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THE INVESTMENT PROCESS AND PRESENT VALUE CALCULATIONS

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It has become fashionable for research-minded actuaries in our profession to cast classical life contingency problems in a stochastic framework. The 1978 Actuarial Research Conference was devoted to this topic. At that conference, Panjer and Bellhouse presented a paper[1] that showed how to compute moments of insurances and annuities when interest rates as well as mortality (or other causes of decrement) are governed by a stochastic process. What distinguishes their contribution from earlier work is that their method is capable of handling realistic interest rate processes - in their paper expressions for both discrete and continuous first- and second-order autoregressive processes are given.

Recent experience has shown that interest rates are prone to large and relatively rapid changes. The increasing importance of external factors on the domestic economy suggests that the experience of the last decade is more likely to be the norm rather than the exception for many years to come. Since the actuary's business is the assessment of risk and uncertainty, it is necessary for him to be able to place the traditional "discounting at interest" and "accumulating at interest" calculations in a stochastic setting. In addition to accounting for the investment environment as accurately as possible, it is important to do likewise for the investment process. In a fluctuating interest rate environment, the interest rate risk to which a fund is subject depends on the principal repayment pattern of its investments. Consider a situation where a given cash flow is available for investment in either "bullet" bonds with principal repaid in one lump sum at the end of ten years or in ten-year "sinking fund" bonds requiring principal to be repaid in equal installments over the term of the bond. The value of the fund at the end of the ten-year period depends not only on how interest rates behave but also on which of the two instruments is used. The sinking fund bond has greater reinvestment risk: it benefits more than the bullet bond if interest rates rise, but suffers more if interest rates fall.

After reading the Panjer-Bellhouse paper in the Proceedings of the Ball State University Conference, I wondered whether their technique could be used when a realistic investment process was assumed. I believe that such an extension is necessary if the theory is to have any practical application I have had little success in trying to do this. Since ARCH is a forum for the presentation of ideas and for the stimulation of research, I have decided to present some thoughts and to invite others with different perspectives to bring them to fruition.

The easiest way to introduce the investment process into the calculation is to define a rollover function r(t) that describes how principal from a unit investment at time 0 is repaid. The amount of principal repaid between times t and t + dt is r(t) dt. Excluding perpetuities from the discussion, it must be true that

$$\int_0^{\infty} r(t)\, dt = 1, \tag{1}$$

since all principal must be repaid ultimately (ignoring the possibility of default). The function r(t) can be chosen to represent the distinctive rollover pattern of a particular type of asset (conventional mortgage, farm mortgage, bullet bond, sinking fund bond, etc.) or the rollover pattern of a portfolio of fixed-income instruments. Discrete rollover can be handled through the use of generalized functions (distributions).

The following analysis examines a fund built up from the investment of external cash flow of density $\mathcal{B}(t)$. Interest on invested funds is assumed to be payable continuously. All funds invested in the interval from t to t + dt earn interest at the force of interest $\delta(t)$. Such funds include not only $\mathcal{B}(t)$, but also interest paid during the interval (t,t+dt) and any principal repayments during the interval. Let $\mathcal{A}(t)$ represent the total density of invested funds. Then,

$$\mathcal{A}(t) = \mathcal{B}(t) + \int_0^t \mathcal{A}(u)\, r(t-u)\, du + \int_0^t \mathcal{A}(u)\, \delta(u) \left[1 - \int_0^{t-u} r(v)\, dv \right] du \tag{2}$$

The first term on the right-hand side is the density of external funds, the second is the density of principal repayments from previous investments, and the third is the density of paid interest. Let $\mathcal{F}(t)$ denote the value of the entire fund at time t.

$$\mathcal{F}(t) = \int_0^t \mathcal{A}(u) \left[1 - \int_0^{t-u} r(v)\, dv \right] du \tag{3}$$

Equation (2) is an inhomogeneous Volterra integral equation of the second kind;

$$\mathcal{A}(t) = \mathcal{B}(t) + \int_0^t \mathcal{K}(t,u)\, \mathcal{A}(u)\, du, \tag{4}$$

with kernel

$$\mathcal{K}(t,u) = r(t-u) + \delta(u) \left[1 - \int_0^{t-u} r(v)\, dv \right]. \tag{5}$$

If (2) could be solved explicitly for general $\mathcal{B}(t)$, $\delta(t)$, and r(t), then equation (3) would serve as the starting point in a stochastic treatment of $\mathcal{F}(t)$. Specifically, $\mathcal{B}(t)$, which might depend on mortality rates or other causes of decrement, could be represented by a stochastic process. Also, $\delta(t)$ could be represented by a stochastic process, as discussed in Panjer and Bellhouse. Unfortunately, apart from a series solution, (2) cannot be solved in general.*

Two special cases can be solved and produce well-known results. First, consider a constant force of interest δ. Then $\mathcal{K}(t,u)$ is a function of t-u only, and Laplace transforms can be used to solve (2). The Laplace transform of a function f(t) is defined as

$$\tilde{f}(k) \equiv \mathcal{L}[f] = \int_0^\infty f(t)\, e^{-kt}\, dt \ . \tag{6}$$

