ESTIMATION OF SELECT AND ULTIMATE MORTALITY RATES BY LEAST SQUARES

Aaron Tenenbein

Abstract

The method of least squares is applied to the estimation of the parameters of the Gompertz mortality curve and to select mortality models discussed by Tenenbein and Vanderhoof (1979).

Estimation of Select and Ultimate Rates by Least Squares

1. Introduction

In many mortality studies on either a select-ultimate basis or a combined basis crude mortality rates are obtained. The process of smoothing these rates is called graduation in the actuarial literature (Miller 1946) and estimation in the statistical literature. The resulting graduated or estimated mortality rates are used as a basis for developing mortality tables.

There are various methods of graduation. The method known as "Graduation by Mathematical Formula" involves the use of a mathematical law such as Gompertz's Law (1825) for the purpose of estimating mortality rates. Under Gompertz's Law we have that:

$$\mu_x = Bc^x \tag{1.1}$$

Various Techniques have been used to estimate the constant B and c. Once B and c are known the mortality rates can be computed.

One method for determining the constant B and c is the method of least squares. Wetterstrand (1978) has discussed the use of least squares for fitting ultimate or combined mortality rates. Tenenbein and Vanderhoof (1979) develop select mortality models and use least squares to estimate the parameters. Aside from these publications there does not seem to be any literature on the use of least squares regression methods for determination of parameters of mortality curves.

The purpose of this paper is to introduce the use of least squares for estimating parameters of mortality curves and discuss the advantages of this method.

2. The Method of Least Squares

Suppose we have a model which expresses the true mortality rates q as a function of various parameters: $\beta_1, \beta_2, \ldots, \beta_k$:

$$Y(q) = F(\beta_0, \beta_1, \ldots, \beta_k) \qquad (2.1)$$

From mortality studies we obtain crude or estimated mortality rates $\hat{q}$ and from these estimates we wish to estimate $\beta_0, \beta_1, \ldots, \beta_k$.

The method of least squares chooses as estimates of $\beta_0, \beta_1, \ldots, \beta_k$, which minimize

$$SSE = \sum_{\substack{\text{all} \\ \text{mortality classes} \\ i}} [Y(\hat{q}_i) - Y(q_i)]^2 \, w_i \qquad (2.2)$$

where w_i is the weight of class i. The weight takes in account the relative error of crude rates in mortality class i . If all the weights are equal, $w_i = 1$, and the method of least squares is known as ordinary least squares (see Draper and Smith (1966)). The resulting values of $\beta_0, \beta_1, \ldots, \beta_k$ which minimize SSE in equation (2.2) are called the least squares estimates of the β_i .

3. Estimation of Ultimate Mortality Rates Using Gompert's Law

Wetterstrand (1978) and Tenenbein and Vanderhoof (1979) discussed the method of least squares for ultimate mortality rates. Following the notation of section 2, we let:

$$Y(q) = \ln[\text{colog } (1-q_x)] \qquad (3.1)$$

$$F(\beta_0,\beta_1) = \beta_o + \beta_1 x \qquad (3.2)$$

$$\beta_0 = \ln B + \ln(c-1) - \ln(\ln c) \qquad (3.3)$$

$$\beta_1 = \ln c \qquad (3.4)$$

where B and c are the Gompertz parameters of equation (2.1). Jordan (1975) shows that Gompertz's Law can be equivalently stated as:

$$Y(q_x) = \ln[\text{colog } (1-q_x)] = \beta_0 + \beta_1 x$$

which is the special case of equation (2.1).

The method of ordinary least squares involves choosing β_0 and β_1 to minimize:

$$SSE = \sum_{\text{all } x} (Y(\hat{q}_x) - \beta_0 - B_1 x)^2$$

Note that the mortality classes are defined by the age of the lives.

If the resulting least square estimates of β_0 and β_1 are b_0 and b_1 respectively, the resulting estimates of the Gompertz parameters can be determined by solving equations (3.3) and (3.4) for B and c . The estimates of B and c are thus:

$$\hat{c} = \exp(b_1) \qquad (3.5)$$

$$\hat{B} = \frac{b_1 \exp(b_0)}{\exp(b_1) - 1} \qquad (3.6)$$

The method of weighted least squares is discussed by Tenenbein and Vanderhoof (1979). For more details see the papers by Wetterstrand, and Tenenbein and Vanderhoof.

3. Estimation of Select Mortality Rates

Tenenbein and Vanderhoof (1979) describe a series of three laws which relate select mortality rates to age x and duration t. In this situation and following the notation of section 2 we have that:

$$Y(q) = \ln[\operatorname{colog}(1-q)]$$

and the three mortality laws can be described as

$$Y(q) = F_i(x,t|\beta_0,\beta_1,..,\beta_k) \tag{3.1}$$

where $\beta_0,\beta_1,..,\beta_k$ are the parameters of the three laws $(i=1,2,3)$. The method of weighted least squares would involve the minimization of:

$$\sum_{\text{all } x} \sum_{\text{all } t} [Y(q) - F_i(x,t|\beta_0,\beta_1,..,\beta_1 c)]^2 w_{xt}$$

For more details see Tenenbein and Vanderhoof (1979).

4. Advantages of the Least Squares Method

The advantage of least squares method over the traditional methods of "Graduation by Mathematical Formula" are:

i) The method is objective as opposed to graphical or adhoc approaches.

ii) The theory of regression analysis can be used to develop standard errors of the estimated parameters.

iii) The computations are facilitated by such computer packages as Statistical Analysis System. See Barr et al. (1976).

The author hopes that least squares regression techniques will be used in the future to graduate mortality rates when a graduation by Mathematical formula is carried out.

BIBLIOGRAPHY

1. Barr, A., Goodnight, J., Sall, J., Helwig, J. A User Guide to SAS. Raleigh, North Carolina: SAS Institute, Inc., 1976.
2. Draper, N. and Smith, H. Applied Regression Analysis, New York: John Wiley and Sons, 1966.
3. Gompertz, B. "On the Nature of the Function Expressive of Human Mortality and on a New Mode of Determining Life Contingencies," Phil. Trans. of the Royal Society of London, A 115 (1825), 513-580.
4. Jordan, C. W. Life Contingencies, Chicago: Society of Actuaries, 1975.
5. Miller, M. Elements of Graduation, Chicago: Society of Actuaries, 1946.
6. Tenenbein, A. and Vanderhoof, I. T., "New Mathematical Laws of Select and Ultimate Mortality", New York University Graduate School of Business Working Paper, 1979, Number 79-29, to be published in the Transactions of the Society of Actuaries, 1980.
7. Wetterstrand, W. B., "Recent Mortality Experience Described by Gompertz's and Markeham's Law", Proceedings of the Thirteenth Annual Actuarial Research Conference, 1978, Ball State University, Muncie, Indiana.