Robustness of Moving Weighted Average Graduation Formulas

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Abstract.

The theory underlying the Moving Weighted Average graduation method is restated in the language of linear algebra which provides for an enrichment of the family of formulas. A comparison of the results of applying six members of the family to 100 simulated data sets is provided.

Introduction.

First, our apologies for not discussing the robustness of Moving Weighted Average (M-W-A) formulas in the technical sense of this conference. It is our intention to explore the richness (i.e. robustness) of the theory of M-W-A formulas.

By M-W-A formulas we shall mean a set of coefficients that serve as the weights in calculating graduated values as weighted combinations of ungraduated values. Our distinction between M-W-A graduations and Whittaker-Henderson graduations is that each M-W-A graduated value is a linear combination of relatively few ungraduated values where as the Whittaker-Henderson graduated value is a linear combination of all ungraduated values. Moreover, most of the M-W-A values will be calculated using the same weights--hence the term "moving". In the Whittaker-Henderson method the calculation of each value will use a different set of weights.

Among actuaries M-W-A formulas are restricted to those derived to reproduce polynomials of a given degree and to minimize the sum of squared differences of a given order--usually denoted R_Z^2. In this paper we will enrich the family of M-W-A formulas and illustrate the application of some of the new members on simulated mortality data. We hope the foundations of M-W-A graduation theory will be better understood. As a consequence a rational decision regarding the robustness of this method in our computer age can be made.

Traditional M-W-A Formulas.

Now we shall formulate the M-W-A method in the statistical framework by use of matrices. We are given a set of random variables,

$$u''_x , \quad x = a, a+1, \ldots, b.$$

We shall assume that their means and variances exist and shall denote them by V_x and σ_x^2 respectively. More significantly, we shall assume that the variances are equal and that these $b - a + 1$ random variables are uncorrelated. I.e.

$$\sigma_x^2 = \sigma^2 \tag{1}$$

and

$$\mathrm{Cor}(u''_x, u''_y) = 0 \qquad x \neq y \tag{2}$$

This model is traditional in the theory of the M-W-A method (see [2], [4]).

A M-W-A formula is a set of $N + K + 1$ coefficients, a_s, $s=-K, -K+1, \ldots, N$, which are used to calculate the linear estimates

$$u_x = \sum_{-K}^{N} a_s u''_{x+s} \qquad x = a+K, a+K+1, \ldots, b-N.$$

These are interpreted as estimates of V_x. A specific M-W-A formula is determined by requiring, (A) these estimators to be unbiased for those distributions with V lying on a d degree polynomial and (B) among these "unbiased" $N + K + 1$ term linear estimators the variance of $\Delta^z u_x$ is minimal.

We note that these specifications for M-W-A formulas have four parameters, i.e., N, K, d, and Z. In the statement of this theory in the study material of the Society of Actuaries, $N = K = n$. Here we shall allow $N \neq K$ to obtain implicitly one solution to the so-called "end value problem" and to show explicitly that the usual symmetry of M-W-A formulas is a consequence of $N = K$.

For the matrix formulation of this material we shall be working in, R^{N+K+1}, the $N + K + 1$ dimensional real vector space. We shall denote column vectors by an arrow over the basic symbol and row vectors by a "T" superscript to denote the transpose. Thus, we have

$$\vec{a}^T = (a_{-K}, a_{-K+1}, \ldots, a_N)$$

$$\vec{u}''^T_{x-K} = (u''_{x-K}, u''_{x-K+1}, \ldots, u''_{x+N}) \qquad (3)$$

and, then

$$u_x = \vec{a}^T \cdot \vec{u}''_{x-K} \quad .$$

We shall follow the subscript pattern used in (3), i.e. when the coordinates of a vector are members of one of the sets u", V, or u, the subscript on the vector will be the smallest of the subscripts of the coordinates.