Using the convolution theorem for Laplace transforms, (2) and (3) may be written as

$$\tilde{\mathcal{A}}(k) = \tilde{\mathcal{B}}(k) + \tilde{\mathcal{A}}(k)\,\tilde{r}(k) + \frac{\delta}{k}\,\tilde{\mathcal{A}}(k)\left[1 - \tilde{r}(k)\right], \tag{7}$$

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*If rollover, paid interest, cash flow, and reinvestment occur only at discrete times, equation (2) can be solved in general by finding the "solving kernel" or "Green's function" of the integral equation. In the continuous form, the Green's function $\mathcal{G}(t,u)$ satisfies

$$\mathcal{A}(t) = \int_0^t \mathcal{G}(t,u)\, \mathcal{B}(u)\, du \ .$$

The Green's function depends only on the investment process and the investment environment, that is, on rollover rates and interest rates only. I used this method in a paper titled "The Matching of Assets and Liabilities" to be published in TSA XXXII. In that paper the discrete form of the Green's function was represented by the matrix $\{\gamma_{k\ell}\}$.

and
$$\tilde{\mathcal{F}}(k) = \frac{1}{k} \tilde{\mathcal{A}}(k) \left[1 - \tilde{r}(k)\right] . \tag{8}$$

Solving (7) for $\tilde{\mathcal{A}}(k)$ and then substituting the result into (8), we obtain

$$\tilde{\mathcal{F}}(k) = \frac{\tilde{\mathcal{B}}(k)}{k - \delta} \tag{9}$$

Finally, inverting the Laplace transform of $\mathcal{F}$, we derive

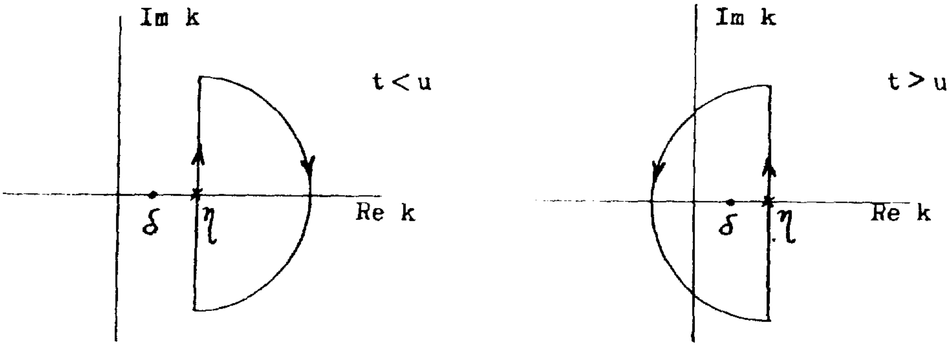
$$\mathcal{F}(t) = \frac{1}{2\pi i} \int_{-i\infty+\eta}^{i\infty+\eta} \frac{1}{k-\delta} \left[\int_0^\infty e^{-ku} \mathcal{B}(u)\, du \right] e^{kt}\, dk \tag{10}$$

The constant η must be chosen so that $\eta > \delta$. Changing the order of integration, we get

$$\mathcal{F}(t) = \int_0^\infty \left[\frac{1}{2\pi i} \int_{-i\infty+\eta}^{i\infty+\eta} \frac{e^{k(t-u)}}{k-\delta}\, dk \right] \mathcal{B}(u)\, du . \tag{11}$$

Integration in the complex k plane is used to evaluate the inner integral in (11). For $t<u$, the contour is completed to the right and the integral vanishes since the integrand is analytic everywhere to the right of the line $\mathrm{Re}\, k = \delta$. For $t>u$, the contour is completed to the left. Since the integrand has a simple pole at $k = \delta$ on the interior of the contour, the value of the integral is $2\pi i$ times the residue of the integrand at $k = \delta$. Thus,

$$\mathcal{F}(t) = \int_0^t \mathcal{B}(u)\, e^{\delta(t-u)}\, du . \tag{12}$$



Equation (12) is, of course, the familiar result. Notice also that $\mathcal{F}(t)$ is independent of r(t), as it should be if interest rates are constant.

The second special case occurs when all invested funds are continuously and completely rolled over and then entirely reinvested at the current rate $\delta(t)$. This is the assumption underlying the treatment of Panjer and Bellhouse and most other papers on the subject. To solve this special case, we first solve the more general problem with $r(t) = (1/t_o)e^{-t/t_o}$ and then consider the limit $t_o \to 0$. With this choice of r(t) the integral equation is solvable because the kernel is separable. Equation (2) becomes

$$\mathcal{A}(t,t_o) = \mathcal{B}(t) + \frac{e^{-t/t_o}}{t_o} \int_0^t \mathcal{A}(u,t_o)\left(1 + \delta(u)t_o\right) e^{u/t_o}\, du \tag{13}$$

