aligned}
(\bar{I}\bar{A})_x &= \lim_{m\to\infty} \left(I^{(m)}\bar{A}\right)_x \\
&= \lim_{m\to\infty} \left[(I\bar{A})_x - \left(\frac{1}{d} - \frac{1}{d^{(m)}}\right)\bar{A}_x \right] \\
&= (I\bar{A})_x - \left(\frac{1}{d} - \frac{1}{\delta}\right)\bar{A}_x \\
&= \frac{i}{\delta}(IA)_x - \frac{i}{\delta}\left(\frac{1}{d} - \frac{1}{\delta}\right)A_x
\end{aligned}
$$

An expression for $(\bar{I}\bar{A})_x$ can be derived directly and simply by using calculus.

$$(\bar{I}\bar{A})_x = (I\bar{A})_x - \bar{A}_x + \sum_{r=0}^{\infty} {}_rE_x \cdot (\bar{I}\bar{A})^{1}_{x+r:\overline{1|}}$$

$$(\bar{I}\bar{A})^{1}_{x:\overline{1|}} = \int_0^1 t\, v^t\, {}_tp_x\, \mu_{x+t}\, dt$$

but $${}_tp_x\, \mu_{x+t} = \frac{l_{x+t}}{l_x} \cdot \frac{\frac{-d\, l_{x+t}}{dt}}{l_{x+t}}$$

$$= \frac{1}{l_x} \cdot -\frac{d}{dt}\left[(1-t)\cdot l_x + t \cdot l_{x+t}\right] = \frac{d_x}{l_x} = q_x$$

$$(\bar{I}\bar{A})^{1}_{x:\overline{1|}} = q_x \cdot \int_0^1 t\, v^t\, dt$$

Integrating by parts: $u = t$, $du = dt$, $dv = v^t dt$, $v = \frac{v^t}{-\delta}$

and noting that.

Let $y = v^t$; then $\frac{d}{dt}(\ln y) = \frac{d}{dt}(t \ln v)$ and $\frac{d}{dt}(y) = y \cdot \ln v$

so $\frac{d}{dt}(v^t) = -\delta v^t$

Thus,

$$(\bar{I}\bar{A})^{1}_{x:\overline{1|}} = q_x \cdot \left[-\frac{t v^t}{\delta}\Big|_0^1 + \int_0^1 \frac{v^t}{\delta}\, dt\right]$$

$$= q_x \cdot \left[-\frac{v}{\delta} + \frac{v^t}{-\delta^2}\Big|_0^1\right] = q_x \cdot \left[-\frac{v}{\delta} + \frac{1-v}{\delta^2}\right]$$

$$= q_x \cdot \left[-\frac{v}{\delta} + \frac{iv}{\delta^2}\right] = q_x \cdot \frac{iv}{\delta} \cdot \left[-\frac{1}{i} + \frac{1}{\delta}\right]$$

Then

$$(\bar{I}\bar{A})_x = (I\bar{A})_x - \bar{A}_x + \sum_{r=0}^{\infty} {}_rE_x \cdot \bar{A}^{\;1}_{x+r:\overline{1|}} \cdot \left(\frac{1}{\delta} - \frac{1}{i}\right)$$

$$= (I\bar{A})_x - \bar{A}_x \cdot \left[1 + \frac{1}{i} - \frac{1}{\delta}\right]$$

$$= (I\bar{A})_x - \bar{A}_x \cdot \left[\frac{1+i}{i} - \frac{1}{\delta}\right] = (I\bar{A})_x - \left(\frac{1}{d} - \frac{1}{\delta}\right)\bar{A}_x$$

It may be difficult to grasp the difference between $\frac{m-1}{2m}$ and $\frac{1}{d} - \frac{1}{d^{(m)}}$. The following development should be helpful in this respect.

