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THE UNIFORM DISTRIBUTION OF DEATHS ASSUMPTION AND PROBABILITY THEORY

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The purpose of this note is to show how certain formulas of life contingencies can be derived almost painlessly under the uniform distribution of deaths assumption.

Let the random variable T denote the remaining life time of someone aged x, and let K be the greatest integer less than or equal to T. For a life insured at x, T is the duration at death and K is the curtate duration at death. Then $U = T - K$ is the fractional part of a year lived in the year of death. Furthermore, for $m = 1, 2, 3, \ldots$ the random variable U_m is defined by the requirement that

$$U_m = (j+1)/m \ , \text{ if } j/m < U \leq (j+1)/m \ ,$$

$j = 0, 1, \ldots, m-1$. By convention, $U_\infty = U$. Net single premiums can be expressed as expected values of functions of these random variables. For example,

$$A_x = E[v^{K+1}]$$

$$\bar{a}_x = E[\bar{a}_{\overline{T|}}]$$

$$(I^{(m)}\bar{A})_x = E[(K+U_m)v^T] \ .$$

Of course, $\overset{\circ}{e}_x = E[T]$, and $e_x = E[K]$.

In the following, the uniform distribution of deaths assumption will be made over each of the intervals $(x, x+1)$, $(x+1, x+2)$, etc. Then for $k = 0, 1, 2, \ldots,$ and $0 < u < 1$,

$$\Pr(K = k \text{ and } U \leq u) = \Pr(k < T \leq k + u)$$

$$= {}_kp_x \; {}_uq_{x+k} = {}_kp_x \; q_{x+k} \cdot u = \Pr(K = k) \Pr(U \leq u)$$

under this assumption. It follows that the random variable U is a) independent of the random variable K, and b) is uniformly distributed between 0 and 1. Furthermore, this implies that U_m is also independent of K and has a discrete uniform distribution over the points $1/m, 2/m, \ldots, 1$.

To derive the formula for $A_x^{(m)}$, we start with the identity

$$v^{K+U_m} = (1+i)^{1-U_m} v^{K+1} .$$

Now by use of the independence of U_m and K, and the uniform distribution of U_m, we obtain

$$\text{(1)} \qquad A_x^{(m)} = E[(1+i)^{1-U_m}] \, A_x = s_{\overline{1}|}^{(m)} \, A_x .$$

Next, we derive the formula for $\ddot{a}_x^{(m)}$. The general identity

$$d \, \ddot{a}_x + A_x = d^{(m)} \, \ddot{a}_x^{(m)} + A_x^{(m)}$$

(both sides are one) can be rearranged as

$$\ddot{a}_x^{(m)} = \frac{d}{d^{(m)}}\,\ddot{a}_x - \frac{1}{d^{(m)}}\,(A_x^{(m)} - A_x)$$

Under the uniform distribution of deaths assumption we may substitute (1) to get

(2)
$$\ddot{a}_x^{(m)} = \frac{d}{d^{(m)}}\,\ddot{a}_x - \frac{s_{\overline{1}|}^{(m)} - 1}{d^{(m)}}\,A_x \,.$$

This formula has an appealing interpretation: $\ddot{a}_x^{(m)}$ is less than $\ddot{a}_x$ for two reasons, loss of interest and missing payments in the year of death. Substituting $A_x = 1 - d\,\ddot{a}_x$, we obtain the formula

(3)
$$\ddot{a}_x^{(m)} = \frac{d}{d^{(m)}}\,s_{\overline{1}|}^{(m)}\,\ddot{a}_x - \frac{s_{\overline{1}|}^{(m)} - 1}{d^{(m)}} \,,$$

which may be more useful for numerical evaluation.

To demonstrate the simplicity of this approach, let us look at the daring act of calculating

(4)
$$(I^{(m)}A)_x^{(n)} = E[(K+U_m)\,v^{K+U_n}] \,,$$

where one of the integers m, n is a multiple of the other. (Note that this includes the cases where one of them is 1 or ∞, or where both are equal.) The starting point is now the identity

$$(K+U_m)v^{K+U_n} = (1+i)^{1-U_n}(K+1)v^{K+1} - (1-U_m)(1+i)^{1-U_n}\,v^{K+1} \,.$$

Taking expectations, under the uniform distribution of deaths assumption we see that

(5) $$(I^{(m)}A)_x^{(n)} = s_{\overline{1}|}^{(n)} \ (IA)_x - f(m,n) \ A_x ,$$

where the factor

(6) $$f(m,n) = E[(1-U_m)(1+i)^{1-U_n}]$$

is a function of only interest. Finding a closed form solution for $f(m,n)$ is a problem of compound interest. For this purpose, observe that

(7) $$f(m,n) = s_{\overline{1}|}^{(n)} - g(m,n) ,$$

where

(8) $$g(m,n) = E[U_m(1+i)^{1-U_n}] .$$

Finally, let us distinguish two cases:

<u>Case 1</u>: If n is a multiple of m ,

$$g(m,n) = (I^{(m)}s)_{\overline{1}|}^{(n)} = \frac{i^{(m)}}{i^{(n)}} \ (I^{(m)}s)_{\overline{1}|}^{(m)}$$

<u>Case 2</u>: If m is a multiple of n , observe that

$$E[U_n - U_m \mid U_n] = \frac{m-n}{2mn} .$$

(Note that in the special case $n = 1$, this is the usual $\frac{m-1}{2m}$.)

Replacing U_m by $U_n - (U_n - U_m)$ is (8), and taking the conditional expectation (given U_n) first, we see that

$$(9) \qquad g(m,n) = (I^{(n)}S)_{\overline{1}|}^{(n)} - \frac{m-n}{2mn} s_{\overline{1}|}^{(n)} .$$

Thus the final result is that

$$(10a) \qquad f(m,n) = s_{\overline{1}|}^{(n)} - \frac{i^{(m)}}{i^{(n)}} (I^{(m)} S)_{\overline{1}|}^{(m)}$$

in case 1, and

$$(10b) \qquad f(m,n) = (1+\frac{m-n}{2mn}) s_{\overline{1}|}^{(n)} - (I^{(n)}S)_{\overline{1}|}^{(n)}$$

in case 2.

To obtain the formula for $(I^{(m)}\ddot{a})_x^{(n)}$, which is only meaningful if n is a multiple of m (case 1), the easiest method is to start with the identity

$$(11) \qquad \ddot{a}_x^{(m)} = d^{(n)} (I^{(m)}\ddot{a})_x^{(n)} + (I^{(m)}A)_x^{(n)}$$

and use the previous results (2) and (5). In the special case $m = 1$, a more direct approach is possible, however. By interpreting the standard increasing annuity as a series of deferred level annuities, we see that

$$(12) \qquad (I\ddot{a})_x^{(n)} = \sum_{k=0}^{\infty} {}_kE_x \, \ddot{a}_{x+k}^{(n)} .$$

From (2) we see that

$$\ddot{a}^{(n)}_{x+k} = \frac{d}{d^{(n)}} \ddot{a}_{x+k} - \frac{s^{(n)}_{\overline{1|}} - 1}{d^{(n)}} A_{x+k} .$$

We substitute this in (12) to obtain

$$(I\ddot{a})^{(n)}_{x} = \frac{d}{d^{(n)}} (I\ddot{a})_{x} - \frac{s^{(n)}_{\overline{1|}} - 1}{d^{(n)}} (IA)_{x}$$

As formula (2), this formula separates the effect of n-thly payments into an interest and a mortality component.