Differentiating (13) with respect to t yields

$$\frac{\partial \mathcal{A}(t,t_o)}{\partial t} = \frac{d\mathcal{B}}{dt} - \frac{e^{-t/t_o}}{t_o^2} \int_0^t \mathcal{A}(u,t_o)\left(1 + \delta(u)t_o\right) e^{u/t_o}\, du + \frac{1}{t_o}\,\mathcal{A}(t,t_o)\left(1 + \delta(t)t_o\right). \tag{14}$$

Using (13), the integral in the second term on the right-hand side of (14) can be expressed in terms of $\mathcal{A}(t,t_o)$ and $\mathcal{B}(t)$. Making that substitution into (14), we derive

$$\frac{\partial\mathcal{A}(t,t_o)}{\partial t} = \frac{d\mathcal{B}}{dt} + \frac{\mathcal{B}(t)}{t_o} + \delta(t)\,\mathcal{A}(t,t_o)\,. \qquad (15)$$

This is a first order linear differential equation for $\mathcal{A}(t,t_o)$.

The general solution of the inhomogeneous differential equation

$$\frac{d\psi}{dy} - \xi(y)\,\psi(y) = \varphi(y)$$

consists of a particular solution plus the general solution of the homogeneous equation. Suppose the initial condition (boundary value) is a specified value of $\psi(0)$. Then,

$$\psi(y) = \psi(0)\, e^{\int_0^y \xi(z)\,dz} + \int_0^y \varphi(x)\, e^{\int_x^y \xi(z)\,dz}\, dx$$

Thus, for our problem,

$$\mathcal{A}(t,t_o) = \mathcal{B}(0)\, e^{\int_0^t \delta(w)\,dw} + \int_0^t \left(\frac{d\mathcal{B}}{du} + \frac{\mathcal{B}(u)}{t_o}\right) e^{\int_u^t \delta(w)\,dw}\, du\,. \qquad (16)$$

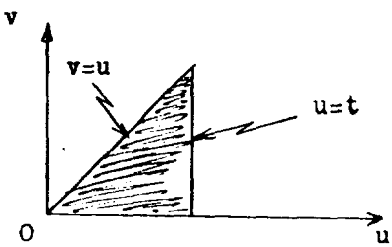
Integrating the $d\mathcal{B}/du$ term by parts produces two "surface" terms: $\mathcal{B}(t)$ and one that cancels the first term on the right-hand side of (16). Hence,

$$\mathcal{A}(t,t_o) = \mathcal{B}(t) + \int_0^t \left(\delta(u) + \frac{1}{t_o}\right) \mathcal{B}(u)\, e^{\int_u^t \delta(w)\,dw}\, du\,. \qquad (17)$$

Substituting (17) into (3), we finally arrive at

$$\mathcal{F}(t,t_0) = \int_0^t \mathcal{B}(u)\, e^{-(t-u)/t_0}\, du + \int_0^t \left[\int_0^u \left(\delta(v) + \frac{1}{t_0}\right)\mathcal{B}(v)\, e^{\int_v^u \delta(w)dw}\, dv\right] e^{-(t-u)/t_0}\, du \tag{18}$$

The region of integration in the second term on the right-hand side of (18) is shown below.



As $t_0 \to 0$, the factor $e^{-(t-u)/t_0}$ appearing in (18) becomes vanishingly small in the entire region of integration outside the strip $t - \epsilon t_0 \leq u \leq t$ for some fixed number ϵ (independent of t_0). In this strip, the above factor varies rapidly from 0 to 1 while the rest of the integrand is approximately constant, taking its value at u=t. Therefore,

$$\begin{aligned}\lim_{t_0 \to 0} \mathcal{F}(t,t_0) &= \left(\mathcal{B}(t) + \int_0^t \delta(v)\mathcal{B}(v)\, e^{\int_v^t \delta(w)dw}\, dv\right) \lim_{t_0 \to 0} \int_0^t e^{-(t-u)/t_0}\, du \\ &\quad + \left(\int_0^t \mathcal{B}(v)\, e^{\int_v^t \delta(w)dw}\, dv\right) \lim_{t_0 \to 0} \frac{1}{t_0} \int_0^t e^{-(t-u)/t_0}\, du \\ &= \int_0^t \mathcal{B}(v)\, e^{\int_v^t \delta(w)dw}\, dv\,. \end{aligned} \tag{19}$$

Equation (19) is the form that Panjer and Bellhouse use in their paper "Theory of Stochastic Mortality and Interest Rates." It is based on _instantaneous_ rollover of all invested funds.

It might be possible to solve equations (2) and (3) for specific forms of r(t) characteristic of bonds, mortgages, and other fixed-income instruments. However, that is certainly not as satisfying as having a general theory. Perhaps the approach embodied in (2) and (3) is not the most useful way to incorporate the investment process into the "discounting" and "accumulating" calculations of actuaries. I am interested in hearing the views of others on this subject.

REFERENCE

1. H.H. Panjer and D.R. Bellhouse, "Theory of Stochastic Mortality and Interest Rates," Proceedings, Thirteenth Annual Actuarial Research Conference, ARCH 1978.2