To formulate requirment (A), that a M-W-A formula be unbiased on d degree polynomials, we shall use D to denote the $d + 1$ dimensional subspace of R^{N+K+1} in which the coordinates of each member lie on a d degree polynomial. Since the expectation operator is linear, we can write

$$E[u_x] = E[\vec{a}^T \cdot \vec{u}''_{x-K}] = \vec{a}^T E[\vec{u}''_{x-K}] = \vec{a}^T \vec{V}_{x-K} .$$

Then specification (A) above requires that

$$\vec{a}^T \vec{V}_{x-K} = V_x \quad \text{for all} \quad \vec{V}_{x-K} \quad \text{in} \quad D. \tag{4}$$

To formulate the "minimum variance" requirement we shall use the vectors

$$\vec{\Delta}^T = ((-1)^z \binom{z}{0}, (-1)^{z-1} \binom{z}{1}, \ldots, \binom{z}{z}) ,$$

and

$$\vec{u}_x^T = (u_x, u_{x+1}, \ldots, u_{x+z}) .$$

Then specification (B) above, requires that $\mathrm{Var}(\vec{\Delta}^T \vec{u}_x)$ is minimal among linear estimators unbiased over D.

To translate this requirement from the statistical statement to an algebraic statement, we shall express the random variable $\vec{\Delta}^T \vec{u}_x$ as a matrix product. Let A_z be the $(N + K + z + 1) \times (z + 1)$ matrix such that

$$\vec{u}_x = A_z^T \vec{u}''_{x-K} ,$$

i.e.,

$$A_z^T = z\left\{\begin{pmatrix} a_{-K} & \cdots & & a_N & \overbrace{0 \ldots 0}^{z} \\ 0 & a_{-K} & & \ddots & \vdots \\ \vdots & & \ddots & & a_N \;\; 0 \\ \underbrace{0 \ldots 0}_{z} & & a_{-K} & & a_N \end{pmatrix}\right\} z \tag{5}$$

Then

$$\vec{\Delta}^T \vec{u}_x = \vec{\Delta}^T (A_z^T \vec{u}''_{x-K}) = (\vec{\Delta}^T A_z^T)\vec{u}''_{x-K} \tag{6}$$

From (5) we can see that the product $\vec{\Delta}^T A_z^T$ can be written as the $N + K + z + 1$ vector

$$(-1)^z \cdot (\Delta^z a_{-k-z}, \Delta^z a_{-K-z+1}, \ldots, \Delta^z a_N) \tag{7}$$

by defining $a_{-K-s} = a_{N+s} = 0$ for $s = 1,2,\ldots,z$. It then follows from assumption (2) that

$$\mathrm{Var}(\vec{\Delta}^T \vec{u}_x) = (\vec{\Delta}^T A_z^T) \begin{pmatrix} \sigma^2_{x-k} & & 0 \\ & \ddots & \\ 0 & & \sigma^2_{x+N+z} \end{pmatrix} (\vec{\Delta}^T A_z^T)^T \tag{8}$$

and from (1) that

$$\begin{aligned} \mathrm{Var}(\vec{\Delta}^T u_x) &= \sigma^2 (\vec{\Delta}^T A_z^T)(\vec{\Delta}^T A_z^T)^T \\ &= \sigma^2 \sum_{s=-K-z}^{N} (\Delta^z a_s)^2 \end{aligned} \tag{9}$$

For the case with $K = N = n$, (9) is equal to $\sigma^2 ({}^{2}{}_{z}{}^{z}) R_z^2$, where R_z is the smoothing coefficient of order z.

We have formulated the specifications for a M-W-A formula as follows:

For a given K, N, d, and z, determine $\vec{a}$ in R^{N+K+1} such that

$$\vec{a}^T \vec{d} = (K+1)\text{st coordinate of } \vec{d} \text{ for all } \vec{d} \text{ in } D \tag{10}$$

and

$$(A_z\vec{\Delta})^T (A_z\vec{\Delta}) \tag{11}$$

is a minimum.

M-W-A Formulas and Linear Algebra.

To find an algorithm for determination of $\vec{a}$, we shall rely on a result from linear algebra as reported in [1]. The result is as follows:

Problem: For a given positive definite $n \times n$ matrix H, a linear subspace $L \subseteq R^n$ and a linear functional $f: L \to R^1$, find a $y \in R^n$ for which

$$y^T b = f(b) \quad \text{for all} \quad b \in L \tag{12}$$

and

$$y^T Hy \quad \text{is minimal.} \tag{13}$$

Solution: There is a unique vector $\hat{y} \in H^{-1} L$ that satisfies (12). This vector, $\hat{y}$, and only this vector, solves the problem.

