The Utility Value of Longevity Risk Pooling: Analytic Insights

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(Based on joint work with Huaxiong Huang)

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Background

The material in these (brief) slides are based on the (draft) working paper: *The Utility Value of Longevity Risk Pooling: Analytic Insights*, by: M.A. Milevsky and H. Huang (December 2017), attached to this deck of slides. Please reference accordingly.

My objective in this particular lecture is to explain how to compute the *value of mortality pooling* (VoMoP), which is also known as the *Annuity Equivalent Wealth* (AEW) in the academic economics literature. The AEW represents the additional amount of wealth that a rational consumer would require or demand to be as well-off (in a utility sense) without access to fairly priced annuities. I denote this value by δ, which is expressed as a percentage of initial retirement wealth.

Getting a feel for δ

For example, a value of $\delta = 0.5$ implies the individual would require 50% more money or wealth at retirement to compensate for the fact they didn't have access to annuities. Or, stated differently, the value of mortality pooling (VoMoP or AEW) to this individual is 50%.

Naturally, the higher the value of δ, the stronger the case for annuitization, the pension economists and actuaries are happier, etc. Note that this value (δ) is obviously a function of age, mortality and annuity pricing assumptions as well as personal (utility) preferences. For what follows I will assume constant relative risk aversion (CRRA) utility, parameterized by γ. That critical parameter captures the retiree's attitude to longevity risk. Please refer to any basic textbook on micro-economics for a proper introduction to utility functions (whether you believe they exist, or not.)

First: The Recipe Algorithm for Discrete Tables

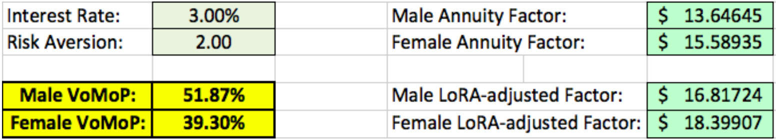
1. Start with any mortality table $q_x, x = 0...\omega$ and valuation rate r.
2. Let a_x denote the annuity factor at age x, using the $\{q_x\}$ vector.
3. Next, simply scale the mortality table by the coefficient of (longevity) risk aversion γ, so every entry in the (modified) table is now $q^*_x = \frac{q_x}{\gamma}$.
4. Compute a longevity-risk adjusted (LoRA) annuity factor, denoted by a^*_x using the scaled mortality rates, but under the same valuation rate. Note that when $\gamma > 1$ the scaled annuity factor should be higher than a_x, because mortality is lower; vice versa when $\gamma < 1$.
5. The Value of Mortality Pooling (VoMoP, a.k.a. AEW) is equal to:

$$\delta = \left(\frac{a_x}{a^*_x}\right)^{\frac{\gamma}{1-\gamma}} - 1.$$

Note: The situation is a bit tricky when $\gamma = 1$. It's best to take limits. See section #3 in the paper (attached to this deck of slides.)

Numerical Example #1

Under the Individual Annuity Mortality (IAM) 1983 (basic) table, which was used to derive many of the original AEW estimates in the economics literature, and a $r = 3\%$ interest rate, the annuity factor at age $x = 65$ is $a_{65} = 13.64645$ for males and $a_{65} = 15.58935$ for females.



Interest Rate:	3.00%		Male Annuity Factor:	$ 13.64645
Risk Aversion:	2.00		Female Annuity Factor:	$ 15.58935
Male VoMoP:	**51.87%**		Male LoRA-adjusted Factor:	$ 16.81724
Female VoMoP:	**39.30%**		Female LoRA-adjusted Factor:	$ 18.39907

Now, assuming a $\gamma = 2$ coefficient of longevity risk aversion (LoRA) in a CRRA utility function and a subjective discount rate equal to the interest rate, the scaled annuity factor is $a^*_{65} = 16.81724$ for males and $a^*_{65} = 18.39907$ for females. Recall that all I have done is divided the q_x values in the IAM1983 table by $\gamma = 2$ and priced the appropriate annuity.

And the answer is...

The value of mortality pooling (VoMoP), which is δ, is equal to:

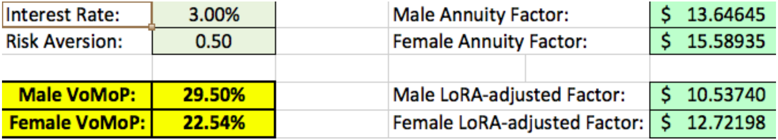
$$\delta = \left(\frac{13.64645}{16.81724}\right)^{-2} - 1 = 0.5187,$$

or 51.87% for a male at age 65, and the equivalent value is $\delta = 0.3930$ or 39.30% for a female at age 65. The VoMoP is lower for females because their mortality rate is lower at all ages, making the annuity (pooling) relatively more expensive, etc. The intuition should be obvious.

These values are similar to those reported by J. Brown, O.S. Mitchell, J.M. Poterba and M.J. Warshawsky in their book *The Role of Annuity Markets in Financing Retirement*, published by Cambridge University Press in 2001. Note. They did the work numerically via dynamic programing. I prefer closed-form analytic approximations. Also, note that occasionally the value $(1 - 1/\delta)$ is reported instead of δ, representing the amount of wealth someone would be willing to forfeit to have access to annuities.

Numerical Example #2

If I reduce the (longevity) risk aversion parameter from 2 to $\gamma = 1/2$ (Note: this is still risk averse) the corresponding objective annuity factors a_{65} do not change, but the modified or LoRA factors are now reduced to $a^*_{65} = 10.53740$ for males and $a^*_{65} = 12.72198$ for females.

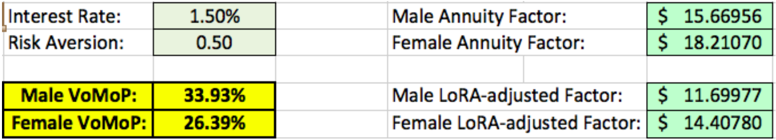


Interest Rate:	3.00%	Male Annuity Factor:	$ 13.64645
Risk Aversion:	0.50	Female Annuity Factor:	$ 15.58935
Male VoMoP:	**29.50%**	Male LoRA-adjusted Factor:	$ 10.53740
Female VoMoP:	**22.54%**	Female LoRA-adjusted Factor:	$ 12.72198

The corresponding values of δ are (only) 29.5% for males and 22.54% for females because the individual's risk aversion is lower. Stated differently, the value of pooling is reduced when you don't *hate* longevity risk as much! Remember that this implicitly assumes (i.) frictionless annuity markets, (ii.) symmetric mortality beliefs and (iii.) no pre-exsiting pension income. There is absolutely no way this holds in the real world, so the above δ is an upper bound.

Numerical Example #3

For the third and final numerical example, I leave the coefficient of (longevity) risk aversion at $\gamma = 1/2$, but reduce the valuation (and pricing) interest rate from 3% to $r = 1.5\%$, which increases the demographic a_{65} as well as the LoRA annuity factor a^*_{65}.



Interest Rate:	1.50%		Male Annuity Factor:	$ 15.66956
Risk Aversion:	0.50		Female Annuity Factor:	$ 18.21070
Male VoMoP:	**33.93%**		Male LoRA-adjusted Factor:	$ 11.69977
Female VoMoP:	**26.39%**		Female LoRA-adjusted Factor:	$ 14.40780

The value of mortality pooling (VoMoP) is now (slightly higher than numerical example #2) at 33.93% for males and 26.39% for females. Note that the results are not as sensitive to interest rates compared to levels of risk aversion, but the fact remains that when interest rates are lower the extra mortality credits are (relatively more) appreciated.

A Technical Caveat: Continuous vs. Discrete Time

Technically speaking, the *continuous* mortality hazard rate μ should be scaled by γ when computing the modified annuity factor a^*, and not the one-period mortality rate q.

Recall the relationship between these two quantities (when the mortality rate is constant in any given year) is:

$$q = 1 - e^{-\mu}.$$

These two variables are approximately equal to each other at young ages, but obviously diverge at higher ages since $q \leq 1$ but $\mu < \infty$. Practically speaking I would convert the mortality rates to monthly or even weekly and then compute the annuity factors, to get closer to continuous time. Also, for joint-life annuities vs. individual-life annuities, the hazard rate would have to be implied from the relevant joint-survival probabilities.

Generating More Numbers: Working in R

Working in **R**, I would price the annuity a_x using the Gompertz-Makeham law of mortality, with parameters (λ, m, b) that best (least squares) fit the mortality table being analyzed. Then, the modified or adjusted annuity factor a_x^* is computed by replacing m with $m + b\ln[\gamma]$ in the valuation equation. The shift in m is equivalent to scaling the mortality rates in the manner described earlier.

A sketch of the proof: See paper for more details

Part 1

Let the pair (w, π) denote an initial (retirement) endowment of total wealth w, plus pension income denoted by π, which measures an annual cash-flow in real terms beginning immediately $(t = 0)$ and continuing until death. The non-pensionized wealth w, is assumed to be invested and growing at a real risk-free rate denoted by r. That one account will be the source of all consumption spending above and beyond what is flowing from the pension income π.

A sketch of the proof

Part 2

Now, in a moment I will formally define the meaning of a *maximized* discounted lifetime utility for the given pair (w, π) but at this juncture I simply introduce and denote it by $U^*(w, \pi)$. I add the subscript $U^*_x(w, \pi)$ when I need to draw attention to the individual's current age x. Note that in the background of $U^*(w, \pi)$ resides an optimal consumption strategy denoted by c^*_t, beginning at time $t = 0$ until death, which dictates how w, is spent. In other words, there are as many possible values of utility $U(w, \pi)$ as there are strategies c_t, but there is only one optimal $U^*(w, \pi)$ and corresponding c^*_t.

A sketch of the proof

Part 3

The individual can convert (some or all) of their initial wealth w into additional pension income by purchasing (fair) annuities at a unit price denoted by a_x. Spending a_x units at age x, will entitle the retiree to one additional unit of pension income for life. Practically speaking for every ϵ of wealth that is annuitized, pension income will increase by $(\epsilon)/a_x$.

In theory, an individual with an initial endowment pair (w, π) can convert their entire holdings into the new pair $(0, \pi + w/a_x)$ if they so choose, but we do not allow the transfer in the other direction, that is to unwind or sell their pension annuities. In other words, they can't convert the pair (w, π) into $(w + \pi a_x, 0)$. But nor would they want to, as Menachem Yaari proved over 50 years ago.

