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SHOULD THE DEFINITION OF COMPOUND INTEREST BE MODIFIED? - ACTUARIAL NOTE

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The problem

This note is labeled actuarial because it deals extensively with the force of interest, a concept which is characteristically of concern to the actuarial community.

During the five years which I have taught the theory of interest, I have never been able to adequately address the tricky question inevitably raised by a brighter-than-the-average student: "How do you explain that the expression $(1+i)^t$ is good for any real value of t?" The traditional definition of compound interest stipulates that the interest earned by a fund during, let's say, a year is added to the principal at the end of the year and earns interest thereafter. The formula $(1+i)^t$ is trivially derived for integral values of t. However the definition is mute on the way interest is earned during the year, so that the answer to the above question is pure speculation.

I have looked through many textbooks in search of an answer, but no one has proved satisfactory. Most do not even touch on the subject. Kellison [3], for one, says: "Strictly speaking, the accumulation function for compound interest has been defined only for integral values of t. However, it is natural to assume that interest is accruing continuously and, therefore, to extend the definition to non-integral values of t." This solution is unsatisfactory because, even admitting that interest accrues continuously (this not being stipulated in the definition), it does not necessarily follow that $(1+i)^t$ is continuous, as we shall see.

The most widespread solution to the problem is to replace the given effective rate i by an equivalent rate which is compounded with a frequency that makes the time involved a whole number of periods, then accumulating in the usual way with this new rate. Let us find, for example, the compound amount of $1000 after 15 years 3 months at $j^{(2)} = 6\%$. The solution presented by Hummel and Seeback[2] is:

"Solution: The first step is to replace the rate, $j_2 = 6\%$, by a rate compounded quarterly, since the time involved is an integer when expressed as quarters. So let i be the interest rate per quarter that is equivalent to $j_2 = 6\%$. Then

$$(1+i)^4 = (1.03)^2 \quad \text{and} \quad (1+i) = (1.03)^{1/2}$$

Accumulating $1000 for 15 years 3 months at rate i gives

$$S = 1000\,(1+i)^{61}$$

Substituting for $(1+i)$ its value from the preceding equation gives

$$S = 1000\left[(1.03)^{1/2}\right]^{61} = 1000\,(1.03)^{30\frac{1}{2}}$$

. . . So it appears that the compound interest formula holds whether n is an integer or not".

Hummel and Seeback's reasoning leads logically to the replacement of i by an equivalent rate of interest compounded continuously (the force of interest, actuarially speaking) which would fit any situation. This last approach has been utilized by Butcher and Nesbitt [1]. They first derive the two following expressions:

$$a(n) = e^{\int_0^n \delta_t \, dt} \qquad (1)$$

$$\lim_{m \to \infty} \left(1 + \frac{i^{(m)}}{m}\right)^m = e^{\delta} = 1 + i \qquad (2)$$

where (1) holds true for any law of interest and any real value of n, and where $\delta = i^{(\infty)}$. Then: "If now, for the continuous theory, we assume that $\delta_t = \delta$, a constant, then ..."

$$a(n) = e^{\delta n} \qquad \text{using (1)}$$

$$= (1 + i)^n \qquad \text{using (2)}$$

Expression (1) being good for any value of n, $(1+i)^n$ has been proved continuous for $n \geq 0$. An immediate question is: "Why should we assume that δ_t is a constant?"

Author's viewpoint

I believe, as do Butcher and Nesbitt (and many others), that a constant force of interest underlying the law of compound interest is the unique solution to prove the continuity of $(1+i)^t$. However the fact is that a compound rate of interest does not imply necessarily a constant force of interest. The present definition implies that $a(n) = (1+i)^n$ for integral values of n. As a corollary, we have that the effective annual rate of interest for a given year n, i_n, is constant and equal to i, for any integral values of n ($n \geq 1$):

