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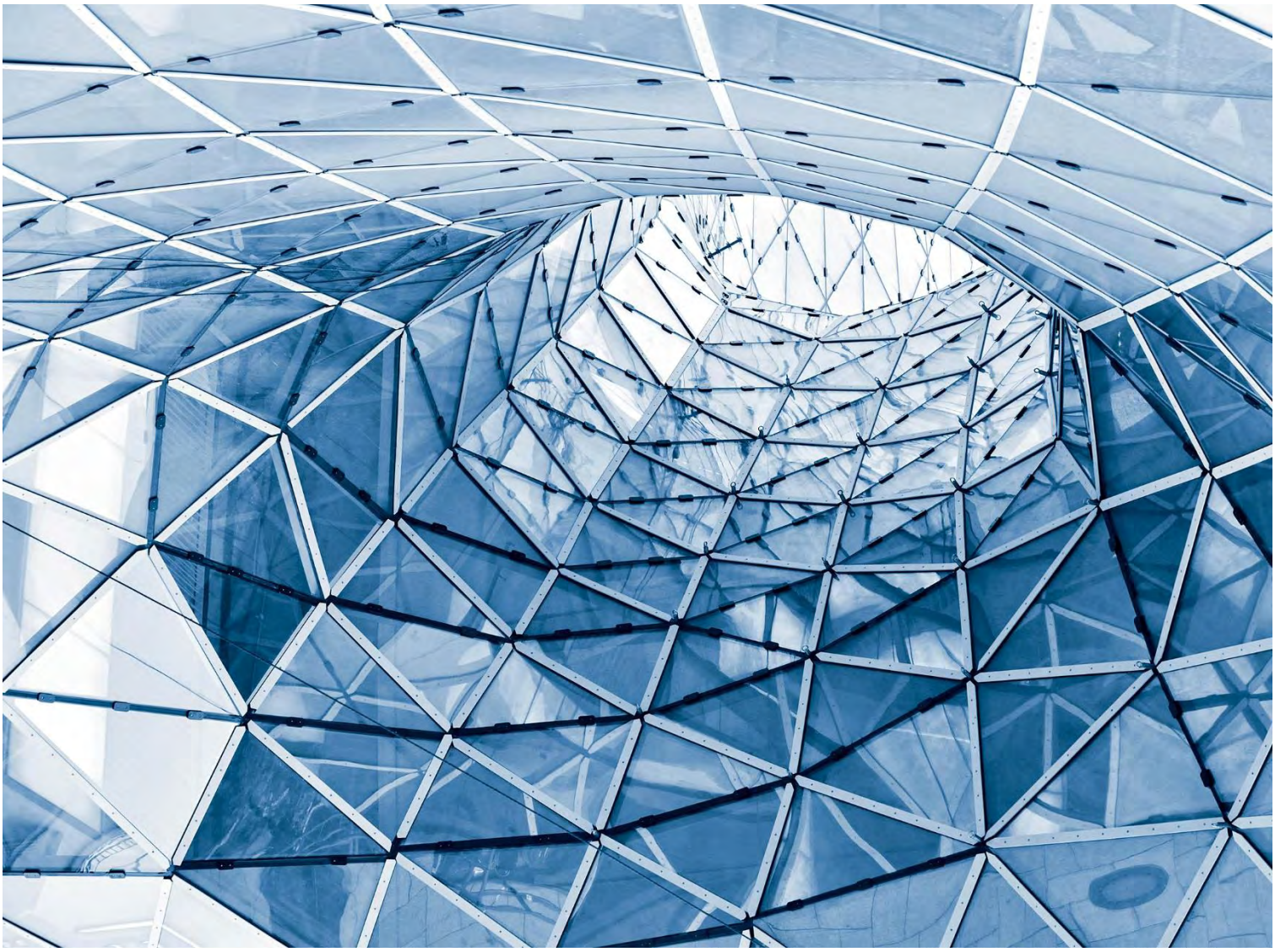
Hedging with Life and General Insurance Products



Hedging with Life and General Insurance Products

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Abstract

In this study, a hybrid portfolio that combines life and general insurance aspects is considered. There has been intense research on hedging of an insurance company's portfolio when the risk is incoming from life insurance only or general insurance only, but little on both in the same portfolio. However, most large multiline insurance holding companies have separate legal entities that can and do issue both products. We assume there are one risk-free asset and one risky asset in the financial market, and the investor may choose the proportion of investments for each of them. In addition to the inflows and outflows from trading, the insurance cash flow account has inflows from premiums and outflows from losses of life and general insurance. Once the portfolio of the insurance company is set, the dynamic programming principle is applied to derive a partial differential equation for the optimal function, and the corresponding optimal trading strategy can be obtained.