For the application of this result to the calculation of the M-W-A formula, we restate (11) by use of the identity

$$A_z \vec{\Delta} = \Delta_{N+K} \vec{a}$$

where Δ_{N+K} is the $(N+K+z+1) \times (N+K+1)$ matrix such that

$$\Delta_{N+K}^T = \begin{pmatrix} (-1)^z \binom{z}{0}, \ldots, & \binom{z}{z} & 0, & \ldots, & 0 \\ 0, (-1)^z \binom{z}{0}, & \ldots, & \binom{z}{z} & 0, & \ldots \\ 0, \quad 0, & & & & \\ & & & & ,0 \\ 0, \quad 0, \ldots, & 0, (-1)^z \binom{z}{0}, & \ldots, & \binom{z}{z} & \end{pmatrix}$$

In words, the ith row of Δ^T_{N+K} has $i-1$ zeros followed by the $z+1$ coefficients of Δ^z and then $N+K-i+1$ zeros. Now we have

$$(A_z\vec{\Lambda})^T (A_z\vec{\Lambda}) = (\Delta_{N+K}\vec{a})^T (\Delta_{N+K}\vec{a}) \tag{14}$$

$$= \vec{a}^T(\Delta^T_{N+K}\ \Delta_{N+K})\vec{a}$$

The relation between the elements of the linear algebra problem and the M-W-A formula determination is as follows:

Linear Algebra	M-W-A Formula
R^n	R^{N+K+1}
H	$\Delta^T_{N+K}\ \Delta_{N+K}$
L	D
f	$f(\vec{V}_{x-K}) = (k+1)$st coordinate
$\hat{y}$	$\vec{a}$

These identifications are clear with the possible exception that $\Delta^T_{N+K}\ \Delta_{N+K}$ is positive definite. It follows from (14) and then (7) that if $\vec{a}^T(\Delta^T_{N+K}\ \Delta_{N+K})\vec{a} = 0$ for some a, then $\Delta^z a_s = 0$ for $s = -K-z, -K-z+1, \ldots, N$ where $a_{-K-s} = a_{N+s} = 0$ $s = 1, 2, \ldots, z$. Hence these $2z + K + N + 1$ values lie on a $z - 1$ degree polynomial which would have $2z$ zeros and hence be the zero polynomial. Thus $\Delta^T_{N+K}\ \Delta_{N+K}$ is positive definite.

From the solution to the algebra problem, we can conclude that the M-W-A formula is given by

$$\vec{a} = (\Delta^T_{N+K}\ \Delta_{N+K})^{-1}\ (\lambda_0\vec{d}_0 + \lambda_1\vec{d}_1 + \ldots + \lambda_d\vec{d}_d) \tag{15}$$

where $\{d_i \; ; \; i = 0, 1, \ldots, d\}$ is a basis for D. The d+1 coefficients, λ_i, can be determined by applying (4) for the basis vectors written as

$$\vec{a}^T \vec{d}_i = \sum_{j=0}^{d} \lambda_j \; \vec{d}_j^T (\Delta_{N+K}^T \; \Delta_{N+K})^{-1} \; \vec{d}_i \qquad i = 0,1,\ldots,d \quad .$$

The inverse matrix of (15) can be easily calculated on a computer, or by interpretation as described in [1].

If $N = K = n$ as in [2], and the bases vector $\vec{d}_i$ is the vector of ith powers, $i = 0,1,\ldots,d$, then

$$\vec{d}_j^T (\Delta_{2n}^T \; \Delta_{2n})^{-1} \; \vec{d}_i = 0 \quad \text{when} \quad i + j \quad \text{is odd.}$$

This implies that if d is an odd integer, the system of d + 1 equations will separate into a non-homogeneous system in the $(d + 1)/2$ unknowns, $\lambda_0, \lambda_2, \ldots, \lambda_{d-1}$, and a homogenous system in λ_1, $\lambda_3, \ldots, \lambda_d$ which will all be zero. It follows, with some interpretation of the inverse matrix that the coordinates of $\vec{a}$ lie on a $2z + d - 1$ degree even polynomial.

We have set the specifications of the classical M-W-A formulas in a linear algebra framework and used a result thereof to determine the formulas. With the exception that we provided for formulas of unequal length, this is the class of formulas developed in [2].

One benefit in allowing formulas of unequal lengths $(K \neq N)$ becomes evident when deriving graduated value for the "end values". The traditional $2n + 1$ term formula of equal length does not provide for graduated values at the n values of the index at either end of the data set. By successively using $K = n - 1$ and $N = n$, $K = n - 2$ and $N = n$, etc. until $K = 0$ and $N = n$, graduated values can be obtained for each index value on the low end. *Mutatis Mutandis*, n graduated values t the high end can be obtained.