A sketch of the proof

Part 4

It should be no great secret to anyone reading this note that:

$$U_x^*(w,\pi) \;\leq\; U_x^*(w-\epsilon, \pi+\epsilon/a_x).$$

Intuitively, (maximized) utility is greater with more (vs. less) annuities. Every dollar ϵ that is spent on annuities will increase maximal utility. This is the essence of the Yaari (1965) argument for converting all wealth into annuities, since $U^*(0,\pi+w/a_x)$ will provide the greatest amount of utility. It's important to remember again that every combination of (w,π) and its corresponding maximized utility $U^*(w,\pi)$ will involve a different consumption strategy c_t, which will be crucial in what follows. With that concept behind us, the *classic* definition of annuity equivalent wealth (AEW, which is our VoMoP) is the variable δ that satisfies the following equation:

$$U_x^*(w+\delta,\pi) \;=\; U_x^*(0,\pi+w/a_x). \tag{1}$$

A sketch of the proof
Part 5

Using the prior notation, let $U^*_x(1,0)$ denote discounted lifetime utility of wealth $w = 1$ without annuities. This can be expressed mathematically as:

$$U^*_x(1,0) \; = \; \int_0^\infty e^{-rt} \, ({}_t p_x) \, u(c^*_t) \, dt, \qquad (2)$$

where c^*_t is the (to be solved for) optimized consumption path, which can actually be written (smoothly) as: $c^*_t = c^*_0 \, ({}_t p_x)^{1/\gamma}$. Note that there is no wealth depletion time since there are no annuities.

See the paper (attached) and in particular section #3.2.

A sketch of the proof

Part 6

Let $U^*_x(0, 1/a_x)$ denote discounted lifetime utility of wealth, assuming wealth $w = 1$ **is entirely** annuitized at age x. Discounted utility is:

$$U^*_x(0, 1/a_x) \;=\; u(1/a_x) \int_0^\infty e^{-rt} \left({}_t p_x\right) dt, \tag{3}$$

where the optimized consumption path is trivially $c^*_t = 1/a_x$, for all t. I won't belabor the point that $U^*_x(0, 1/a_x) \geq U^*_x(1, 0)$ but simply conclude by noting that δ will satisfy the following equation:

$$U^*_x(1 + \delta, 0) \;=\; U^*_x(0, 1/a_x) \tag{4}$$

...and the rest is algebra. Finally, for the case in which $\pi > 0$, please see Table #1 of the paper (on page 15). Spoiler alert: The δ is (much) lower.

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Abstract

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The consensus among scholars is that (some) longevity risk pooling is the optimal strategy for drawing down wealth in retirement and a robust literature has developed around its measurement via *annuity equivalent wealth*. However, most of the published work is conducted numerically and authors usually report only a handful of limited values. In this paper we derive some (nice) closed-form expressions for the value of longevity risk pooling with constant relative risk aversion preferences. For example, under logarithmic utility and when lifetimes are exponentially distributed the value converges to $\sqrt{e} \approx 65\%$, when the interest rate is the inverse of life expectancy. In general the various formulae we derive match previously published numerical results, when properly calibrated to discrete time and tables. More importantly, we focus attention on the incremental utility from annuitization when the retiree is already endowed with pre-existing pension income such as Social Security benefits. Indeed, due to the difficulty in working with the so-called *wealth depletion time* in lifecycle models, we believe this is an area that hasn't received proper attention from annuity researchers (and promoters.)

1 Motivation and Outline

It is well known – since the original work by Yaari (1965) – that longevity risk pooling in the form of life annuities (a.k.a. actuarial notes, or instantaneous tontines) is the optimal strategy for drawing down wealth in retirement. In an idealized (utopian) world with no market imperfections, fair insurance pricing and zero bequest motives, the optimal allocation to annuities (anytime, anywhere) is 100%. Even when loadings, imperfections and bequest motives are added to the basic model, the work by Davidoff et al. (2005) argues for a substantial allocation to life annuities and the value of longevity risk pooling.

To this end, a large and very robust literature has developed around the measurement of so-called annuity equivalent wealth (AEW), a metric which has been used in a variety of widely cited-papers starting with Kotlikoff and Spivak (1981) and continued by Brown (2001). But, most of the AEW-based research is conducted or solved numerically and authors usually report only a handful of limited values in their papers – for understandable reasons, we might add. Nevertheless a diligent reader is hard pressed to solve (or replicate) their own stochastic dynamic program if they seek personalized numbers. This is becoming increasingly important in a scholarly environment which places (greater) value on replicability and transparency. In addition to the academic need, industry practitioners have also struggled to explain the *value* of longevity risk pooling to a wider public.

Motivated by the need for computational simplicity (and replicability) in this paper we derive a variety of analytic expressions for the AEW under constant relative risk aversion (CRRA) consumption preferences. We provide a complete characterization under exponential mortality as well as results for more general models of mortality, such as Gompertz-Makeham (GM). And, since any mortality table (relevant to retirees) can be calibrated to the GM law with little economic error, our expressions can also be used to estimate the value of pooling in any environment, which should be of practical use to pension actuaries.

We note up front that the output from our analytic expressions (roughly) match previously published numbers which were based on discrete time and tables, so there are no surprises, conflicts or inconsistencies with the prior annuity economics literature. Rather, our pedagogical simplification enables users to calculate and report the utility *value of longevity risk pooling* under different mortality assumptions, retirement ages, discount rates and levels of risk aversion.

From a risk theory or conceptual point of view, our work highlights the (often small) incremental utility that comes from annuitization when the individual is already endowed with (substantial) pre-existing pension income in the form of Social Security benefits or income from a Defined Benefit (DB) pension plan. Methodologically, we compute the annuity equivalent wealth without having to resort to (tricks for) discounting the present value of pension benefits, adding them to wealth or ignoring them altogether; which is often the case in the economics literature.

The benefit of additional longevity risk pooling – and the proper computation of the AEW with background pension income – is an item that hasn't received as much attention from annuity researchers. For example, the book by Sheshinski (2007) makes no reference to this matter, mainly because he is concerned with the optimal strategy and annuity valuation in the absence of pre-existing pensions. The more financially oriented lifecycle and portfolio choice literature, such as Cocco and Gomes (2012) for example, aren't concerned with computing AEW. They focus on explaining observed behavior, so these issues aren't addressed (explicitly) either.

1.1 Plan and Agenda

The remainder of this paper is organized as follows. In the next section (#2) we review the conceptual framework, formally define the annuity equivalent wealth and explain its relation to the value of longevity risk pooling, a term which is more familiar to actuaries and insurance specialists. In section (#3) we restrict our attention to the case in which the remaining lifetime random variable is exponentially distributed, which means that the mortality hazard rate is constant. Although this is a (very) unrealistic assumption (especially to actuaries), it is used in many insurance economic papers. In the same section (#3) we offer a range of numerical results or case studies in the presence of pre-existing pension income. Section (#4) moves on to the more general mortality models. We obtain a closed-form expression for the value of pooling in the absence of pre-existing pension income as a function of the ratio of two annuity factors. If the annuity factors themselves can be expressed analytically, then so can the value of longevity risk pooling. Our expressions and results are compared with the (numerical) values presented in Brown et al. (2001), and are in fact consistent with their results in the absence of pre-existing pension income.

One of the main theoretical or conceptual contributions in this paper is to argue that when pre-existing (fixed) pensions are included in a lifecycle model one has to be (very) careful about how to define annuity equivalent wealth and the value of longevity risk pooling. This is due to the *wealth depletion time* which effectively destroys our ability to scale. Although the concept of a *wealth depletion time* is explained in Leung (2002, 2007) or in Lachance (2012), within the context of the Yaari (1965) lifecycle model, it doesn't appear to be well known[1]. We will explain the implication of this to the utility valuation of AEW. Finally, section (#5) concludes the paper with a summary of the main results (and expressions.)

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[1] For example, Leung (2002) writes that: *"Previous studies utilizing Yaari's (1965) model of uncertain lifetime...have failed to recognize that terminal wealth depltion is an intrinsic and important property of the model...erroneous and misleading results will be obtained if it is ignored in the investigation"*, pg. 582

2 Annuity Equivalent Wealth: Derived

2.1 Notation and Terminology

Let the pair (w, π) denote an initial (retirement) endowment of liquid (non annuitized) wealth w, plus pension income denoted by π, which measures an annual cash-flow in real terms beginning immediately ($t = 0$) and continuing until death. The non-pensionized wealth w, is assumed to be invested and growing at a real risk-free rate denoted by r. That one account will be the source of all consumption spending above and beyond what is flowing from the pension income π. The economic value of the total endowment is: $w + PV(\pi)$, where the term $PV(\pi)$ denotes the actuarial present value of pre-existing pension annuity income. And, while it is quite clear to pension and financial economists that *ceteris paribus* it is better to have less w and more $PV(\pi)$, the question of how exactly to optimize consumption given a particular combination of (w, π) obviously affects the utility value of additional pension income.

In a moment we will formally define the meaning of a *maximized* discounted lifetime utility for the given pair (w, π) but at this juncture we simply introduce and denote it by $U^*(w, \pi)$. We add the x subscript to $U^*_x(w, \pi)$ if and when we need to draw attention to the individual's current age x. Remember that in the background of $U^*(w, \pi)$ resides an optimal consumption strategy denoted by c^*_t, beginning at time $t = 0$ until death, which dictates how w, is spent over time. In other words, there are as many possible values of utility $U(w, \pi)$ as there are strategies c_t, but there is a unique $U^*(w, \pi)$ and corresponding strategy c^*_t.

Given the centrality of this expression – and to clarify notation – for example, $U_{65}(100, 3)$ without the asterisk could represent one possible utility value at the age of $x = 65$, of someone who starts her retirement with an initial wealth of $w = 100$ dollars plus a pension entitlement of $\pi = 3$ dollars who withdraws 4% of her initial investable wealth, that is $c_t = 4 + 3 = 7, t \geq 0$, dollars each year until death (or the money runs out). It's a feasible consumption strategy, but not the optimal one. Indeed, c^*_t is more subtle.