1. Introduction

There has been intense research on hedging of an insurance company's liability portfolio when the risk is incoming from life insurance only [4, 7, 8] or general insurance only [2, 5, 6]. In this report, we consider two insurance products, a life insurance and a general insurance, in a single liability portfolio. Most large multiline insurance holding companies have separate legal entities that can and do issue both products.

The financial market in this study has one risk-free asset and one risky asset. We set up a portfolio with one life insurance product and one general insurance product. The life insurance product that we consider pays its premium as a single payment up-front, and the general insurance product has continuously paid premiums or up-front premiums. The up-front premium is used as an initial wealth of this portfolio, and it is invested in the financial market. The portfolio manager will decide the portions invested into each of the two asset investments supporting the two products (one stock and one bond). Our goal is to find the optimal allocation strategy[1] to eliminate the risks from the insurance products in the portfolio, as well as the risk from the risky asset.

There are three risks to consider: the risk from the trading assets, the risk from the general insurance, and the mortality risk from the life insurance product. The risk from the trading assets will be modeled with a Brownian motion, and the risk from the general insurance will be modeled using a limit of the classical Poisson process. The mortality risk will be modeled with a stochastic process introduced by

[1] The optimization that we consider in this project is to find the trading strategy that maximizes the expected utility function.

Wang [7]; however, it can be considered as a deterministic process if a perfect diversification of the mortality is assumed.

Once the dynamics of the portfolio value are set, the dynamic programming principle is applied to derive the partial differential equation, and the corresponding optimal trading strategy should be obtained.

2. Model

2.1 The Financial Market

The financial market to consider is the so-called Black-Scholes model: There is one risky asset (stock) and one risk-free asset (U.S. Treasury bond). Suppose that at time *t* the risky asset price is *S*. For a small change in time *dt*, the asset price will change into *S+dS*. The return on the risky asset $\frac{dS}{S}$ can be decomposed into two parts. One is deterministic, and the other is random. It is modeled with the following stochastic differential equation:

$$\frac{dS}{S} = \mu dt + \sigma dB^1.$$

The first term on the right-hand side, μdt, indicates the deterministic return with a rate of return $\mu > 0$. The second term explains the unpredictable return with a volatility $\sigma > 0$, which measures the standard deviation of the returns. B^1 is a standard Brownian motion, and $dB^1$is a random sample from a normal distribution, with mean 0.

Assuming a deterministic rate of return *r* to be $0 < r < \mu$, the risk-free asset price *P* is modeled with the following differential equation:

$$\frac{dP}{P} = r\, dt.$$

2.2 General Insurance Product

The general insurance claims process can be complicated because of its uncertain frequency and uncertain severity. We will consider the model that is introduced by Promislow and Young [6] for simplicity. Let *C* be the claims process of a general insurance product, and we assume it follows

$$dC = adt - bdB^2,$$

where *a* and *b* are positive constants and B^2 is a standard Brownian motion in a probability space (Ω, F, P) that is independent of B^1. It is a limit of the classical compound Poisson model [3]. For example, if the claims process frequency follows a Poisson distribution with the mean α and the claims size follows an exponential distribution with the mean β, then $a = \alpha\beta$and $b = \sqrt{2\alpha\beta^2}$.

We first assume that the premium for this general insurance is being paid continuously at a constant rate $c_1 = (1 + \theta)a$ with a positive θ, which is a relative security loading [5]. We can modify the premium for the general insurance to be an up-front payment if we set $\theta = -1$, which will be discussed in Section 4.