Some New M-W-A Formulas.

Previously we have used the result from linear algebra to obtain an algorithm for the determination of the M-W-A formulas defined by the classical conditions. Now we want to turn this procedure around and to define the concept of a M-W-A formula in this algebra framework. Our motivation is to broaden the family of M-W-A formulas.

Definition: For a given positive definite $n \times n$ matrix H, a linear subspace $L \subseteq R^n$, and a linear functional $f: L \to R^1$, the vector $\vec{a} \in R^n$ which satisfies $\vec{a}^T b = f(b)$ for all $b \in L$ and which minimizes $\vec{a}^T H \vec{a}$ is a M-W-A formula.

To interpret the n-dimensional vector $\vec{a}$ as a M-W-A formula we visualize successively calculating the inner product of a with n coordinates at a time of a long vector of observed values. Thus with this definition we have accepted an invariance in H, L, and f over the span of the observed values.

At this point the generalizations of the M-W-A formulas will be limited to the natural extensions of the traditional ones given in [2]. The starting point is L, the subspace of vectors which are "smooth". This has been a set of polynomials--now we can think of it as the subspace of all linear combinations of some set of functions other than powers, say exponential functions.

To obtain a true M-W-A formula, we must choose an L that is satisfactory at all index values--i.e. invariant. Otherwise we would generate different $\vec{a}$ s for different x's. While this would give satisfactory results, it would be more in the spirit of a Whittaker-Henderson formula than of a M-W-A formula.

After choosing L, then f can be defined. The value of f(b) is the "smoothed value" for a "true vector". Thus f defines what function of the coordinates we are estimating. Usually this is one of the coordinates. It can be a linear combination of the coordinates--or a financial function in some cases.

The third step is the choice of H. Traditionally, $H = A^T A$ where A^T is such that $A^T \vec{u}$ is a vector of a fixed order of differences of successive values of u. The order of differences is related to the degree of the polynomials in L, (see page 14, [2]), i.e. $\Delta^z b$ or $\Delta^{z+1} b$ is zero for all b in L. Similarly, we can base H on a linear operator which annihilates the chosen L.

An Example.

To illustrate and to explore the process outlined in the previous section, we have applied six different M-W-A formulas to 100 simulated mortality study data sets. Summary statistics of the results are shown in Tables 4 and 5.

Each of the six formulas was a 15 term formula with $K = N = 7$ and $f(b)$ equal to the eighth, or middle coordinate value. These are very traditional. We chose to smooth the estimates of the force of mortality. The choices for L and the linear operator used to develop H for each of the formulas are given in Table 1. For L Table 1 shows the names of the subspace of functions. E.G. "Makeham $c = 1.10$" means that the set of functions of the form $A + B(1.10)^x$ are the subspace L.

For H Table 1 shows the linear operator whose coefficients are the non-zero elements of each row of a 17×15 matrix (except for formula III which would be 16×15) such that the product of its transpose and itself provide the 15×15 matrix H.

TABLE 1

Formula	L	H
I	First Degree Polynomials	$(E - 1)(E - 1) = \Delta^2$
II	Makeham $c = 1.10$	$(E - 1)(E - 1) = \Delta^2$
III	Makeham $c = 1.10$	$(E - 1) = \Delta$
IV	Makeham $c = 1.08$	$(E - 1)(E - 1.08) = \Delta(E -$
V	Makeham $c = 1.10$	$(E - 1)(E - 1.10) = \Delta(E -$
VI	Makeham $c = 1.12$	$(E - 1)(E - 1.12) = \Delta(E -$

The choice of the numbers 1.08, 1.10, and 1.12 was motivated by the interval of values (1.08, 1.12) given in [3] for the Makeham constant c. The M-W-A coefficients are given in Table 2 at the end of the paper.

Each of the 100 simulated data sets consisted of the deaths and the number of lives exposed at each age from 30 through 80. The exposures, which are shown below, and the one year mortality rates used to simulate the deaths were the same for all data sets.

Age Interval	Exposure at each Age within the Interval
30-39	10,000
40-49	5,000
50-59	2,000
60-69	1,000
70-80	500

The total exposure for a single data set is 185,500 lives. The decreasing exposures were used to compensate for the increasing mortality rates and thus to have roughly equal variances as postulated in the M-W-A theory.