We now move on to voluntary annuitization. The individual can convert (some or all) of his/her initial liquid wealth w into additional pension income by purchasing (fair) annuities at a unit price denoted by a_x. This implies that the actuarial present value of the pre-existing pension income is $PV(\pi) = a_x \pi$. Spending a_x units at age x, will entitle the retiree to one additional unit of pension income for life. Pension annuity income and voluntary annuity purchases are priced and valued the same. They are substitutes, notwithstanding some of the frictions one might encounter in replacing one with the other. For background on the economic role of pension and longevity annuities as retirement insurance, see Bodie (1990) or Blake (1999). Of course, the exact functional form (or pricing equation) for a_x will depend on the assumed law of mortality and valuation rate – as well as real world transaction frictions and insurance loadings – all of which we will address in subsequent sections.

Our notation and terminology for a liquid wealth plus pension income endowment pair (w, π) – and the clear distinction between the two – is standard. It is explicitly used by Cannon and Tonks (2008) for example, and implicitly used in Brown (2001) and other papers that measure the utility value of annuities. Practically speaking, for every \$1 of liquid investable wealth (w) that is annuitized, leaving $(w-1)$ to invest, pension income will increase by $1/a_x$. And yes, of course, an individual with an initial endowment pair (w, π) can convert his/her entire holdings into the new pair $(0, \pi + w/a_x)$ if they so choose, but we do not allow the transfer in the other direction; that is to unwind or sell their pension annuities. In other words, they can't convert the pair (w, π) into $(w + \pi a_x, 0)$. But nor would they want to, as Menachem Yaari argued over 50 years ago in Yaari (1965.)

2.2 Definition and Distinction

We now get to the crux of the matter and define the so-called *annuity equivalent wealth* (AEW). Even though we haven't yet written down a formal expression for discounted lifetime utility – the value function in the language of dynamic programming – or proven the optimality of pooling longevity risk, it should be obvious to anyone reading this that:

$$U_x^*(w, \pi) \;\leq\; U_x^*(w - \epsilon, \pi + \epsilon/a_x), \tag{1}$$

under plausible assumptions about utility preferences. Intuitively, (maximized) utility is greater with more (vs. less) annuities. Every ϵ that is spent on annuities will increase maximal utility. This again is the essence of the Yaari (1965) argument for converting all wealth into annuities, since $U^*(0, \pi + w/a_x)$ will provide the greatest amount of utility, assuming no loading, no fees and no anti-selection. We will return to these (harsh) assumptions in the concluding remarks. In the background of equation (1), every initial combination of (w, π) and its corresponding maximized utility $U^*(w, \pi)$ will involve a different consumption strategy c_t^*, which is central to what follows later.

The formal definition of annuity equivalent wealth, as used in the work by Kotlikoff and Spivak (1981), or in Brown (2001), is the quantity $\hat{w}$ that satisfies the following equation:

$$U_x^*(\hat{w}, \pi) \;=\; U_x^*(0, \pi + w/a_x), \tag{2}$$

where w is the initial liquid wealth of the retiree. The markup of $\hat{w}$ over w, reflects the amount someone (with wealth w) would be willing to pay (WTP) to be able to annuitize their entire liquid investable wealth. Brown (2001) argues that consumers with larger imputed $\hat{w}/w$ values, who should value annuities more highly, were actually more likely to purchase additional annuities at retirement. So, there is some empirical validity to $\hat{w}$ as well. In what follows we will focus on $\hat{w}/w - 1$, and refer to this as the value of longevity risk pooling *in-the-large,* for reasons that will soon become clear.

In some sense, the entire agenda of this paper is concerned with solving for the value of δ that satisfies the utility equation:

$$U_x^*(w(1+\delta), \pi) \;=\; U_x^*(0, \pi + w/a_x), \tag{3}$$

We use the phrase value of pooling *in the large*, because it assumes they convert all of their wealth into the annuity. We will later contrast this quantity δ with a parallel metric *in the small* which measures the incremental value of converting only $1 into additional annuity income. And, while the CRRA assumption might tempt readers into believing that the value of pooling for one incremental dollar is the same as the value of pooling for the entire w, the fact is the homogeneity breaks down due to the wealth depletion time. More on this to come. Also, by the term *content*, we mean in the classic sense of maximal utility. We will not (be tempted to) veer into a discussion of behavioral economics as it relates to the *framing* of annuities – investment vs. consumption – or the behavioral reasons for the so-called annuity puzzle. Rather, we carry the torch of Daniel Bernoulli and refer readers to Brown et al. (2008) for that aspect of annuity demand (or lack thereof.)

The value of pooling $\delta = \hat{w}/w - 1$ will depend on a large number of explicit and implicit variables that are buried in the utility calculations, such as risk aversion and subjective discount rates, the specific mortality table being used (subjectively or objectively) and the interest rate used for pricing the annuity factor a_x. If we wanted to clutter-up the δ, we would add all those parameters as explicit arguments. But, that dependence is well known to researchers. What is less known (or at least emphasized) is that the value of pooling δ also depends on the amount of pre-existing pension income π in the initial endowment pair.

Of course, authors such as Brown (2001) make it very clear that the fraction of total wealth that is pre-annuitized will impact the AEW and that it's lower for individuals with pre-existing pension income. Nevertheless, it is often obscured within the dynamic programming calculations and rarely treated as a separate component or argument in the AEW value. The exact mechanism by which π reduces the AEW is, for the most part, glossed over. This is a subtle point which we will flush-out.

For example, it is incorrect to capitalize or compute the actuarial present value of the annuity income π, add this to initial wealth w and then use the aggregate (new) number $w + PV(\pi)$ to compute AEW values. It will overstate the AEW. Methodologically, it is equally erroneous to ignore the pre-existing pension income π and compute the AEW for the w alone, under the misguided assumption that utility is CRRA and hence everything scales. On a more subtle level, adding the pre existing annuity income into the felicity function's consumption argument – and effectively moving from a CRRA formulation into a HARA world – also ignores the issue by forcing a non-zero discretionary consumption at the boundary. Stated formally:

$$U_x^*(w(1+\delta), \pi) \;\neq\; U_x^*(w(1+\delta) + \pi a_x, 0) \neq\; U_x^*((w + \pi a_x)(1+\delta), 0) \tag{4}$$

The pre-existing pension income π alters the optimal consumption path c_t^* in a very critical way that feeds back into utility. And, since this is a large part of our story, we will occasionally use the subscript δ_π to remind readers that the AEW (also) depends on the pattern of pension income and not only its actuarial present value.

Ceteris paribus, we know that: $\delta_{(\pi+\epsilon)} - \delta_\pi > 0$. In the Yaari (1965) setting it is always worth paying for a little bit more of annuity income. The issue and impetus for this paper is to arrive at an expression for δ_π so that this marginal or incremental benefit can be properly measured quickly, easily and under a wide variety of parameters. By its very definition, δ_π assumes that the entire w wealth is used to purchase annuities. But what is the *incremental* value of longevity pooling?

2.3 Pooling in the small

In the presence of pre-existing pension income π – whether it be Social Security benefits or voluntarily purchased annuity income – we are forced to redefine or at least *refine* the meaning of *annuity equivalent wealth* and the corresponding value of longevity risk pooling.

Indeed, when the consumer or retiree is (only) endowed with liquid investable assets w – and assuming they exhibit constant relative risk aversion utility preferences – the δ value scales in wealth. The value of $\hat{w}$ for someone with $w = \$1$ (divided by w) is identical to the AEW of a wealthier retiree with an initially endowed $w = \$1$ million (divided by w.) In fact, this is precisely why virtually all of the authors in the annuity economics literature express and report AEW numbers relative to a dollar value. Authors arbitrarily assume an initial wealth of $w = \$1$, because it makes no difference to δ. The value of longevity risk pooling to Bill Gates or Warren Buffet is the same as for 50 Cent (a well-known rapper who recently declared bankruptcy), assuming they (i.) have CRRA utility preferences, (ii.) no exogenous pension income and (iii.) no bequest motives – all of which are debatable, but not really the point.

However, once an exogenous or pre-existing pension income is *properly* introduced into the life-cycle consumer optimization problem, the homogeneity or scaling simply breaks down – even under CRRA utility. This is not usually the case with the (very) convenient and popular homogenous specification of utility and might be one of the reasons it has been overlooked in the annuity economics literature. Ignoring this fact will *overstate* the AEW and the value of pooling.

This is the reason we have decided to define an additional metric for the value of longevity risk pooling, denoted simply by v which captures the value for one *incremental* dollar of wealth and not total wealth. In some sense, it's back to a marginal analysis. Technically speaking the AEW *in-the-small* addresses the following issue. Assume a retiree has $w = \$100,000$ in investable retirement assets and $\pi = \$10,000$ in pre-existing pension income which induces a total discounted lifetime utility denoted by $U_x^*(w, \pi)$ at retirement age x.

Now, they are about to spend (only) \$1 from their $w = \$100,000$ to purchase incremental pension income beyond the $\pi = \$10,000$ they already are entitled to. This will obviously leave them with only \$99,999 in investable retirement wealth plus revised pension income of $\$10,000 + 1/a_x$, where a_x is the (same) standard annuity factor at age x. How much additional wealth would the retiree – who didn't spend the additional \$1 to purchase additional pension annuity income – require to induce the same level of utility? Needless to say, the answer will depend on the coefficient of relative risk aversion (as usual), but will actually change if the initial wealth were $w = \$1,000,000$ or only \$50,0000, or the pension π was different.

For clarity we will report and display both δ values (in-the-large) and v values (in-the-small). We solve for v by equating levels of optimal utility, we solve the following equation:

$$U_x^*(w + v, \pi) \;=\; U_x^*(w - 1, \pi + 1/a_x). \tag{5}$$

This is all for notation and terminology and now we move on to the computations.

2.4 Computing Utility

We are at the point in the narrative were we can present a formal expression for discounted lifetime utility: $U_x^*(w, \pi)$. Let $u(c)$ denote a constant relative risk aversion (CRRA) utility (a.k.a. felicity) function parameterized by risk aversion γ, and a subjective discount rate ρ. Formally, $u(c) = c^{1-\gamma}/(1-\gamma)$. The maximal utility is:

$$U_x^*(w, \pi) \;= \max_{c_t} \int_0^\infty e^{-\rho t}\,({}_t p_x)\,u(c_t)dt, \tag{6}$$

with a dynamic budget constraint determined by:

$$dW_t \;=\; (rW_t \,+\, \pi \,-\, c_t)dt, \;\; W_0 = w. \tag{7}$$

The objective function (6) or constraint (7) doesn't require much justification. In fact this lifecycle framework will soon celebrate its centenary anniversary, although recently Bommier (2006) questioned the underlying risk neutrality assumption and proposes a different form altogether. We follow Yaari (1965), Levhari and Mirman (1977), Davies (1981), Butler (2001) or Pashchenko (2013) where this framework is used to extract testable implications for rational behavior with lifetime uncertainty and annuities. The pension income π flows into the (one) account earning r, and then consumption c_t is extracted from the same account. By definition the optimal consumption function is c_t^* and the difference: $(c_t^* - \pi)$ is the net spending rate from liquid wealth, whereas $(c_t^* - \pi)/W_t$ is the spending rate as a fraction of current wealth. Indeed, saving might continue into retirement and wealth might continue to grow temporarily, for someone with a sufficiently low discount rate ρ. We will not allow any borrowing (against future pension income) so that wealth $W_t \geq 0$ at all times.