The surplus process U^1 for the general insurance contract has an inflow from the premiums. For a small change in time (*dt*) the change in inflow is $c_1 dt$. The outflow of the surplus process is from the claims process, and the change in outflow is *dC* for a small change in time. The change in the surplus process dU^1 for a small change of time *dt* is given by the following stochastic differential equation:

$$\begin{cases} dU^1 = c_1 dt - dC = \theta a dt + b dB^2, \\ U_0^1 = u_0, \end{cases}$$

where $u_0 \geq 0$ is an initial surplus.

2.3 Life Insurance Product

For a life insurance product sold, we consider a level-term insurance. Suppose the policy holder pays an up-front premium and a prespecified benefit *G* will be paid to the heir at the time of death of the policy holder, provided death occurs before the end of a specified term *T* (in years).

Let *N* be the fraction of individuals still alive at *t* $(0 \leq N \leq 1)$, and we use the model that is introduced by Wang [7] for its dynamics:

$$\begin{cases} dN = -\lambda N dt + \epsilon\sqrt{N} dB^3 - dL, \\ N = 1 \text{ at time } 0, \end{cases}$$

where $\epsilon \geq 0$ is small, $\lambda \geq 0$ is the hazard rate of the insured population, B^3 is a standard Brownian motion that is independent of B^1 and B^2, and *L* is the local time of *N* at 1. If we assume a perfect diversification of the mortality[2], then this model is simplified as follows:

$$dN = -\lambda N dt.$$

We will develop an optimal trading strategy based on this model and extend it to the stochastic mortality model in the future.

We normalize so that the total portfolio value at time 0 is 1 in the next section. Since the premium is paid at time 0, the change in surplus process for this life insurance product dU^2 has only the change in outflow due to the death of the policy holder:

$$dU^2 = G dN = -\lambda G N dt.$$

2.4 The Portfolio

Let *V* be the value of the portfolio. We assume it is a self-financing portfolio; that is, no extra fund is being added or withdrawn from this portfolio. We also do not consider other additional costs such as operational costs or taxes. The initial wealth $V = v_0$ at time 0 from the premium of the life insurance product is invested into the risky asset and the risk-free asset in the financial market.[3] Let π be the proportion of *V* that is invested in the risky asset, then $1 - \pi$ will be the proportion of the wealth that is invested in the risk-free asset. For a small change in time *dt,* the change in the value of the portfolio *dV* can be decomposed to four changes: from the risky asset, the risk-free asset, the general insurance surplus, and the life insurance surplus. It can be modeled with the following stochastic differential equation:

$$dV = \pi V \frac{dS}{S} + (1 - \pi) V \frac{dP}{P} + dU^1 + dU^2,$$
$$V = v_0 \text{ at time } 0.$$

Using the dynamics of the components introduced in the previous sections, we have

[2] If the insurance company is able to sell enough contracts, then it can eliminate the fluctuations between actual and expected mortality rates.

[3] Here we assume the initial surplus of the general insurance u_0 is 0.

$$dV = \pi V(\mu dt + \sigma dB^1) + (1-\pi)V(rdt) + \theta a dt + bdB^2 - \lambda GNdt$$
$$= [rV + \pi(\mu - r)V + \theta a - \lambda GN]dt + \sigma\pi V dB^1 + bdB^2.$$

3. The Hamilton-Jacobi-Bellman (HJB) equation

The methodology that is used to derive a model partial differential equation is the dynamic programming principle, commonly used in stochastic control problems. To apply it, the problem must observe the principle of optimality (Bellman principle [1]); that is, whatever the initial state is, all remaining decisions must be optimal regarding the state following from the first decision.

Suppose the portfolio manager wants to maximize the expected utility of the terminal portfolio value, and define the optimal function *H*(*t,v,n*) to be

$$H(t,v,n) = sup_{\pi \in A} E[u(V_T)|V_t = v, N_t = n],$$

where *A* is the set of admissible policies and $u: R \to R$ is a utility function, which is increasing, concave, and smooth.