The one year mortality rates used in the simulation were determined by a Makeham force of mortality fit to q_{30}, q_{55}, and q_{80} of the New Basic Male Table, (Exhibit 3, [5]). The constants were $A = 0.00022154$, $B = 0.00003935$, and $C = 1.10168484$.

One data set is shown in Table 3, at the end of the paper.

For each of the six formulas, the four statistics

$$\sum_{37}^{70} |\Delta^3 \text{ Graduated } \mu|^k, \quad \text{and} \quad \sum_{37}^{73} |\text{True } \mu\text{-Graduated } \mu|^k, \; k = 1,2$$

were calculated for each of the 100 data sets. The values of the minimum, maximum, mean and standard deviation for each of these sets of 100 observations are shown in Tables 4 and 5 at the end of the paper.

The statistics $\sum_{37}^{70} |\Delta^3 \text{ Graduated } \mu|^k$, $k = 1,2$, which are called measures of smoothness, can be viewed as measures of fit to the underlying function in the traditional sense and in the real world--where the true values are not known but are assumed to be in some family of functions. For these measures that family of functions would be second degree (or less) polynomials. These statistics should be small for those M-W-A formulas based on differences. These statistics are summarized in Table 4.

Since this is a simulated environment rather than the real world we can measure the fit of the graduated values to the known underlying function precisely. These measures $\sum_{37}^{73} |\text{True } \mu\text{-Grad } \mu|^k$, $k = 1,2$ should be small for those M-W-A formulas based on a Makeham function subspace--especially the one with c near the underlying fitted value. These statistics are summarized in Table 5.

From a comparison of the means and standard deviations for the sums of the absolute deviations (say by use of classical confidence intervals), we conclude that the results for the five formulas based on a subspace, D , of Makeham's functions are indistinguishable relative to those for a first degree polynomial subspace, D. Given the Makeham function subspace, neither the choice of c nor the construction of H s had an impact on the sum of the absolute deviations.

Summary.

We have extended the theory underlying M-W-A formulas and consequently enriched the family. In an application to simulated mortality data the importance of the choice of a subspace of "smooth functions near" the underlying function can be observed.

The extended theory may provide a rational basis for the selection of the invariant subspace and the minimized sum of squares. We have not completed it in this paper.

Since presentation of this paper at the Iowa City Conference, a paper [6] on M-W-A formulas by Ornulf Borgan has appeared in the Scandinavian Actuarial Journal, 1979, No. 2-3. The reader will find a more extensive development of the statistical properties of the generalized M-W-A formulas in that paper.

The opening question of the robustness of the M-W-A formula family in the computer age is left for the readers and the users to the answer.

TABLE 2

M-W-A Formula Coefficients

Index	I	II	III	IV	V	VI
-7	.01032	.01339	.02835	.01296	.01354	.01409
-6	.02709	.03413	.05154	.03307	.03439	.03563
-5	.04696	.05734	.06976	.05565	.05758	.05941
-4	.06708	.07921	.08325	.07711	.07936	.08148
-3	.08514	.09701	.09227	.09481	.09700	.09908
-2	.09933	.10894	.09709	.10699	.10877	.11047
-1	.10836	.11407	.09802	.11273	.11379	.11483
0	.11146	.11227	.09540	.11178	.11195	.11215
1	.10836	.10407	.08960	.10456	.10379	.10310
2	.09933	.09059	.08103	.09201	.09042	.08893
3	.08514	.07339	.07013	.07552	.07336	.07131
4	.06708	.05436	.05741	.05681	.05445	.05218
5	.04696	.03555	.04341	.03785	.03570	.03363
6	.02709	.01901	.02872	.02070	.01917	.01767
7	.01032	.00665	.01401	.00744	.00674	.00604

