THEORY OF STOCHASTIC MORTALITY AND INTEREST RATES

by

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ABSTRACT

Statistical properties of interest, annuity and insurance functions are examined when mortality and interest are treated as having a random component. Several special models of interest rate fluctuation are examined in detail. It is hoped that such theory will be useful in developing fluctuation reserves and premium margins.

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1. INTRODUCTION

The theory of life contingencies has been historically developed from a deterministic point of view. In calculating individual annuity and insurance values, actuaries have ignored chance fluctuations in mortality, morbidity, interest, expenses, etc. They have traditionally, however, implicitly considered chance fluctuations by using conservative assumptions for each of the factors entering the formula. In recent years Hickman (1964), Pollard and Pollard (1969), Hickman and Gayda (1971), Taylor (1972), Boyle (1973) and Panjer (1978) have considered the role played by the time of decrement (death, disability, etc.) as a random variable in the calculation of actuarial functions. As a result, it is known that most actuarial functions are expected values of interest functions when time of decrement is considered as a random variable. Pollard (1971, 1976) and Boyle (1973) have considered interest rate fluctuation by treating the interest rate (or equivalently the force of interest) as a random variable. They each consider very special models for interest fluctuations. Boyle examined the case in which the force of interest in any year is a Normal variable but uncorrelated with the force of interest in any other year (a somewhat unrealistic assumption). Pollard, on the other hand, used a second-order autoregresive stochastic process to model the force of interest.

In this paper we attempt to develop some general theory for both continuous and discrete models. The cases considered by Boyle and Pollard will be seen to be special cases of examples in this paper.

Let $v(t)$ denote the present value of 1 payable in t years $(t \geq 0)$. Let $\delta(t)$ denote the force of interest at time t in the continuous case and let δ_t denote the force of interest in the t-th year in the discrete case.

Suppose that $\{\delta(t);\ t \geq 0\}$ is a continuous parameter stochastic process and that $\{\delta_t;\ t = 1,2,\ldots\}$ is a discrete parameter stochastic process.

Then the transformed variable $\Delta(t) = \int_0^t \delta(s)ds$ is also a continuous parameter stochastic process and the transformed variable $\Delta(t) = \sum_{s=1}^{t} \delta_s$ is a discrete parameter stochastic process. As a result, for appropriate values of t, it can be seen that $\Delta(t) = -log\ v(t)$ and that $v(t) = exp\{-\Delta(t)\}$.

Let $M_z(u)$ denote the moment generating function (mgf) of the random variable z; that is

$$M_z(u) = E[exp(uz)]$$

where E is the expectation (over z) operator. Using this notation, one can express various moments of the random variable $v(t)$, for fixed t, in terms of the moment generating function of the random variable $\Delta(t)$. The expected value of $v(t)$ is then

$$E[v(t)] = E[exp\{-\Delta(t)\}] = M_{\Delta(t)}(-1)$$

which is the moment generating function of $\Delta(t)$ evaluated at $u = -1$. In general, the r-th raw moment is

$$E[v(t)^r] = E[exp\{-r\Delta(t)\}] = M_{\Delta(t)}(-r)$$

and the cross product moments are, for fixed t and s,

$$E[v(t)^r v(s)^q] = E[exp\{-r\Delta(t)-q\Delta(s)\}] = M_{r\Delta(t)+q\Delta(s)}(-1).$$

In particular, for $r = q = 1$,

$$E[v(t)v(s)] = M_{\Delta(t)+\Delta(s)}(-1).$$

Now consider the annuity certain functions $\bar{a}_{\overline{t}|} = \int_0^t v(s)ds$ and $a_{\overline{t}|} = \sum_{s=1}^{t} v(s)$. We first examine the continuous case. The expectation of $\bar{a}_{\overline{t}|}$ is

$$E[\bar{a}_{\overline{t}|}] = E[\int_0^t v(s)ds] = \int_0^t E[v(s)]ds = \int_0^t M_{\Delta(s)}(-1)ds.$$

The corresponding second order raw moment is

$$E[(\bar{a}_{\overline{t|}})^2] = E[\int_0^t\int_0^t v(r)v(s)dr\,ds] = \int_0^t\int_0^t E[v(r)v(s)]dr\,ds$$

$$= \int_0^t\int_0^t M_{\Delta(r)+\Delta(s)}(-1)dr\,ds$$

and the cross product moment is

$$E[\bar{a}_{\overline{t|}}\ \bar{a}_{\overline{s|}}] = E[\int_0^t\int_0^s v(r)v(w)dr\,dw] = \int_0^t\int_0^s M_{\Delta(r)+\Delta(w)}(-1)dr\,dw$$

In the discrete case similar results hold, namely

$$E[a_{\overline{t|}}] = \sum_{s=1}^{t} M_{\Delta(s)}(-1),$$

$$E[(a_{\overline{t|}})^2] = \sum_{s=1}^{t}\sum_{r=1}^{t} M_{\Delta(r)+\Delta(s)}(-1),$$

and $E[a_{\overline{t|}}\ a_{\overline{s|}}] = \sum_{w=1}^{t}\sum_{r=1}^{s} M_{\Delta(r)+\Delta(w)}(-1).$