By applying the dynamic programming principle, we obtain the following HJB partial differential equation (PDE) for the optimal function *H*[4]:

$$H_t + [rV + \theta a - \lambda Gn]H_v - \lambda n H_n + \frac{1}{2}b^2 H_{vv} + max_\pi \left[(\mu - r)vH_v\pi + \frac{1}{2}\sigma^2 v^2 H_{vv}\pi^2\right] = 0.$$

Since the maximum function is a quadratic function in π and the concavity of the utility function *u* is inherited to the optimal function *H*, the maximum exists, and we have the optimal investment process

$$\pi = -\frac{(\mu - r)H_v}{\sigma^2 v H_{vv}}.$$

This gives a closed form PDE for *H*:

$$H_t + [rV + \theta a - \lambda Gn]H_v - \lambda n H_n + \frac{1}{2}b^2 H_{vv} - \frac{[(\mu - r)H_v]^2}{2\sigma^2 v H_{vv}} = 0,$$
$$H(T,v,n) = u(v).$$

We can simplify the derived PDE further by using a reduction of dimension, namely, introducing a new variable $Y = \frac{V}{N} = e^{\lambda}V$. The optimal function $H(t,y) = H(t,v,n)$ satisfies the following HJB PDE:

$$H_t + H_y\left[(\lambda + r)y + \theta a e^{\lambda t} - \lambda G\right] + \frac{1}{2}b^2 e^{2\lambda t} H_{yy} - \frac{\left[(\mu - r)H_y\right]^2}{2\sigma^2 H_{yy}} = 0,$$
$$H(T,y) = u\left(e^{-\lambda T}y\right).$$

4. Numerical Examples

In this section, we solve (2) using a backward in time finite difference method. We adopt an exponential utility function as

$$u(v) = -e^{-v},$$

so that the terminal condition is

$$H(T,y) = -e^{-e^{-\lambda T}y}.$$

Since the terminal condition is given, we can solve the PDE (2) backward in time to obtain the solution at time 0, *H*(0,*y*). After computing *H*(0,*y*), we estimate the optimal strategy π using (1).

[4] H_q is a partial derivative of *H* with respect to *q = t, v, n.*

We take the expiration time for the life insurance to be *T* and consider the domain to be $(t, v) \in [0, T] \times [0,2]$. We also take *G* = 1. The parameters in Table 1[5] are used as a basis case.

Parameter	a	b	θ	r	σ	λ
Value	1.5	0.5	0.1	0.04	0.2	0.1

Table 1: Parameters

First, we consider *T* = 1 for simplicity, and the result (π) is plotted in Figure 1 when $\mu = 0.05$. For a very small value of the initial wealth, the model indicates to borrow money and invest all into the stock ($\pi \gg 1$). It also shows that the optimal trading strategy (π) is a convex and decreasing function in terms of the initial wealth (v_0).

We can interpret the meaning of π as follows: Suppose the life insurance payoff is $1 million, and the expected number of claims for the general insurance is 18 with the expected claim size $\frac{1}{12}$ million. The computed value of $\pi = 0.2194$ when $v_0 = 1$ means 21.94% of the initial wealth should be invested in the risky asset, and the remaining 78.06% should be invested in the risk-free asset.

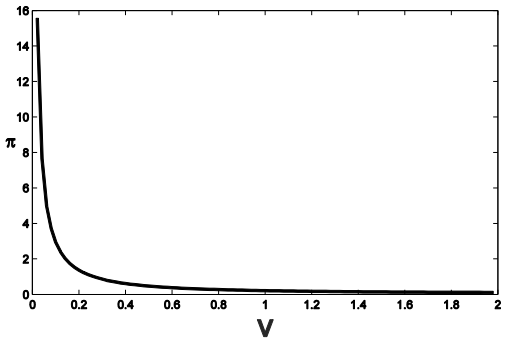

Figure 1: Optimal strategy at time 0 when $\mu = 0.05$

4.1 Effect of the Return Rate μ

To see the effect of the return rate of the stock, μ, we compare the values of the optimal strategy π at time 0 for various values of μ when $v_0 = 1$ in Table 2. As expected, as the value of the return rate of the stock μ decreases, the model suggests less investment in the stock because of the lower return.

μ	π
0.08	0.8950
0.07	0.6724
0.06	0.4487
0.05	0.2225
0.041	0.0225

Table 2: Optimal strategy for various return rates of the stock

[5] *A* = 1.5 and *b* = 0.5 imply that the expected number of claims is 18 per year, and the expected claims size is $\frac{1}{12}$ for the general insurance product.