Also, to be consistent with the literature (and inspired by Occam's razor) we assume for most of what follows that $\rho = r$ and the subjective discount rate is equal to the interest rate earned in the account. The only reason to prefer early vs. late consumption, in our model, is due to mortality beliefs and the inter-temporal elasticity of substitution, $1/\gamma$ in our model.

Moving on, without any loss of generality we can decompose or break-up the integral in equation (6) into two arbitrary parts as follows:

$$U^*_x(w,\pi) \ = \max_{c_t} \ \left[\int_0^\tau \ e^{-rt} \, ({}_tp_x) \, u(c_t) dt \ + \ \int_\tau^\infty \ e^{-rt} \, ({}_tp_x) \, u(c_t) dt \right], \tag{8}$$

where the break takes place at time τ. It might seem odd to split up the objective function in this manner, but in fact when $\pi > 0$, there is a qualitative change in optimal consumption at some point during the horizon ($t = 0...\infty$). That is, liquid wealth is actually depleted and the optimal consumption rate $c^*_t = \pi$ from that point onward. That is the τ value we select. Until that *wealth depletion time* consumption is sourced from from both pension income and wealth. But after $t \geq \tau$ consumption is exactly equal to the pension, the individual has run out of liquid (non-annuitized) funds and $W_t = 0$, for $t \geq \tau$.

To be crystal clear we are not imposing this on the problem. It actually *is* the optimal policy, as elaborated on by Leung (2002, 2007) and carefully explained in the (textbook) by Charupat, Huang and Milevsky (2012). We can't emphasize enough how critical this (seemingly minor) point is to the calibration of lifecycle models in general and the computation of AEW values in particular. At the risk of flogging a dead (work) horse, if one assumes all pension income is capitalized and discounted to time zero, or if the pension income is added to optimal consumption as a scaling afterthought, the *wealth depletion time* will be lost in the backward induction algorithm.

Technically, the value of $c^*_t = \pi$ for $t \geq \tau$ and therefore $u(c^*_t)$ from the point of $t = \tau$ onward is constant. This enables us to take the next step and express the objective function (yet again) and optimal utility as:

$$U^*_x(w,\pi) \ = \max_{\tau,\, c_t} \ \left[\int_0^\tau \ e^{-rt} \, ({}_tp_x) \, u(c_t) dt \right] \ + \ u(\pi) \, a_x(\tau), \tag{9}$$

where $a_x(\tau)$ is the (deferred) annuity factor at age x, but beginning or starting income at time τ. It is sometimes called an advanced life delayed annuity (ALDA). As far as notation is concerned, when the deferral period is $\tau = 0$, we will resort to the simpler expression a_x instead of the cumbersome $a_x(0)$. More importantly, when $w = 0$ this all collapses to:

$$U^*_x(0,\pi) \ = \ u(\pi) \, a_x, \tag{10}$$

If the consumer has no investable funds ($w = 0$) and they are living-off pension π income only, then discounted (optimal) lifetime utility is simply: $u(\pi) \, a_x$. See Cannon and Tonks (2008), and specifically chapter #7, for a derivation of (the well known, at least to annuity economists) equation (10), under more general (Epstein Zin, for example) preferences.

Finally, the optimal consumption before the wealth depletion time τ, when consumption is not equal to the pension π, is:

$$c_t^* \;=\; c_0^*({}_tp_x)^{1/\gamma} \;=\; \left(\frac{\pi}{({}_\tau p_x)^{1/\gamma}}\right)({}_tp_x)^{1/\gamma}, \;\; when\; \pi > 0 \tag{11}$$

where the optimal initial consumption rate c_0^* is related to (terminal) consumption π, via the relationship $c_0^* = \pi/({}_\tau p_x)^{1/\gamma}$, as long as there actually is some pension income $\pi > 0$. In contrast to equation (11), when $\pi = 0$, the relevant consumption rate must be sufficient to last forever (in theory), so that $w = c_0^* \int_0^\infty e^{-rt}({}_tp_x)^{1/\gamma}dt$, which leads to the corresponding:

$$c_t^* \;=\; \left(\frac{w}{\int_0^\infty e^{-rt}({}_tp_x)^{1/\gamma}dt}\right)({}_tp_x)^{1/\gamma}, \;\; when\; \pi = 0. \tag{12}$$

We simply note that the integral in the denominator of equation (12) is an annuity factor (of sorts) assuming that the survival probability $({}_tp_x)$ is shifted or distorted by $1/\gamma$. For example, when $\gamma = 1$ and utility is logarithmic, the optimal consumption function c_t^* in equation (12) collapses to the hypothetical annuity consumption w/a_x times the survival probability $({}_tp_x)$, which is clearly less than what a true annuity would have provided. The individual who converts all liquid wealth w into the annuity would consume $c_t^* = w/a_x$ for ever, but the non-annuitizer must continue to reduce consumption in proportion to their survival probability as a precautionary measure. Although it might seem as if equation (11) or equation (12) is plucked-out of thin air, neither of these are new or novel. See Cannon and Tonks (2008) or Charupat, et al. (2012), chapter #13 for example. Rather, our objective here is to use these analytic expressions to solve for and extract δ, which has not been done previously in the literature.

2.5 Solving for δ when $\pi = 0$.

Using the prior notation, let $U_x^*(w, 0)$ denote discounted lifetime utility of wealth w without annuities. This can be expressed mathematically as:

$$U_x^*(w, 0) \;=\; \int_0^\infty e^{-rt}\,({}_tp_x)\,u(c_t^*)\,dt, \tag{13}$$

where c_t^* is the optimized consumption path we displayed in equation (12). Note that there is no wealth depletion time since there are no annuities. In contrast let $U_x^*(0, w/a_x)$ denote discounted lifetime utility of wealth, assuming wealth w **is entirely** annuitized or pooled at age x. Discounted utility is:

$$U_x^*(0, w/a_x) \;=\; \int_0^\infty e^{-rt}\,({}_tp_x)\,u(w/a_x)\,dt, \tag{14}$$

where the optimized consumption path is trivially $c_t^* = w/a_x$, for all t. Technically $\tau = 0$ because all wealth has been annuitized at time zero.

We won't belabor the point that $U^*_x(0, w/a_x) \geq U^*_x(w, 0)$ but simply conclude by noting that δ_0 will satisfy the following equation:

$$U^*_x(w(1+\delta_0), 0) \ = U^*_x(0, w/a_x), \tag{15}$$

and the rest is algebra. Basically, we compute the inverse function $y = U_x^{(*,-1)}(z, \pi)$ for $z = U^*_x(y, \pi)$ with respect to the wealth variable w. The quantity we are looking for is:

$$\delta_0 \ = \ \frac{1}{w} U^{(*,-1)}(U^*_x(0, w/a_x), 0) \ - 1, \tag{16}$$

and more generally, in the presence of pension income $\pi > 0$, the expression generalizes to:

$$\delta_\pi \ = \ \frac{1}{w} U^{(*,-1)}(U^*_x(0, w/a_x + \pi), 0) \ - 1, \tag{17}$$

where the (inverse) value function in the denominator of equation (17) includes and requires the calculation of the wealth depletion time τ. Both of these, δ_0 and δ_π are are concerned with complete annuitization, or what we called AEW *in-the-large.* If we focus on the incremental dollar of annuitization, the relevant expression becomes:

$$v \ = \ U^{(*,-1)}(U^*_x(w-1, 1/a_x + \pi), \pi) \ - w, \tag{18}$$

where v is (what we call) the value *in-the-small.* The rest of this paper is about making specific assumptions on the underlying mortality function (${}_tp_x$), pre-existing pension income π, and then doing the (tedious) work of inverting the value functions to obtain expressions for equations (16) - (18).

3 Exponential Remaining Lifetime

3.1 Start with the Corners

If the entire w is annuitized at time zero, the optimal consumption rate is: $c^*_t = (w/a_x + \pi)$, which in the case of an exponentially distributed lifetime collapses to: $c^*_t = w(r+\lambda) + \pi$, because the annuity factor is: $a_x = 1/(r+\lambda)$. The maximal utility possible when the entire endowment (w, π) is annuitized, according to equation (14), is:

$$U^*_\lambda(0, w/a_x) \ = \ \int_0^\infty e^{-rt}(e^{-\lambda t}) \frac{(w(r+\lambda))^{1-\gamma}}{1-\gamma} dt \ = \ \frac{(w(r+\lambda))^{1-\gamma}}{(1-\gamma)(r+\lambda)}, \tag{19}$$

where the subscript λ in the U_λ reminds readers we are operating under an exponential remaining lifetime in which the (only) parameter that matters is the hazard rate λ. The first item in the integrand captures the subjective discounting of utility, the second is the survival probability and the third is the instantaneous utility, which is constant (because consumption equals the annuity).

Equation (19) represents the gold standard or nirvana by which all other strategies will be measured. It represents the highest possible level of utility, assuming the retire converts the entire w to additional annuity units. At the other extreme of equation (19) is the (obstinate) individual who refuses to annuitize any wealth at all, and finances all consumption from liquid wealth. For this individual the maximal achievable level of utility can be simplified to:

$$U_\lambda^*(w,0) \;=\; \int_0^\infty e^{-(r+\lambda)t}\,\frac{(c_t^*)^{1-\gamma}}{1-\gamma}dt \;=\; \frac{(w(r+\lambda/\gamma))^{1-\gamma}}{(1-\gamma)(r+\lambda/\gamma)}. \tag{20}$$

Note from equation (12) that $({}_tp_x)^{1/\gamma} = e^{-(\lambda/\gamma)t}$ and the optimal consumption function $c_t^* = w(r+\lambda/\gamma)e^{-(\lambda/\gamma)t}$. By comparing equation (19) to equation (20) and the concavity of the power function, one can see explicitly that as long as the mortality rate $\lambda > 0$, the utility from annuitization $U_\lambda^*(0, w/a_x)$ is greater than the utility from self-annuitization $U_\lambda^*(w,0)$, or any other systematic withdrawal or drawdown plan.