$$(I^{(m)}\bar{A})^{\;1}_{x:\overline{1|}} = \sum_{t=1}^{m} \frac{1}{\delta} \cdot \frac{q_x}{m} \cdot v \cdot t\left[(1+i)^{\frac{m+1-t}{m}} - (1+i)^{\frac{m-t}{m}}\right]$$

$$= \frac{vq_x}{m\delta} \cdot \left[\left(\sum_{t=1}^{m} (1+i)^{\frac{t}{m}}\right) - m\right]$$

$$\doteq \frac{vq_x}{m\delta} \cdot \left[-m + \sum_{t=1}^{m}\left(1 + \frac{t}{m} \cdot i + \frac{t}{m} \cdot \frac{t-m}{m} \cdot \frac{1}{2} \cdot i^2 \right.\right.$$

$$\left.\left. + \frac{t}{m} \cdot \frac{t-m}{m} \cdot \frac{t-2m}{m} \cdot \frac{1}{6} \cdot i^3\right)\right]$$

$$= \frac{vq_x}{m\delta} \cdot \left[-m + \sum_{t=1}^{m}\left\{1 + \frac{t}{m} \cdot i + \frac{t^{(2)} + t(1-m)}{2m^2} \cdot i^2 \right.\right.$$

$$\left.\left. + \left(t^3 - 3mt^2 + 2m^2 t\right) \cdot \frac{i^3}{6m^3}\right\}\right]$$

Consider

$$\frac{1}{6m^3} \cdot \sum_{t=1}^{m} t^3 - 3m t^2 + 2m^2 t$$

Note

$$t^{(3)} + 3t^{(2)} + t = t(t-1)(t-2) + 3t(t-1) + t$$
$$= t^3 - 3t^2 + 2t + 3t^2 - 3t + t = t^3$$

Thus one has

$$\frac{1}{6m^3} \sum_{t=1}^{m} \left(t^{(3)} + 3t^{(2)} + t - 3m t^{(2)} - 3m t + 2m^2 t\right)$$

$$\frac{1}{6m^3} \cdot \sum_{t=1}^{m} \left[t^{(3)} + (1-m)\cdot 3t^{(2)} + (2m^2 - 3m + 1)\cdot t \right]$$

$$= \frac{1}{6m^3} \cdot \left\{ \frac{1}{4} t^{(4)} + (1-m) t^{(3)} + \frac{1}{2}(2m^2 - 3m + 1) t^{(2)} \right\} \Big|_{t=1}^{t=m+1}$$

$$= \frac{1}{6m^3} (m+1)(m) \left\{ \frac{1}{4}(m-1)(m-2) + (1-m)(m-1) \right.$$
$$\left. + \frac{1}{2}(2m-1)(m-1) \right\}$$

$$= \frac{1}{6m^3} (m+1)(m)(m-1) \left\{ \frac{1}{4} m - \frac{1}{2} + 1 - m + m - \frac{1}{2} \right\}$$

$$= \frac{m^2 - 1}{4 \cdot 6 \cdot m}$$

Since

$$\sum_{t=1}^{m}\left(1+\frac{t}{m}\cdot i+\frac{t^{(2)}+t(1-m)}{2m^2}\cdot i^2\right)$$

$$=t+\frac{t^{(2)}}{2m}\cdot i+\frac{\frac{1}{3}t^{(3)}+\frac{1}{2}t^{(2)}(1-m)}{2m^2}\cdot i^2$$

$$=m+\frac{m+1}{2}\cdot i+\frac{1}{2m}\left\{\frac{1}{3}(m+1)(m-1)-\frac{1}{2}(m+1)(m-1)\right\}\cdot i^2$$

$$\left(I^{(m)}\bar{A}\right)^{1}_{x:\overline{n}} \doteqdot \frac{m+1}{2m}\cdot\frac{iv}{\delta}\cdot q_x-\frac{m^2-1}{12m^2}\cdot\frac{iv}{\delta}\cdot q_x\left(i-\frac{i^2}{2}\right)$$

$$\doteqdot \bar{A}^{1}_{x:\overline{n}}\cdot\left(\frac{m+1}{2m}-\frac{m^2-1}{12m^2}\cdot\delta\right)$$

since $\delta=\ln(1+i)\doteqdot i-\frac{i^2}{2}$

Thus, the error in using $\frac{m+1}{2m}$ is very close to $-\frac{m^2-1}{12m^2}\cdot\delta$ and for $\frac{m-1}{2m}$ is $\frac{m^2-1}{12m^2}\cdot\delta$.