4.2 Effect of the Volatility σ of the Risky Asset

Figure 2 shows that the proportion in the risky asset (π) decreases as the value of the volatility (σ) increases. The volatility of the underlying risky asset is a measure of a risk. If it gets higher, but with a fixed rate of return μ in the risky asset, then investment in the risky asset is not preferable compared to investment in the risk-free asset. Conversely, if the volatility gets smaller, then investment in the risky asset yields the same return with less risk, and the portfolio manager would find it more attractive to invest in the risky asset.

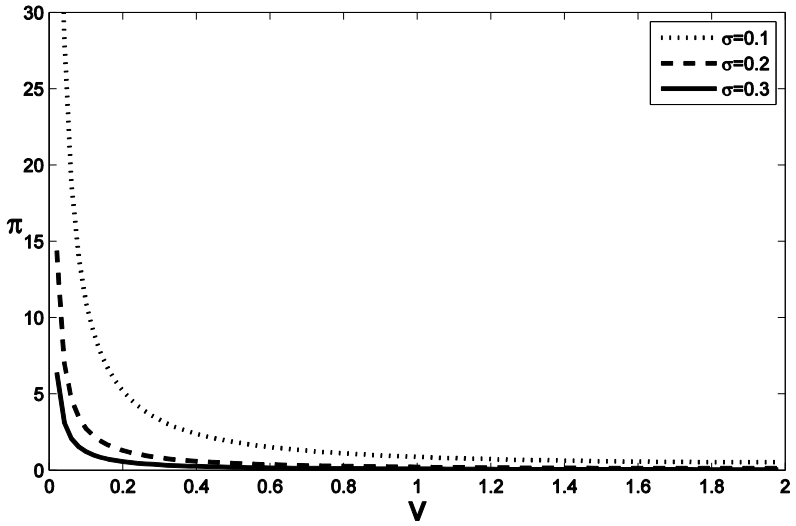

Figure 2: Optimal strategy at time 0 for various values of σ

4.3 Effect of the Risk-Free Interest Rate *r*

As observed in Figure 3, as the risk-free interest rate *r* increases, the investment in the risky asset decreases. With a fixed rate of return μ and a fixed risk measure σ in the risky asset, a higher risk-free

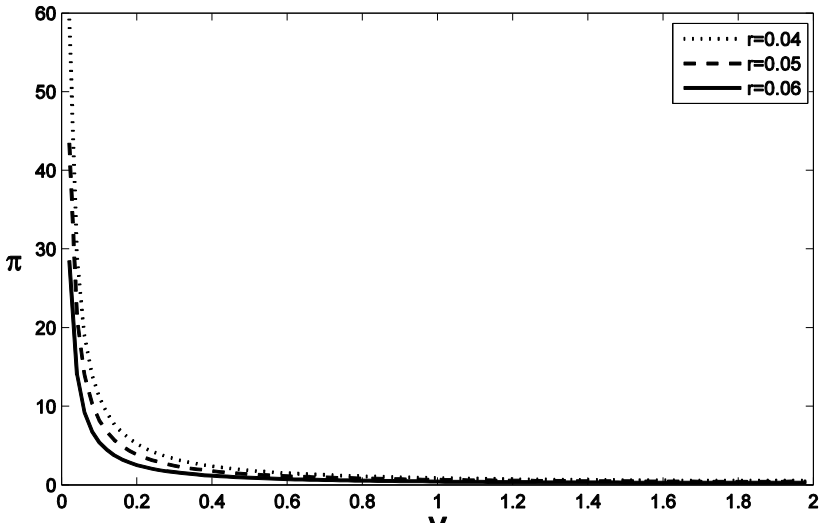

Figure 3: Optimal strategy at time 0 for various values of r

interest rate means the portfolio manager does not have to take risk in the risky asset to yield a higher return.

4.4 Effect of the Relative Security Loading θ

Figure 4 indicates that as the relative security loading θ increases, the investment in the risky asset decreases. If a higher security loading is imposed, then it yields a higher stable inflow from the premiums. That means the portfolio manager does not have to take much risk in the risky asset. It is more observable if the initial wealth is smaller.