Equating the utility from equation (19) to the utility in equation (20), the value of longevity pooling δ_0 (in-the-large), when the individual has no pre-existing annuity income, will solve the following equation:

$$\frac{((1+\delta_0)(r+\lambda/\gamma))^{1-\gamma}}{r+\lambda/\gamma} = \frac{(r+\lambda)^{1-\gamma}}{r+\lambda}, \tag{21}$$

which after logarithms, cancelations and simplifications leads to one of our main (closed-form) expressions:

$$\delta_0 = \left(\frac{r+\lambda/\gamma}{r+\lambda}\right)^{\gamma/(1-\gamma)} - 1, \quad \text{when } \lambda > 0,\ \pi = 0 \tag{22}$$

under any combination of r, λ, γ assuming $\gamma \neq 1$; otherwise one has to take limits as $\gamma \to 1$.

To be clear, equation (22) only applies to the case in which the future remaining lifetime is assumed to be exponentially distributed with expected remaining lifetime $1/\gamma$, and in addition there is no pre-existing pension income, so that $\pi = 0$. Nevetheless, the structure of equation (22), will continue to make an appearance in all our expressions for the value of longevity risk pooling. It will have the same (ratio) format, regardless of the specific (continuous or discrete) law of mortality.

The value of longevity risk pooling in equation (22) increases in the mortality rate λ (and declines in the remaining life expectancy), but declines in the interest rate r. This characteristic of δ generalizes to non-decreasing laws of mortality, such as Gompertz-Makeham. As far as exponentially mortality is concerned, the derivative of δ_0 can be written as:

$$\frac{\partial \delta_0}{\partial r} \;=\; \frac{\gamma}{1-\gamma}\;\left(\frac{\lambda(1-1/\gamma)}{(r+\lambda)^2}\right)^{\gamma/(1-\gamma)-1}, \tag{23}$$

which is negative for $\gamma > 1$. Indeed, when interest rates are relatively lower (i.e. $r = 2\%$ vs. $r = 4\%$) the value of longevity pooling is greater to the consumer. And, the older you are (i.e. larger λ) the more you benefit from pooling. All this according to equation (23).

Note also that when the mortality rate λ happens to be such that $r\gamma = \lambda$, and the ratio of interest rates to mortality rates happens to equal the coefficient of relative risk aversion, the value of longevity pooling reported in equation (22), can be pushed further to yield:

$$\delta_0 = \left(\frac{2}{1+\gamma}\right)^{\gamma/(1-\gamma)} - 1, \quad \text{when } \lambda = r\gamma, \ \pi = 0 \tag{24}$$

This (oddly enough) does not depend on λ or the interest rate r. For example, when $\gamma = 2$ the value of longevity risk pooling is then $\delta_0 = 125\%$ in equation (24), which means that the AEW is \$2.25 per initial \$1 of wealth in the absence of any pre-existing pension income. And, when $\gamma = 1.25$ the value of longevity risk pooling is $\delta = 80\%$ in equation (24), both in the simplified case in which $\lambda/r = \gamma$. For comparison purposes, Kotlikoff and Spivak (1981) on page #378, report values for $\gamma = 1.25$ which in their case are: $\delta_0 = 97\%$ for males and $\delta_0 = 85\%$ for females at the age of 75, assuming $\rho = r = 1\%$. So these numbers – under exponential remaining lifetime assumptions – aren't very far from numbers based on more realistic mortality tables.

Interestingly, if we compute the limit of $(2/(1+\gamma))^{\gamma/(1-\gamma)} - 1$ as γ goes to 1 and the utility function converges to logarithmic, the value of longevity risk pooling in equation (24), converges to:

$$\delta_0 = \sqrt{e} - 1 \approx 64.9\%, \qquad \text{when } r = \lambda, \gamma \to 1. \tag{25}$$

which just to be clear (again), assumes that the mortality rate $\lambda = r$. Perhaps this is a stretch, but an interesting expression nevertheless which provides an *actuarial economic* interpretation to $\sqrt{e}$.

Now, these simple expressions are nice and helpful at the extremes of all-or-nothing annuities, but what is the value of longevity pooling δ_π when the individual has pre-existing pension income? We now move on to maximal utility for the generalized endowment (w, π), to obtain an expression that can be compared against equation (22). Obviously it won't be as clean and will also involve initial w as well as pension income π. More importantly, the value of δ_π will not be as large as δ_0, when the discounted economic value of the initial endowments (w, π) are identical.

3.2 Wealth Depletion Time

The first order of business is to derive the wealth depletion time (WDT) denoted by τ, which is instrumental and a key milestone for computing the maximal utility value $U^*(w, \pi)$. We start with the basic budget constraint which dictates that:

$$w \; = \; \int_0^\tau (c_t^* - \pi) e^{-rt} dt \; = \; \pi \int_0^\tau \left(\frac{e^{-\lambda t/\gamma}}{e^{-\lambda\tau/\gamma}} - 1 \right) e^{-rt} dt. \tag{26}$$

Note that the transition from the c_t^* to the expression in the second integrand comes directly from equation (11) and the boundary condition (at the WDT) which states $c_\tau = \pi$.

We are solving for τ in the above equation, which after a bit of calculus leads to:

$$\left(\frac{r}{r+\lambda/\gamma}\right) e^{\frac{\lambda}{\gamma}\tau} + \left(\frac{\lambda/\gamma}{r+\lambda/\gamma}\right) e^{-r\tau} = \frac{rw}{\pi} + 1, \tag{27}$$

where the variable we are most interested in isolating, τ, appears in two different exponents and is difficult to extract in closed form. It can however easily be solved iteratively starting with $\tau = 0$ and incrementing (month by month) until reaching the value of the right-hand side. For example, when $w = 100, \pi = 10, r = 0.03, \gamma = 2$ and the mortality rate $\lambda = 1/20$, the wealth depletion time is $\tau = 28.24$ years. But, if the pension is doubled to $\pi = 20$ units, the WDT drops to $\tau = 20.08$ years. Note that for positive values of γ, λ, r, the left-hand side of the above equation is monotonically increasing in $\tau \geq 0$, which lends itself nicely to quick (root finding) numerical methods for locating the unique value of τ for which the left-hand side is equal to $\frac{rw}{\pi}$. In fact, when $\tau = 0$, the value of the left-hand side is one, which is consistent with the fact that $\tau = 0$ can be a wealth depletion time if-and-only-if $w = 0$, whenever $\pi > 0$.

3.3 Special Case Again: $\lambda/\gamma = r$

When the mortality rate $\lambda = \gamma r$, a condition we mentioned earlier in the context of a simplified value for δ_0, the left hand side of equation (27) collapses to: $\frac{1}{2}\left(e^{r\tau} + e^{-r\tau}\right) = \cosh(r\tau)$, which is one of the well-known hyperbolic trigonometric functions, This enables us to solve for the wealth depletion time (WDT) explicitly as:

$$\tau = \frac{1}{r} \ln\left[\frac{rw}{\pi} + 1 + \sqrt{\left(\frac{rw}{\pi} + 1\right)^2 - 1}\right], \quad \text{when } \lambda/\gamma = r. \tag{28}$$

We now have an explicit (analytic) expression for the wealth depletion for an exponential remaining lifetime. More importantly, τ doesn't depend on either the level of risk aversion γ or the mortality rate λ, as long as $\lambda/\gamma = r$. For example, with an initial pension income of: $\pi = 3$, initial liquid wealth of $w = 60$ and interest rate of $r = 2.5\%$, the value of $rw/\pi + 1 = 1.5$, which results in $\tau = \ln(1.5 + \sqrt{1.25}))/0.025 = 38.5$ years. This might seem like a (very) long time, but if indeed the individual lives to this age he/she would (rationally) exhaust all wealth and live-off the \$3 of pension income. The consumption rate up to the wealth depletion time is: $c_t = (3/(e^{(0.025)(38.5)})^{0.025t}$, according to equation (11), which is equal to: $c_t^* = 3$, when $\tau = 38.5$.

Now that we have the wealth depletion time τ, either explicitly from equation (28), or implicitly from equation (27), as well as the optimal consumption function c_t^*, we move-on to compute maximal utility.

3.4 Maximal Utility

Recall that when the mortality rate λ is constant, the immediate annuity factor can be expressed as $a_x = 1/(r+\lambda)$ and the deferred annuity factor is $a_x(\tau) = e^{-(r+\lambda)\tau}/(r+\lambda)$. The maximal value of utility, going back to the original formulation in equation (9), is:

$$U^*_\lambda(w,\pi) \;=\; \frac{\pi^{1-\gamma}}{1-\gamma}\left[\int_0^\tau e^{-(r+\lambda)t}\left(\frac{e^{-\lambda t/\gamma}}{e^{-\lambda\tau/\gamma}}\right)^{1-\gamma} dt + \frac{e^{-(r+\lambda)\tau}}{r+\lambda}\right], \tag{29}$$

where the optimal consumption function comes from equation (11). Note that τ appears three times in the above expression, once in the upper bound of integration, then in the denominator of the integrand and finally in the deferred annuity factor. Once again, after a bit of calculus, equation (29) can be re-written as:

$$U^*_\lambda(w,\pi) \;=\; \frac{\pi^{1-\gamma}}{1-\gamma}\left(\frac{e^{2r\tau}-1}{2r} + \frac{1}{r+\lambda}\right)\left(\frac{1}{e^{r\tau}}\right)^{\frac{\lambda}{r}+1} \tag{30}$$

The value of longevity risk pooling *in-the-large* or *in-the-small*, will be the value of δ_π and v that respectively solve:

$$U^*_\lambda(w(1+\delta_\pi),\pi) \;=\; U^*_\lambda(0,\pi + w(r+\lambda)) \tag{31}$$

and

$$U^*_\lambda(w+v,\pi) \;=\; U^*_\lambda(w-1,\pi+r+\lambda) \tag{32}$$

With all the analytics in hand, we are now ready for some numerical examples.

3.5 Numerical Examples

Table #1 contains detailed results from two separate parametric examples. As stated in the objective, our annuity equivalent wealth (AEW) computations do not involve any dynamic programing algorithms, discretization or approximation schemes and can be easily replicated or reapplied to other parameters.