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} \tag{3}$$
$$= i$$

using the common definition of effective rate and Kellison's symbols. Now let us examine what a constant annual effective rate of interest implies on the underlying force of interest. It does not imply much. Replacing (1) in (3):

$$i_n = i = \frac{e^{\int_0^n \delta_t \, dt} - e^{\int_0^{n-1} \delta_t \, dt}}{e^{\int_0^{n-1} \delta_t \, dt}}$$

leading to

$$1 + i = e^{\int_{n-1}^{n-1} \delta_t \, dt}$$

and

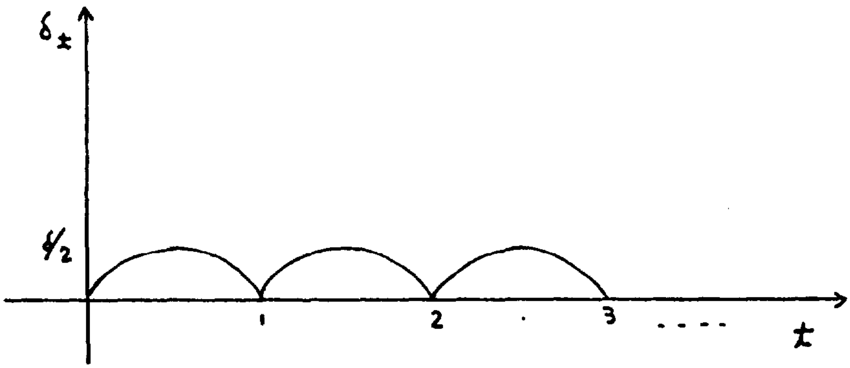
$$\int_{n-1}^{n} \delta_t \, dt = \ln(1+i) = \text{constant} \tag{4}$$

All that is required of δ_t is that it sum up to a constant when integrated from one integral duration to the next one.

Actually a number of functions δ_t may solve equation (4), a constant being obviously one of them. A cyclical function could also be used. Let us, for example, consider the following function:

$$\delta_t = \frac{\delta}{2} \left| \sin \pi t \right|$$

The graph of δ_t is as follows:



One can easily verify that the surface under any semi-circle is equal to $\ln(1+i)$ so that (4) holds true. We would then get the following "elegant" formula for $a(n+k)$, where n is integral and k fractional:

$$a(n+k) = e^{\int_0^{n+k} \frac{\ln(1+i)}{2} |\sin \pi t| \, dt}$$

$$= \left[\prod_{t=1}^{n} \left(e^{\int_{t-1}^{t} \frac{\ln(1+i)}{2} |\sin \pi t| \, dt} \right) \right] \cdot e^{\int_n^{n+k} \frac{\ln(1+i)}{2} |\sin \pi t|}$$

$$= (1+i)^n \cdot e^{\frac{\ln(1+i)}{2}\left[1 - \cos((n+k)\cdot\pi)\right]}$$

In summary, an effective rate of interest i does not necessarily imply an equivalent constant force of interest δ, which, in turn, would be necessary to prove that $(1+i)^t$ is true for all real values of t.

Kellison's text states that it is normal to assume that interest accrues continuously. Let us suppose that this also means that interest accrues at a constant continuous rate. Then Kellison would be fully justified in saying that $(1+i)^t$ is good for any value of t (although it would have been better to insert the concept of constant growth rate in the basic definition). Where I disagree is when he proves later that δ_t is constant under compound interest and uses the continuity of $(1+i)^t$ to prove it, which continuity is based on the constancy of δ_t! We thus have arrived at my crucial point: the vicious circle trap. $(1+i)^t$ is continuous if and only if δ_t is constant, but δ_t is constant if and only if $(1+i)^t$ is continuous since

$$\delta_t = \frac{d\,a(t)/dt}{a(t)} \tag{5}$$

and a non-continuous function would not be differentiable. This vicious circle suggests that it might be appropriate to insert in the definition of compound interest the concept of a continuous and constant growth rate.

