ON THE NUMERICAL EVALUATION OF SURVIVAL PROBABILITIES

W. Santermans and M. J. Goovaerts

INTRODUCTION

In his book [1] H. Seal pointed out a new direction for evaluating numerically survival probabilities.
Let $P_n(t)$ ($\{\equiv \text{prob } N(t) = n\}$) denote the probability that N(t) claims, with N(t) = 0,1,2,...., occur in an interval of time extending from now to epoch t, which according to H. Seal is written as (o,t).
Let further B(y) denote the distribution function of a claim size Y. In his book H. Seal deals with the following special choices of $P_n(t)$

i) $$P_n(t) = e^{-t} \frac{t^n}{n!} \quad \text{(the Poisson case)} \qquad (1.1.)$$

ii) $$P_n(t) = \binom{h+n-1}{n} \left(\frac{h}{t+h} \right)^h \left(\frac{t}{t+h} \right)^n \quad \text{(the negative binomial case)} \ (1.2.)$$

iii) $$P_n(t) = \frac{\Gamma(\rho + h)\Gamma(\rho + k)}{\Gamma(\rho)\ \Gamma(\rho + h + k)} \ \frac{(h)_n (k)_n}{(\rho + h + k)_n} \cdot \frac{1}{n!} \qquad (1.3.)$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$

$h = at$

$k = bt$

$\rho = ct + 1$ with $ab = c$

(the generalized Waring case)

The last two special cases can be considered as a weighted Poisson process with an infinitely divisible mixing distribution. This property is well known for the negative binomial case (gamma mixing distribution).
Recently this property was derived for the generalized waring distribution (generalized gamma mixing distribution) (see ref [2])

In order to be able to test his numerical procedure the following claim size distributions are considered by H. Seal :

i) $b(x)(=B'(x)) = e^{-x}$ (the exponential case) (1.4.)

ii) $b(x) = (\frac{\lambda}{2\pi x^3})^{\frac{1}{2}} \exp(-\frac{\lambda(x-\mu)^2}{2\mu^2 x})$ (the inverse normal) (1.5.)

The equation satisfied by the probability U(w,t) surviving at least t time intervals given that the company has a risk reserve of w at the beginning of the interval (o,t) can be written down as follows :

$$U(w,t) = F(w + (1 + \eta)t,t) - (1 + \eta) \int_0^t U(o, t - \tau) f(w + (1 + \eta)\tau,\tau)d\tau \qquad (1.6.)$$

were one assumes that risk loading on the pure unit bevel annual premium, payable continuously throughout the year is 100η %. In addition to following notations have been introduced :

$$f(x,t) = \sum_{n=1}^{\infty} P_n(t)\, b^{n*}(x) \qquad (1.7.)$$

$$F(n,t) = \sum_{n=o}^{\infty} P_n(t)\, B^{n*}(x) \qquad (1.8.)$$

2) SOME COMMENTS

In case $P_n(t)$ is a mixed Poisson process with an infinitely divisible mixing distribution one has :

$$U(o,t) = \frac{1}{(1+\eta)t} \int_0^{(1+\eta)t} F(y,t)dy \qquad (2.1.)$$

Hence, because of the result obtained in ref [2] formula (2.1.) also applies in the case of a generalized Waring claim number process.

As is pointed out in ref. 1 the generalized Waring distribution is obtained by taking a mixture of

$$P_n(\lambda\upsilon) = e^{-\lambda\upsilon} \frac{(\lambda\upsilon)^n}{n!}$$

with mixing distributions :

$$\frac{h^h}{\Gamma(h)} e^{-h\lambda} \lambda^{h-1} \qquad 0 < \lambda < \infty, h > 0 \qquad (2.2.)$$

and :

$$\frac{\Gamma(\rho + k)}{\Gamma(\rho)\Gamma(k)} h^{-k} \upsilon^{k-1} (1 + \frac{\upsilon}{h})^{-(\rho + k)} \qquad k > 0, \rho > 0 \qquad (2.3.)$$