Both cases (A and B) in table #1 are based on an economy in which the (risk-free) interest rate is 2.5%, a retiree (who we call Xi) has an initial economic endowment of \$100 and rational preferences that can be described using a constant relative risk aversion (CRRA) utility function. To be very clear, the \$100 captures the sum of two terms: (i.) liquid non-annuitized investable wealth, and (ii.) pre-pensionized income. The rows in table #1 model different combinations of pension income plus investable wealth, but all are associated with an actuarial present value of exactly \$100. The economic equivalence is important when we illustrate or compare the utility value of pooling. Comparing the AEW of someone with \$100 in investable wealth and \$20 in pension income, to someone with a \$100 nest egg and \$1 in pension income is synonymous with comparing (rich) apples to (poor) oranges.

Table #1

What is the Utility Value of (More) Longevity Pooling

AEW in the large: $U^*_\lambda(w(1+\delta),\pi) \ = \ U^*_\lambda(0,\pi + w/a_x)$: **Solve for** δ**.**

AEW in the small: $U^*_\lambda(w+v,\pi) \ = \ U^*_\lambda(w-1,\pi + 1/a_x)$: **Solve for** v**.**

Case A: Interest Rate $\rho = r = 2.5\%$, CRRA $\gamma = 2.0$, Mortality Rate $\lambda = 5.0\%$.

Endowment ($)	**Pensionized ($)**	**Depletion**	**Consume ($)**	v **($)**	δ **(%)**
$w = 100$, $\pi = 0.0$	$PV(0.000) = 0.0$	$\tau = \infty$	$c_0 = 5.000$	1.986	**125.0%**
$w = 86\frac{2}{3}$, $\pi = 1.0$	$PV(1.000) = 13\frac{1}{3}$	$\tau = 72.8$ yrs.	$c_0 = 6.171$	1.668	**114.8%**
$w = 73\frac{1}{3}$, $\pi = 2.0$	$PV(2.000) = 26\frac{2}{3}$	$\tau = 50.7$ yrs.	$c_0 = 7.104$	1.432	**104.2%**
$w = 60$, $\pi = 3.0$	$PV(3.000) = 40$	$\tau = 38.5$ yrs.	$c_0 = 7.854$	1.232	**93.0%**
$w = 46\frac{2}{3}$, $\pi = 4.0$	$PV(4.000) = 53\frac{1}{3}$	$\tau = 29.8$ yrs.	$c_0 = 8.437$	1.049	**80.9%**
$w = 25$, $\pi = 5.625$	$PV(5.625) = 75$	$\tau = 18.6$ yrs.	$c_0 = 8.974$	0.743	**57.7%**
$w = 10$, $\pi = 6.750$	$PV(6.750) = 90$	$\tau = 10.9$ yrs.	$c_0 = 8.854$	0.468	**35.7%**
$w = 1$, $\pi = 7.425$	$PV(7.425) = 99$	$\tau = 3.28$ yrs.	$c_0 = 8.060$	0.110	**11.0%**
$w = 0$, $\pi = 7.500$	$PV(7.500) = 100$	$\tau = 0$	$c_0 = 7.500$	N.A.	N.A.

Case B: Interest Rate $\rho = r = 2.5\%$, CRRA $\gamma = 1.25$, Mortality Rate $\lambda = 3.125\%$.

Endowment ($)	**Pensionized ($)**	**Depletion**	**Consume ($)**	v **($)**	δ **(%)**
$w = 100$, $\pi = 0.0$	$PV(0.000) = 0.00$	$\tau = \infty$	$c_0 = 5.000$	1.243	**80.2%**
$w = 82.23$, $\pi = 1.0$	$PV(1.000) = 17.77$	$\tau = 71.3$ yrs.	$c_0 = 5.943$	1.035	**72.0%**
$w = 64.45$, $\pi = 2.0$	$PV(2.000) = 35.55$	$\tau = 47.9$ yrs.	$c_0 = 6.618$	0.869	**63.2%**
$w = 46.67$, $\pi = 3.0$	$PV(3.000) = 53.33$	$\tau = 34.2$ yrs.	$c_0 = 7.058$	0.716	**53.4%**
$w = 28.89$, $\pi = 4.0$	$PV(4.000) = 71.11$	$\tau = 23.7$ yrs.	$c_0 = 7.232$	0.555	**41.8%**
$w = 10$, $\pi = 5.063$	$PV(5.063) = 90.00$	$\tau = 12.5$ yrs.	$c_0 = 6.923$	0.330	**24.6%**
$w = 1$, $\pi = 5.568$	$PV(5.568) = 99.00$	$\tau = 3.79$ yrs.	$c_0 = 6.122$	0.078	**7.8%**
$w = 0$, $\pi = 7.625$	$PV(5.625) = 100.0$	$\tau = 0$	$c_0 = 5.625$	N.A.	N.A.

Assumes the Remaining Lifetime (T_x) is Exponentially Distributed with Mortality Rate λ.

The Expected Remaining Lifetime and Standard Deviation of Lifetime (T_x) are both $1/\lambda$ years.

The value of a \$1-per-year life annuity is: $a_x = 1/(r+\lambda)$ when (T_x) is Exponentially Distributed.

Let's start at the very top of Case A, when the retiree (Xi) has \$100 in liquid wealth but no pre-existing pension income, which is why (in the second column) the actuarial present value of any pre-existing pension income is zero. For this initial endowment, the optimal consumption strategy for Xi is spending 5 dollars (in the first year) and rationally reduce consumption by $r + \lambda/\gamma = 5\%$ each year, as per the optimal consumption function and instructions in equation (12). Xi will never exhaust or deplete the investable funds (since $5e^{-0.05*t}$ is always positive) regardless of how long Xi lives.

To our main point, let's compare Xi to his or her utility twin (called Yi) who uses the entire 100 dollars to voluntarily purchase the (actuarially fair, unloaded) life annuity at time zero. By fully annuitizing, Yi's discounted utility of lifetime consumption would be higher. Indeed, this is the foundation of annuity (and much of pension) economics and the famous Yaari (1965) result. Nobody can beat Yi.

Back to utility. Xi would require an additional 125% of initial wealth – that is a total of 225 dollars – to be as well off (or happy) in utility terms as Yi who used the entire \$ 100 to purchase a fairly priced life annuity. For the record, case A assumes a mortality rate of 5%, which implies that \$100 would entitle Yi to $100(0.025{+}0.05) = 7.5$ dollars of annuity income for life. Yi would (immediately) consume more and experience more discounted utility. A win-win situation, which isn't observed in all cases.

For purposes of replication, the maximal utility (for Yi) if the entire \$100 were annuitized is: -1.777 utiles according to equation (19), whereas the maximal utility (for Xi) is exactly -4.0 utiles. Stated differently, giving Xi who has \$100 of investable wealth and no annuity income an additional 125 dollars will increase discounted utility from -4 to -1.777, which is the classic definition of annuity equivalent wealth (AEW). The compensating number can be expressed as a percentage (125%), or on a per dollar basis (2.25). A pension actuary would say that the value of longevity pooling is 125% of retirement wealth, and we adopt that language.

So far there is nothing new here, although some readers might be surprised at the relatively high value of the AEW (or pooling), compared to results reported in Brown et al (2001) for example. They use a similar 2.5% interest rates and obtain pooling values of (only) 65%, when the coefficient of relative risk aversion of the representative consumer is equal to 2.0, which is what we used here.

The main explanation for our (higher) AEW or pooling values is that we have assumed an exponentially distributed remaining lifetime, with a mean value of 20 years and a standard deviation of 20 years. We really had no choice since it is forced by the 5% mortality rate underlying the exponential remaining lifetime. This (for example) leads to a 10% probability of spending 45 years in retirement and living beyond age 110. This (yes, unrealistic) mortality assumption does results in a (much) higher value for the longevity pooling, which is why our numbers are higher than Brown et al (2001).

Note that they used projections for the 1930s Social Security Administration cohort for which life expectancy was 15 years at the age of retirement. In some sense, our exponential remaining lifetime assumptions are biased in favor of higher AEW and pooling values due to the *fatter longevity tail* of the exponential distribution. But our rationale (in this section) was to illustrate how these values decline when pre-existing annuities are included in the initial endowment. In the next section (#4) we will display AEW numbers and the value of pooling for more realistic mortality assumptions. There we aren't as fortunate to have analytic expressions for all quantities of interest, but the numbers (better) match those in Brown et al. (2001).

Moving on, the second to last column in table #1 displays the results of the annuity equivalent wealth (AEW) *in-the-small.* The number is intended to answer the question: How much additional wealth would Xi the retiree (who recall has no annuity income, but \$100 in liquid wealth) require to be as satisfied as the utility twin Yi who only converted \$1 into annuity income? The hypothetical twin doesn't convert the entire \$100 dollars in annuities, only 1 dollar. We are focusing on the marginal impact. Our numbers are based on the v that satisfies equation (32).

To be clear, we are now comparing the utility of someone (Yi) who has \$99 of liquid wealth plus (0.025+0.05)=0.075 dollars in lifetime income, to someone (Xi) who has \$100 in liquid wealth and absolutely no pension or annuity income. The actuarial present value of the 0.075 dollars of annuity income is equal to 1 dollar, so the initial economic endowment of both twins remains the same.

According to equation (32), Xi would now require or demand an additional \$1.986 (for a total wealth of \$101.986) to obtain the same level of discounted lifetime utility, compared to twin Yi with \$99 of liquid wealth and \$0.075 of annual annuity income. Notice that as a percent of the one dollar used by Yi to purchased annuities, the \$1.986 is much more than the 125%, which we associated with annuity equivalent wealth *in-the-large.*

The reason for this apparent non-homogeneity, despite the assumed CRRA utility structure, is that (even) when a small sum is converted into annuity income, the discounted value of lifetime utility, or the value function no longer scales in initial wealth. This is precisely because of the wealth depletion time which is an integral (albeit obscured) part of the optimal consumption strategy.

The sixth row of the table (still in Case A) illustrates a consumer (still Xi) who is entitled to a pension of \$5.625 for life, but who has liquid wealth (savings) of \$25 at retirement. Note that the multiple of pension income to investable wealth is 4.4 to one, which is consistent with a typical (American) retiree who is entitled to social security benefits of \$32,000 per year and might have (an average of) \$140,000 in retirement savings. Within table #1, this retiree also has an initial economic endowment of \$100 dollars, but 75% is already pre-allocated to pension annuities, whereas only 25% is liquid and available for (further) annuitization.