TABLE 3

Age	Exposure	Deaths	$\hat{\mu}$	Age	Exposure	Deaths	$\hat{\mu}$
30	10,000	9	.90041 -3	56	2,000	18	.90407 -2
1	10,000	7	.70025 -3	7	2,000	18	.90407 -2
2	10,000	10	.10005 -2	8	2,000	16	.80322 -2
3	10,000	10	.10005 -2	9	2,000	22	.11061 -1
4	10,000	17	.17014 -2	60	1,000	8	.80322 -2
5	10,000	16	.16013 -2	1	1,000	23	.23269 -1
6	10,000	14	.14010 -2	2	1,000	14	.14099 -1
7	10,000	20	.20020 -2	3	1,000	13	.13085 -1
8	10,000	13	.13008 -2	4	1,000	18	.18164 -1
9	10,000	16	.16013 -2	5	1,000	22	.22246 -1
40	5,000	11	.22024 -2	6	1,000	22	.22246 -1
1	5,000	11	.22024 -2	7	1,000	32	.32523 -1
2	5,000	8	.16013 -2	8	1,000	29	.29429 -1
3	5,000	16	.32051 -2	9	1,000	28	.28399 -1
4	5,000	16	.32051 -2	70	500	18	.36664 -1
5	5,000	19	.38072 -2	1	500	17	.34591 -1
6	5,000	11	.22024 -2	2	500	15	.30459 -1
7	5,000	17	.34058 -2	3	500	21	.42908 -1
8	5,000	14	.28039 -2	4	500	22	.44997 -1
9	5,000	25	.50125 -2	5	500	26	.53401 -1
50	2,000	12	.60181 -2	6	500	28	.57629 -1
1	2,000	11	.55152 -2	7	500	26	.53401 -1
2	2,000	15	.75283 -2	8	500	41	.85558 -1
3	2,000	11	.55152 -2	9	500	39	.81210 -1
4	2,000	16	.80322 -2	80	500	34	.70422 -1
5	2,000	18	.90407 -2		500		

Notation: .xyz - k means $(.xyz) \times (10^{-k})$

TABLE 4

SUMMARY STATISTICS
MEASURES OF SMOOTHNESS
FOR
100 DATA SETS

FORMULA D ; H	MINIMUM	MAXIMUM	MEAN	STD DEV
	$\sum_{37}^{70} \|\Delta^3$ Graduated $\mu\|$			
1st Degree ; Δ^2	.00117	.00245	.00184	.00030
c = 1.10; Δ^2	.00084	.00193	.00133	.00023
c = 1.10; Δ	.00219	.00523	.00330	.00060
c = 1.08; $\Delta(E - 1.08)$	.00091	.00201	.00143	.00024
c = 1.10; $\Delta(E - 1.10)$	.00085	.00193	.00135	.00023
c = 1.12; $\Delta(E - 1.12)$	.00079	.00185	.00128	.00022
	$\sum_{37}^{70} \|\Delta^3$ Graduated $\mu\|^2$			
1st Degree ; Δ^2	.854 -7	.478 -6	.241 -6	.908 -7
c = 1.10; Δ^2	.426 -7	.275 -6	.128 -6	.529 -7
c = 1.10; Δ	.272 -6	.186 -5	.765 -6	.327 -6
c = 1.08; $\Delta(E - 1.08)$	.538 -7	.298 -6	.146 -6	.580 -7
c = 1.10; $\Delta(E - 1.10)$	.438 -7	.277 -6	.130 -6	.533 -7
c = 1.12; $\Delta(E - 1.12)$	.352 -7	.259 -6	.118 -6	.499 -7

Notation: .xyz - k means $(.xyz) \times (10^{-k})$

TABLE 5

SUMMARY STATISTICS
MEASURES OF FIT TO TRUE μ
FOR
100 DATA SETS

FORMULA D ; H	MINIMUM	MAXIMUM	MEAN	STD DEV
	$\sum_{37}^{73}$ |True μ-Graduated|			
1st Degree; Δ^2	.01084	.08700	.03322	.01608
c = 1.10; Δ^2	.00722	.06263	.02689	.01192
c = 1.10; Δ	.00683	.06154	.02513	.01181
c = 1.08; Δ(E - 1.08)	.00560	.06716	.02704	.01250
c = 1.10; Δ(E - 1.10)	.00722	.06261	.02686	.01191
c = 1.12; Δ(E - 1.12)	.00720	.05828	.02708	.01150
	$\sum_{37}^{73}$ |True μ-Graduated μ|^2			
1st Degree; Δ^2	.657 -5	.428 -3	.866 -4	.980 -4
c = 1.10; Δ^2	.233 -5	.238 -3	.545 -4	.541 -4
c = 1.10; Δ	.207 -5	.227 -3	.482 -4	.504 -4
c = 1.08; Δ(E - 1.08)	.118 -5	.268 -3	.563 -4	.594 -4
c = 1.10; Δ(E - 1.10)	.233 -5	.238 -3	.544 -4	.540 -4
c = 1.12; Δ(E - 1.12)	.301 -5	.211 -3	.544 -4	.510 -4

Notation: .xyz - k means $.xyz \times 10^{-k}$

References

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