In the case of an exponential claimsize distribution (2.2.) can be verified to hold by evaluating U_W (o,t) and U_G (o,t), where U_W (o,t) is obtained by taking a mixture of :

$$\frac{1}{(1 + \eta)t} \int_0^{(1 + \eta)t} F(y,\lambda\upsilon)dy \qquad (2.4.)$$

with mixing distributions (2.2.) and (2.3.) ($F(y,\lambda\upsilon)$ denotes the distribution function of the total claim amount based on a Poisson process with parameter $\lambda\upsilon$). On the other hand U_G (o,t) is obtained by evaluating the r.h.s of (2.1.) for F (y,t) denoting the total claim amount based on a generalized Waring distribution. An elementary calculation shows that U_G (o,t) = U_W (o,t) which proves that for an exponential claimsize distribution formula (2.1.) holds also in the generalized Waring case. Consequently the algorithm of H. Seal based on Laplace inversion can be applied, rather than first solving the integral equation

$$U(o,t) = F((1 + \eta)t,t) - (1 + \eta) \int_0^t U(0,t - \tau)f((1 + \eta)\tau,\tau)d\tau \qquad (2.5.)$$

obtained by putting w = 0 in (1.6.). In fact the numerical solution of (2.5.) is a rather difficult task. The accuracy of the numerical results obtained in this way seems to be rather insufficient. On the other hand the algorithm based on the Laplace inversion seems to be very accurate. To illustrate both facts we recall the results for the Poisson and negative binomial distribution also obtained by H. Seal (ref. [1]) both in case of an exponential and an inverse normal claimsize distribution. We also add now the results for the generalized Waring case based on the algorithm of H. Seal by means of the Laplace inversion. In order to show the insufficient accuracy of the method obtained by solving the integral equation (2.5.) we add table 2

Table I, U (o,t) calculated using the algorithm of H.Seal based on Laplace inversion.

P_n (t)	B (y)		t = 1	2	3	4	5
Poisson	Exponential	U (o,t)	0.535819	0.407000	0.344760	0.306690	0.280406
		U (10,t)	0.999544	0.998562	0.996612	0.993951	0.990610
	Inverse-normal	U (o,t)	0.375629	0.357394	0.302806	0.269828	0.247203
		U (10,t)	0.999845	0.999717	0.999260	0.998462	0.997308
Negative binomial h = 2	Exponential	U (o,t)	0.579975	0.476480	0.429822	0.403108	0.385736
		U (10,t)	0.998764	0.993575	0.982555	0.967312	0.949422
	Inverse-normal	U (o,t)	0.530914	0.439019	0.400501	0.379078	0.365382
		U (10,t)	0.999555	0.996659	0.988407	0.975318	0.958902
Generalized Waring a = 2 b = 4 c = 8	Exponential	U (o,t)	0.613964	0.472286	0.400549	0.355995	0.324969
		U (w,t)	0.996543	0.992554	0.986942	0.980415	0.973195
	Inverse-Normal	U (o,t)	0.571057	0.433827	0.367440	0.326680	0.298441
		U (w,t)	0.997781	0.995560	0.992135	0.987903	0.982952

<u>Table II</u>, U (o,t) and U (10,t) calculated using H. Seal's method for solving an integral equation.

Generalized Waring a = 2 b = 4 c = 8	Exponential	U (o,t)	1.	0.6801	0.4688	0.3130	0.1971
		U (10,t)	1.	1.	1.	0.9856	0.9610
	Inverse Normal	U (o,t)	1.	0.6103	0.4029	0.2614	0.1553
			1.	1.	1.	0.9924	0.9722

REFERENCES

[1] H. Seal "Survival Probabilities; The goal of risk Theory" Springer Verlag, (eds) 1979, pp X + 101

[2] M.J. Goovaerts and M. Van Wouwe "The generalized Waring Distribution as a Mixed Poisson with a Generalized gamma mixing distribution", Bulletin K.V.B.A., to be published.