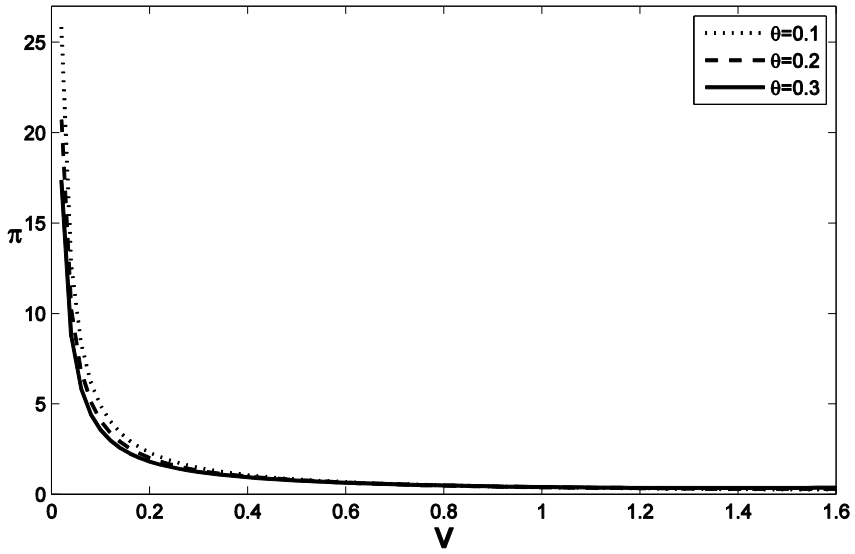

Figure 4: Optimal strategy at time 0 for various values of theta

4.5 Effect of the Mortality Rate λ

To see the effect of the mortality rate, we set $T = 10, a = 1.0, b = \sqrt{0.2}$, and $\mu = 0.06$. This is when the expected number of claims is 10, and the expected claim size is 0.1. Table 3 shows that as the mortality rate increases, the proportion to invest in the risky asset decreases (the relative change is 4.07%). When the insurance company expects to have more payments to make due to a higher mortality rate, it needs to have a more reliable income from the risk-free asset.

Mortality rate	$\lambda = 0.001$	$\lambda = 0.01$
π	0.35773	0.34373

Table 3: Optimal strategy for different mortality rates

4.6 Effect of the Initial Surplus

The proportion to invest in the risky asset π is a decreasing function in terms of v_0 in general, as observed in the previous figures. Since the existence of the initial surplus will have an effect of increasing the initial wealth, we observe less value of π when there is a positive initial surplus. Table 4

shows an example that compares the value of π with and without the initial surplus u_0. As expected, the model suggests less investment in the risky asset when there are more resources to cover the risks.

Initial surplus	$u_0 = 0$	$u_0 = 0.2$
π	0.34373	0.29712

Table 4: Optimal strategy for different initial surplus

4.7 Effect of the Terminal Time of the Life Insurance

We compare the values of the optimal strategy π for different terminal times of the life insurance in Table 5. With all the other parameters fixed, two different terminal times are considered: *T* = 1 and *T* = 10. The result suggests to put less money in the risky asset with the longer expiration date.

Expiration date	*T* = 1	*T* = 10
π	0.2384	0.1721

Table 5: Optimal strategy for different terminal time

4.8 Effect of the Risk from the General Insurance

To see the effect of the risk from the general insurance, we consider three different cases as in Table 6: First, we set the expected number of claims to be 10 and the expected claims size to be 0.1 as a base case (case A). If we increase the expected number of claims to 20 (case B), then the proportion to invest in the risky asset decreases (from 0.34374 to 0.22190). If we increase the expected claims size to 0.2 (case C), then the proportion to invest in the risky asset also decreases (from 0.34373 to 0.11079). Case B and case C have the same number of total expected payments, but the effect is more significant in case C since it has a higher variability (b). This result shows that with more risk from the general insurance, the portfolio manager would want to have a more reliable income from the risk-free asset, as in the case with a higher mortality rate.