The extra utility from annuitizing the remaining \$25 will be lower, and the last column in table #1 indicates that the value of pooling for this situation is (only) 57.7%. This number is less than half of the value of pooling when the individual has no pre-existing annuity income (recall the 125%), but it still quite substantial. To put this number in a sentence, the annuity equivalent wealth of \$25 in investable wealth when you have \$5.625 in pension income is 25(1.577) = \$39.425. This once again is what we term the value of longevity risk pooling *in-the-large.*

The second to last column in the same row indicates that the parallel AEW *in-the-small* is 0.743 dollars. This means that Xi with 25 dollars would require an additional 0.743 dollars (or 25.743) to make them as well-off, in a utility sense, as Yi who has 24 dollars of investable wealth and 5.625 + 0.075 = 5.7 dollars of annuity income. Either way, whether we focus on the absolutely (large) or marginal (small) the value of longevity risk pooling is non-zero, but getting lower and closer to the insurance loadings (and anti-selection costs) that one might observe in practice. More on this in the conclusion.

Moving on to the optimal consumption strategy for this individual (\$25 of investable wealth and \$5.625 of annuity income), the wealth depletion time (WDT) is 18.6 years. As we explained in the earlier section, Xi will rationally exhaust wealth after 18.6 years of retirement and (from that point onward) only consume the pension income of \$5.625. The initial consumption rate will be \$8.437 per year, of which \$5.625 is pre-existing pension income and \$2.812 comes from liquid investable wealth. Using terminology common among financial advisors, this is a drawdown or retirement spending rate of 2.812/25 or 11%, and fully rational given the (higher) pension income.

Moving to the very bottom of the Case A section, notice that when 99% of the retiree's initial endowment is already pre-annuitized, both the AEW *in-the-large* and the AEW *in-the-small* is equal to 11%. Their definitions now coincide. Giving-up one dollar (and buying more annuities) leads to the same utility as annuitizing the entire one dollar.

Table #1 also offers numbers (under Case B) when the mortality rate λ and the coefficient of relative risk aversion γ are reduced. For the second case, the CRRA value is set at $\gamma = 1.5$ and the mortality rate is set to $\lambda = 3.125\%$, which is synonymous with a life expectancy of 32 years (versus the 20 years in Case A.) We selected these precise numbers for Case B, because we wanted to take advantage of the complete analytic representation for τ and δ, that only work when $\lambda/\gamma = r$.

A lower or reduced value for the coefficient of relative risk aversion γ results in a lower value of longevity risk pooling δ_0, δ_π or v. Note also that optimal consumption c_t^* is reduced or lower when γ is higher, due to the higher (risk adjusted) probability of living to an advanced age. As in Case A which was reported above, all of these numbers were obtained using the analytic expressions in the prior section, although the value of v within equation (32) was extracted using a simple root-finding algorithm.

Before we conclude this numerical section (under exponential lifetimes) and move on to more realistic mortality tables and curves, it's worth commenting on the non-monotonic level of consumption as one goes down the rows in the two cases. The initial consumption rate (in the absence of any annuities) is \$5 per year for life, but then increases as the fraction of pre-existing pension income is increased. Eventually it does reach a maximum when 70% to 75% of the initial endowment is pre-pensionized. It then begins to decline with additional annuitization. This shouldn't be viewed as troubling or inconsistent. A higher initial consumption rate doesn't necessarily lead to additional utility primarily because consumption is forced to decline (more) rapidly over time. Stated differently (and perhaps counter-intuitively), maximizing discounted lifetime utility doesn't necessarily maximize the initial consumption or spending rate.

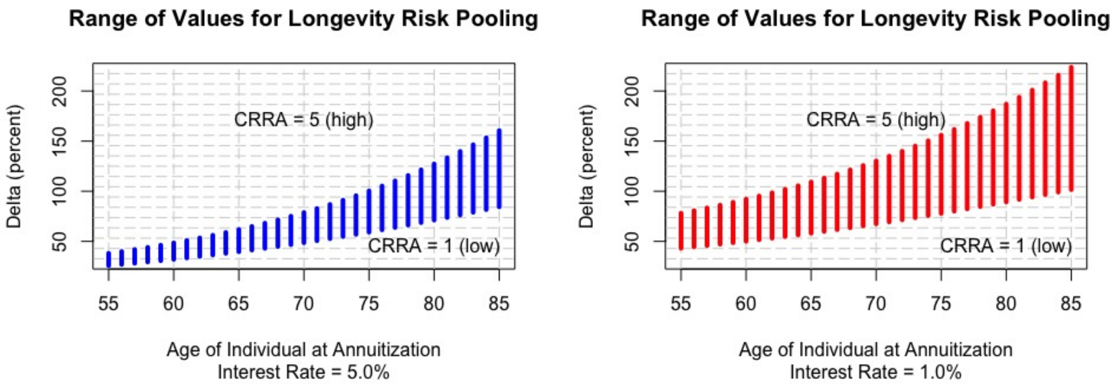
That said, within the context of behavioral finance and economics, it might be (very) hard to convince Xi who is (rationally and optimally) spending \$8.974 per year (sixth row of Case A, for example) that they should annuitize the \$25 of liquid wealth (like Yi), so that their consumption rate can immediately drop to \$7.5 per year. Xi might not be persuaded that such an action is equivalent (in utility terms) to having 57.7% more liquid wealth. This is quite different from (2nd row, Case A) where the individual who is persuaded to annuitize his or her entire wealth can immediately benefit from a jump in consumption from \$6.171 per year to \$7.5 dollars per year, for life. Alas, perhaps it is not surprising that (in the real world) annuitization can be a hard sell.

In the next section #4, we offer some analytic insights for the case in which remaining lifetime is no longer exponentially distributed and the force of mortality increases over time and/or is fit to a proper life table.

4 General Mortality Models

We do not want to get bogged down in the details of an appropriate model for current and future mortality rates λ_t and instead refer the readers to the book by Pitacco, Denuit, Haberman and Olivieri (2009) for appropriate analytic representations. In this section we continue to use the notation a_x to represent the general annuity factor under an interest rate r, assuming the current age of the individual is x. The annuity factor can actually be expressed analytically when the underlying mortality basis is assumed to obey a Gompertz-Makeham law, and this is presented in the book by Charupat et al. (2012), page #289, for example. Alternatively, under a discrete mortality table (with or without projection scales), the expression a_x is easily available and computed using any basic spreadsheet. Either way, using the same methodology we described and followed in section #3, when $\pi = 0$, one obtains the following analytic expression for the value of longevity risk pooling:

Figure 1: *Displayed δ_0 values are function of initial annuitization age x, risk aversion γ and interest rate. Blue (left figure) is for $r = 5\%$, red (right figure) is for $r = 1\%$, both with risk aversion coefficients $\gamma = 1$ (lower) to $\gamma = 5$ (upper). Mortality is fit to 1930 SSA cohort.*



$$\delta_0 = \left(\frac{a_x}{a_{x^*}}\right)^{\frac{\gamma}{1-\gamma}} - 1, \tag{33}$$

where the (new) variable x^* in the denominator of equation (33) represents a modified or risk-adjusted age, similar to the reduction of the mortality rate λ/γ that appeared earlier in equation (22). One can think of x^* as an age set-back. In particular, if we assume Gompertz mortality with parameters (m, b), the adjusted age $x^* = x - b \ln \gamma$, where γ is the coefficient of relative risk aversion. For example, if $\gamma = 2$, the current age $x = 65$ and the dispersion coefficient $b = 11.5$, (which is approximately the standard deviation of the remaining lifetime), then the risk-adjusted age $x^* = 57.03$. But if the risk aversion coefficient is increased to $\gamma = 3$, then $x^* = 52.376$.

Here is another way to express and think about this. The annuity factor in the numerator of the fraction in equation (33) is computed using the retiree's **biological** age, whereas the denominator is computed at a risk-adjusted **economic** age.

4.1 In a Gompertz-Makeham mortality world

The general methodology for computing δ_0, captured by equation (33) is applicable under any mortality basis in which the implied force of mortality is non-decreasing and follows from equation (16). The derivation or proof is relatively simple in the context of Gompertz-Makeham (GM) mortality. In particular, the justification for the age-shift from x to x^* in the annuity factor, is as follows. Recall that the optimal consumption function c_t^* can be written in terms of either the initial consumption rate c_0^* or the terminal (pension) consumption rate π, via the adjusted survival probability $({}_tp_x)^{1/\gamma}$. This comes directly from equation (12) or equation (11). Those analytic expressions for consumption are exponential in t, easily flow through the integral calculations, which lead to the value of longevity risk pooling.

Table #2

Comparing Analytics to Numerics: Annuity Equivalent Wealth

CRRA	**Gompertz Mortality (analytic)**	**Results in Brown et. al. (2001)**
$\gamma = 1.0$	$1 + \delta_0 = 1.499$	1.502
$\gamma = 2.0$	$1 + \delta_0 = 1.650$	1.650
$\gamma = 5.0$	$1 + \delta_0 = 1.872$	1.855
$\gamma = 10$	$1 + \delta_0 = 2.050$	2.004

Gompertz (Unisex) Mortality with: $m = 81, b = 11.5$ and $r = \rho = 2.5\%$ discount rate.

Life Expectancy of 15.4 years at age 65, based on the 1930s SSA cohort.

In the case of GM mortality the risk-adjusted survival probability can be expressed as:

$$({}_tp_x)^{1/\gamma} \;=\; e^{-\frac{1}{\gamma}\int_0^t \lambda_s ds}, \tag{34}$$

where the GM mortality hazard rate can be written as:

$$\lambda_s \;=\; \lambda + \frac{1}{b}\exp\left(\frac{x+s-m}{b}\right). \tag{35}$$

If we divide both sides and scale by γ, the relevant integrand can be expressed as:

$$\frac{\lambda_t}{\gamma} \;=\; \frac{\lambda}{\gamma} + \frac{1}{b}\exp\left(\frac{x - b\ln\gamma + s - m}{b}\right) \;=\; \frac{\lambda}{\gamma} + \frac{1}{b}\exp\left(\frac{x + s - (m + b\ln\gamma)}{b}\right) \tag{36}$$

where λ is replaced with λ/γ and the initial age x is replaced with $x^* = x - b\ln\gamma$. Alternatively, the initial age remains x and the modal parameter m is replaced with $m^* = m + b\ln\gamma$.