Some professors of finance, when questioned on this subject of the continuity of $(1+i)^t$, stated that a compound rate of interest does not at all imply a constant force of interest. To be more explicit, $1000 invested at 5% effective could earn $0.10 the first day. $0.01 the second day, $0.25 the third day, nothing the fourth day, and so on (a day being considered a moment). The important point, they said, is that the interest sum up to $50 after one year. But then, how can one affirm, under such a random scenario, that $1000 will have accumulated to $1000 $(1.05)^{t}$ after 3 months, unless by pure coincidence? This is impossible. If I extend the reasoning of these professors, continuous compounding of interest at a constant rate is a special case of the basic compound interest theory. Perhaps they are right, but this has never been mentioned in any textbook. If they are right, then the continuous expression $(1+i)^t$ is also a special case and does not represent a general expression for compound interest. My personal point of view is that we should modify the basic definition.

Proposed Definition

A modified definition could be as follows:

"A fund earns annual compound interest i when it grows continuously at a constant rate δ such that the effective rate of interest in any year is i".

From a teaching point of view or for those planning to write a textbook, this definition means that the concept of effective rate, including the force of interest (which is the infinitesimal effective rate espressed on an annual basis) should be carefully analyzed at the outset. The procedure to derive $(1+i)^t$ could be summarized as follows:

Step 1

Derive the expression

$$a(n) = e^{\int_0^n \delta_t \, dt} \tag{1}$$

from the basic definition of δ_t, i.e.,

$$\delta_t = \frac{d\,a(t)/dt}{a(t)} \tag{5}$$

which definition is pure reasoning and applies to any law of interest.

Step 2

Use the two basic concepts of the definition (constant δ and constant effective rate i) to derive (6) and (7):

$$a(n) = e^{\delta n} \tag{6}$$

$$i_n = i = \frac{e^{\delta n} - e^{\delta(n-1)}}{e^{\delta(n-1)}} = e^{\delta} - 1$$

leading to $$e^{\delta} = 1 + i \tag{7}$$

Step 3

Replace (7) in (6):

$$a(n) = (1 + i)^n \qquad \forall n$$

The annual compounding of interest would be a corollary.

The proposed definition seems to differ greatly from the traditional one; actually it does not so much since it merely stipulates that interest is compounded continuously at a constant rate δ rather than being compounded yearly at a constant rate i. Moreover the proposed definition has three advantages. First, it is in line with the now-almost-universal practice of banks of quoting their accounts' earnings on the basis of two equivalent rates: one compounded daily (close approximation to δ), the other on a yearly basis (i). Second, the definition makes compound interest more consistent with simple interest from a teaching point of view. Under simple interest, the <u>amount</u> of interest earned at each moment is constant; under compound interest, the <u>effective rate</u> of interest earned at each moment is constant. Third, the definition helps to erase the impression students acquire, from the traditional definition, that compounding is a characteristic of compound interest only. Actually interest is compounded under any law of interest, on an infinitesimal scale. This is expressed in the following equation, which is a rearrangement of (5):

$$a(t+dt) = a(t) + a(t)\,\delta_t\,dt$$

This reflects the fact that, for any law of interest, the interest earned at each moment is credited to the principal instantaneously. Let us consider the simple interest case: δ_t is continuously decreasing $\left[\frac{i}{1+ti}\right]$ but since $a(t)$ is continuously increasing $\left[1+ti\right]$ due to the compounding effect, there is an offset such that the <u>amount</u> of interest earned at each moment is constant $\left[i\,dt\right]$.

The proposed definition is more academic than practical in the sense that the financial world, authors of textbooks and students are all accustomed to viewing compound interest basically as being compounded at the end of each year. This is unquestionably a meaningful aspect but I nevertheless hope that the concept of constant continuous growth will be added to the present definition. The wording could then be: "A fund earns annual compound interest when the interest earned during a year is added to the principal at the end of the year so as to earn additional interest and when the fund grows continuously at a constant rate".

This note will not change the mathematics of finance but it should, however, modify some sections of finance textbooks dealing with continuous aspects. It should also avoid forever the embarrassment of answering the question: "How do you explain that the expression $(1+i)^t$ is good for any real value of t?"

A British author once said: "Our most convinced answers are only questions ..." So let me conclude with a question: "Should we consider the continuous compouding of interest at a constant rate only as a special case of the basic theory of coumpound interest, in which case $(1+i)^t$ cannot be said to be the fundamental accumulation function of compound interest, or should we modify the basic definition of compound interest?" The reader knows my answer.

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