Case	Mean number of claims	Mean claim size	a	b	π
A	10	0.1	1.0	0.4472	0.34374
B	20	0.1	2.0	0.6325	0.22190
C	10	0.2	2.0	0.8944	0.11079

Table 6: Optimal strategy for different general insurance risks

5. Conclusion

In this report we derived a formula to estimate the optimal trading strategy of a portfolio with a general insurance product and a life insurance product. The initial cash account value of this portfolio is invested in the financial market with one risky asset and one risk-free asset. The derived formula for the optimal trading strategy depends on partial derivatives of an optimal function, whose solution satisfies a Hamilton-Jacobi-Bellman–type partial differential equation.

Numerical examples show that the computed optimal trading strategy follows the general theory of the financial market:

- Invest more in the risky asset if the return rate of the risky asset is higher.
- Invest less in the risky asset if the volatility of the risky asset is higher.
- Invest less in the risky asset if the risk-free interest rate is higher.
- Invest less in the risky asset if the continuous premium rate from the general insurance is higher.
- Invest less in the risky asset if the mortality rate is higher.
- Invest less in the risky asset if the expiration date of the life insurance is longer.
- Invest less in the risky asset if the risk from the general insurance is higher.

6. Future Research

We can extend the model in this report to include a more realistic models, for example, explore how the optimal strategy changes as time progress with a periodic installment of the premium for general insurance. Monte Carlo simulation should be a suitable tool for the computations. Another extension can be done in terms of the mortality model. We assumed a simple deterministic mortality model with a constant force of mortality in this study, but it should be interesting to study with other mortality models. Gompertz's law of mortality or a stochastic mortality model can be employed.

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Appendix A

Let us consider the premium for the general insurance to be paid periodically at prespecified times $T_0 = 0, T_1, \dots, T_n$. The surplus process in Section 2.2 is now

$$\begin{cases} dU^1 = -dC = -adt + bdB^2, \\ U_0^1 = u_0. \end{cases}$$

We also normalize so that the initial surplus is 1 ($u_0 = 1$). It has an effect of having the initial wealth as $v_0 = 2$. First, we set the domain to be $(t, v) \in [0, T_1] \times [0,4]$ and solve (2) with $\theta = -1$ using the same method in Section 4. Once we reach the time T_1, we can adjust the *initial* wealth as the value of *V* at time T_1 plus the periodic premium, and then solve (2) on the domain $[T_1, T_2] \times [0,4]$ to compute the optimal strategy π at time T_1. By repeating this, one can find the optimal strategy for each event time.

Appendix B

In Section 2.4, we assume the portfolio has one life insurance product and one general insurance product. In this appendix we assume the portfolio manager can allocate different proportion of life insurance and general insurance products in the portfolio.

Let *V* be the value of the portfolio. Suppose the portfolio manager decides to have α $(0 \le \alpha \le 1)$ life insurance product and $1-\alpha$ general insurance product in the portfolio. Then the value process follows

$$dV = \pi V \frac{dS}{S} + (1-\pi)V\frac{dP}{P} + (1-\alpha)dU^1 + \alpha dU^2,$$
$$V = \alpha v_0 \text{ at time } 0.$$

Note that the initial wealth depends on α, since the portfolio is self-financing and the initial wealth is set up as the premium of the life insurance product v_0.

The corresponding HJB equation becomes

$$H_t + H_y\left[(\lambda+r)y + (1-\alpha)\theta a e^{\lambda t} - \alpha\lambda G\right] + \frac{1}{2}(1-\alpha)^2 b^2 e^{\lambda t} H_{yy} - \frac{\left[(\mu-r)H_y\right]^2}{2\sigma^2 H_{yy}} = 0,$$
$$H(T,y) = u\left(e^{-\lambda T}y\right),$$

and the optimal strategy π has the same formula as in (1).

The computation result of π is presented in Table 7 using the parameters in Table 1. As α increases, we observe that the allocation to the risky asset π decreases. The main reason for that phenomenon is that π for different α is computed for different initial values, since the initial wealth depends on α too. As α increases, the initial wealth αv_0 also increases. Table 7 shows that π is a decreasing function in *V*, hence π decreases as α increases.

α	π
0.1	2.3807
0.3	0.7426
0.5	0.4353
0.7	0.3105
0.9	0.2415

Table 7: Optimal strategy for different values of α