More generally, if we want to compute the value of δ_0 for any given discrete mortality table, the crude key (or rough trick) is to adjust the underlying mortality rate q_x by the coefficient of relative risk aversion γ, which would (roughly) approximate the process of computing λ/γ. If more precision is desired, then it would probably be best to locate the best fitting biometric (λ, m, b) values and then use the analytic Gompertz-Makeham annuity factor with the adjusted modal value: $m + b\ln\gamma$ and with λ/γ.

4.2 Results: Continuous Gompertz vs. Discrete Brown

Table #2 compares results for the annuity equivalent wealth computed in the paper by Brown et al. (2001), vs. the value as per equation (33) and a Gompertz specification with $m = 81$ and $b = 11.5$. The two coefficients were selected to (best) fit the discrete mortality table used in Brown et al (2001). The interest and valuation rate was assumed to be 2.5%, in real terms. We report value of $1 + \delta_0$ under a variety of γ values.

The numbers in the first column of this table (pretty much) match the values computed by Brown, Mitchell, Poterba and Warshawsky (2001), pg. 143, using mortality data from the Social Security Administration (SSA) for the 1930s cohort. For example, if the individual is extremely ($\gamma = 10$) risk averse, he/she would require 2.05 to achieve the same (maximal) level of utility as someone who spends \$1.00 and uses the funds to acquire annuity income $1/a_{65}$. The numbers reported in Brown et al. (2001) are \$2.004 per initial \$1. Finally, figure #1 provides a full spectrum of results and different retirement age (55 to 85) and levels of risk aversion. It tells the same story.

4.3 Discrete Mortality: Gender Specific

Here is another example using a different set of mortality tables and rates. Under the Individual Annuity Mortality (IAM) 1983 (basic) table, which was also used in the literature for many of the original AEW estimates, and an $r = 3\%$ interest rate, the annuity factor at age $x = 65$ is $a_{65} = 13.64645$ for males and $a_{65} = 15.58935$ for females. Assuming a $\gamma = 2$ coefficient of risk aversion in a CRRA utility function and a subjective discount rate ρ equal to the interest rate r, the modified annuity factor is $a_{65^*} = 16.81724$ for males and $a_{65^*} = 18.39907$ for females. For this we have kept it simple and divided the q_x values in the IAM1983 table by $\gamma = 2$ and priced the appropriate (modified) annuity at age $x = 65$ using the scaled mortality. The value of annuity equivalent wealth (AEW) which is δ, is equal to: $\delta_0 = \left(\frac{13.64645}{16.81724}\right)^{-2} - 1 = 0.5187$, or 51.87% for a male at age 65, and the equivalent value is $\delta = 0.3930$ or 39.30% for a female at age 65. The AEW is lower for females because their mortality rate is lower at all ages, making the annuity (pooling) relatively more expensive, etc.

Up until now we have focused on values of $\gamma \geq 1$, but even when $\gamma < 1$ there is still value to longevity risk pooling (although it will obviously be lower), provided that $\gamma > 0$. So, if we reduce the (longevity) risk aversion parameter from 2 (in the prior paragraph) to $\gamma = 0.50$, the corresponding (objective) annuity factors a_{65} do not change. But, the modified or adjusted factors are now (increased) to $a_{65^*} = 10.53740$ for males and $a_{65^*} = 12.72198$ for females. The corresponding values of δ are (only) 29.5% for males and 22.54% for females, according to equation (33). Stated differently, the value of pooling is reduced when you don't *dislike* longevity risk as much.

For the third and final numerical example, we leave the coefficient of (longevity) risk aversion at $\gamma = 0.5$, but reduce the valuation (or pricing) interest rate from 3% to $r = 1.5\%$, which increases a_{65} as well as a_{65^*}. The value of longevity risk pooling is now (slightly higher than the prior numerical example) at 33.93% for males and 26.39% for females. Note that the results are not as sensitive to interest rates. The δ benefits really are driven by risk aversion γ and by the assumed mortality tables.

5 Conclusion

Against the backdrop of declining defined benefit (DB) pension coverage and increasing reliance on defined contribution (DC) investment plans, there is a growing awareness that *longevity risk pooling* is being lost in transition. Practicing actuaries, as well as insurance and pension economists are well aware of the benefits of longevity risk pooling, but the general public, the media and often regulators do not appreciate the social welfare benefits of annuitization.

In this paper we take a step in the direction of helping *explain* and quantify the benefits of pooling to a wider public by deriving some closed-form and easily digestible expressions for the utility-based value of longevity pooling. This utility value is simply the percentage increase in what economists label *annuity equivalent wealth* (AEW). The algorithmic process for computing AEW value is well-known and used by economists, going back to the work by Kotlikoff and Spivak (1981), Brown et al. (2001), or the reference on annuity markets by Cannon and Tonks (2008). What is less known – and we believe one of the contributions of this paper – is that by assuming some analytic representations for the remaining lifetime random variable, one can obtain (nice, easy to use) closed-form expressions for the value of longevity pooling. These formulae are simply unavailable (or highly obscured) when the mortality basis is a discrete and cumbersome mortality table.

In terms of quotable results, under logarithmic utility preferences and the assumption that the remaining lifetime of a retiree is exponentially distributed with a mean value of: $1/\lambda$ (for example 20 years) *and* assuming the risk-free interest rate is coincidentally equal to the instantaneous mortality rate (for example 2.5%), then the value of longevity risk pooling is (remarkably) $\sqrt{e}-1$, which is approximately 65%. Note that in Brown et al. (2001), using a dynamic programing algorithm and a variety of mortality tables and interest rate assumptions, the reported result for the AER was 1.5, or a 50% value from pooling. To be clear, our $\delta_0 = \sqrt{e}-1$ result only holds when the inverse of life expectancy (λ) is equal to the risk-free rate (r). And, while this result might be trivial or impractical or both, the fact is that it provides a lower bound on the value of longevity risk pooling.

More generally when the life expectancy is not equal to the current interest rate, the value of longevity pooling can be expressed in an equally simplified manner as power function of the ratio of two annuity factors. Under *any* combination of interest rates r and mortality rates λ, the value of longevity risk pooling is:

$$\delta_0 = \left(\frac{r+\lambda/\gamma}{r+\gamma}\right)^{\frac{\gamma}{1-\gamma}} - 1 := \left(\frac{a_\lambda}{a_{\lambda/\gamma}}\right)^{\frac{\gamma}{1-\gamma}} - 1.$$

The first annuity factor in the numerator is the actual one based on biological age and mortality rate: $a_\lambda = (r+\lambda)^{-1}$, and the second annuity factor in the denominator assumes an age set-back that depends on the degree of risk aversion, $a_{\lambda/\gamma} = (r+\lambda/\gamma)^{-1}$.

For example, in the case of exponential mortality a $\gamma = 2$ would imply a doubling of life expectancy from $1/\lambda$ to $2/\lambda$, etc. So, the value of longevity risk pooling at the age of 65 is equal to the actuarial annuity factor for a 65-year-old divided by the actuarial annuity factor for a 45-year-old (all to the power of 2). The retiree is 65, but the pricing is done as if they are 45, etc.

In fact, this simplified representation of the value of longevity risk pooling δ_0 extends to all continuous (biologically reasonable) mortality assumptions, which would include Gompertz-Makeham which can be fit to any reasonable mortality table. To that end we presented expressions which can be used in many circumstances and offer some economic and actuarial intution.

Besides cute expressions, in terms of lifecycle modeling we have shown that in the presence of pre-existing pension annuities, not only is the value of longevity pooling lower (known to economists) but the methodology used must be carefully adjusted for the rational wealth depletion time (WDT). It is no longer possible to express the value of pooling as a function of a simple ratio of annuity factors and in fact the discounted value function no longer scales in initial wealth. And, while we are still able to obtain some analytic results under exponential mortality – and reported those in table #1 – we are forced to parse the definition of pooling value to differentiate between annuitizing one more dollar vs. the entire initial wealth. In contrast to what someone might expect, the value of pooling one more dollar (v using our notation) is worth more than the value of pooling the entire wealth (δ, using our notation), even in the presence of pre-existing annuities. We referred to this as the annuity equivalent wealth *in-the-large* vs. *in-the-small.*

5.1 Final Words

We conclude with a sobering note reminding readers that all of these results and expressions were derived assuming the biological annuity factors a_x applicable for valuing pre-existing pension annuity income are identical to the cost of buying additional annuity income. In other words, we live in a utopia with no adverse selection costs and no transaction costs. Mandated groups (e.g. Social Security in the U.S. or the Canadian Pension Plan, CPP) pay the same price for their pension annuities as retail individuals would pay in the open market. This is simply not the case in practice. See, for example, estimates provided by Finkelstein and Poterba (2004), for the magnitude of these costs in what was then the largest annuity market in the word, the U.K.

It's quite clear from the numerical results that if one already has 75% of retirement wealth pre-annuitized, and one has to pay a loading of 20% to 40% to acquire additional annuity income, the value of pooling can actually be negative. The rational consumer would be better of spending-down their (non annuitized) assets and living-off the pension annuity income at the wealth depletion time. A δ_0 of 50% might not be large enough to pay for itself.

Of course, our research has omitted any discussion of more general Epstein-Zin preferences, in which obtaining closed-form expressions is quite hopeless. Although, according to estimates reported in Cannon and Tonks (2008), the annuity equivalent wealth values would be even higher than our numbers. On the other hand recent and highly visible work by Rechling and Smetters (2015) indicates that in the presence of stochastic mortality the value of pooling is lower, although this has been questioned in recent work by others, see for example Bauer (2017). On a separate path, Feigenbaum et al. (2013) use an overlapping generation model to argue that full annuitization is not welfare maximizing even with a deterministic mortality rate. The annuity debate (in academia) continues.

In sum, we believe that building a strong case or consensus for annuitization in the future will crucially depend on a strategy of being able to explain the value to individual retirees who are limited by behavioral and cognitive obstacles. This is in contrast to a research strategy of extending the economic lifecycle model to include more general (behavioral) preferences, longevity insurance products and asset dynamics. In some sense this group is preaching to the (very small) choir of economists within the existing literature. We hope that some of the simple, analytic and digestible expressions we presented in this paper might be lead to contributions beyond the academic literature. At the very least $\sqrt{\epsilon}$ now has an *actuarial economic* interpretation.

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