Similar results can be obtained for annuity-due values and for series of non-level payments. For example, consider an annuity payable at the rate $B(s)$ at time s, $(0 \le s \le t)$. Then the series has present value $\int_0^t B(s)v(s)ds$ with mean present value

$$E[\int_0^t B(s)v(s)ds] = \int_0^t B(s)\ E[v(s)]ds = \int_0^t B(s)\ M_{\Delta(s)}(-1)ds$$

and cross product moment

$$E[\{\int_0^t B(w)v(w)dw\} \cdot \{\int_0^s B(r)v(r)dr\}]$$

$$= \int_0^t\int_0^s B(w)B(r)\ M_{\Delta(r)+\Delta(w)}(-1)dr\,dw.$$

Thus, moments of interest functions used by actuaries can be expressed in terms of moment generating functions. Explicit knowledge of the mgf will allow direct evaluation of such moments. Such moments are taken with respect to interest rate fluctuation. We now consider application of these results to insurance and annuity situations in which the time of decrement is also a random variable.

2. APPLICATION TO LIFE CONTINGENCIES

For the sake of brevity, we consider here only the net single premiums for a whole life insurance and a life annuity.

Consider a life aged x. Let t be a random variable denoting the time of death of the life (aged x at time 0). Then the pdf of t is

$$f(t) = {}_tp_x\ \mu_{x+t}\,, \qquad t \geq 0\ .$$

For a fixed force of interest $\delta = -\log v$

$$E[v^t] = \bar{A}_x^{\delta} ,$$

$$E[\bar{a}_{\overline{t|}}] = \bar{a}_x^{\delta} ,$$

$$V[v^t] = \bar{A}_x^{2\delta} - (\bar{A}_x^{\delta})^2 ,$$

$$V[\bar{a}_{\overline{t|}}] = \frac{1}{\delta^2} V[v^t] .$$

where E is the expectation and V is the variance operator with respect to t and the actuarial functions are computed at the indicated force of interest.

Now consider the fluctuating interest situation. We are interested in the statistical properties of $v(t)$, $a(t) = \int_0^t v(s)ds$ and perhaps $\int_0^t B(s)v(s)ds$. They have expected values $E_1E_2\ [v(t)]$, $E_1E_2\ [a(t)]$ and $E_1E_2\ [\int_0^t B(s)v(s)ds]$ respectively where E_2 is the expectation over $\Delta(t)$ for fixed t and E_1 is the expectation over t. By applying the results of section 1, these values can be rewritten as

$$E_1[M_{\Delta(t)}(-1)] = \int_0^{\infty} M_{\Delta(t)}(-1)\ f(t)dt ,$$

$$\begin{aligned} E_1[\int_0^t M_{\Delta(s)}(-1)ds] &= \int_0^{\infty}\int_0^t M_{\Delta(s)}(-1)\ f(t)ds\ dt \\ &= \int_0^{\infty} M_{\Delta(s)}(-1) \int_s^{\infty} f(t)dt\ ds \\ &= \int_0^{\infty} M_{\Delta(s)}(-1)\ {}_sp_x\ ds , \end{aligned}$$

and similarly

$$E_1\left[\int_0^t B(s)M_{\Delta(s)}(-1)ds\right] = \int_0^\infty B(s)M_{\Delta(s)}(-1)\ {}_sp_x\, ds\ ,$$ respectively.

If we let $\delta_1^*(t) = -\frac{d}{dt} \log M_{\Delta(t)}(-1)$, we have

$$M_{\Delta(t)}(-1) = exp\{-\int_0^t \delta_1^*(s)ds\}.$$

Substituting back into the equation for the expected values, we write the expected values as

$$\int_0^\infty exp\{-\int_0^t \delta_1^*(s)ds\} \cdot {}_tp_x\ \mu_{x+t}dt = \bar{A}_x^{\delta_1^*}\ ,$$

$$\int_0^\infty exp\{-\int_0^t \delta_1^*(s)ds\} \cdot {}_tp_x\ dt = \bar{a}_x^{\delta_1^*}\ ,$$

and $\int_0^\infty B(t)\ exp\{-\int_0^t \delta_1^*(s)ds\} \cdot {}_tp_x\ dt$,

respectively. Thus the expected present values can be expressed as present values with a deterministic force of interest $\delta_1^*(t); t \geq 0$.

Consider now the variance of $v(t)$. It can be similarly decomposed as

$$V[v(t)] = E_1V_2\ [v(t)] + V_1E_2\ [v(t)]$$

$$= E_1[M_{\Delta(t)}(-2) - M_{\Delta(t)}(-1)^2] + V_1[M_{\Delta(t)}(-1)]$$

$$= E_1[M_{\Delta(t)}(-2) - M_{\Delta(t)}(-1)^2] + E_1[M_{\Delta(t)}(-1)^2] - E_1[M_{\Delta(t)}(-1)]^2$$

$$= E_1[M_{\Delta(t)}(-2)] - E_1[M_{\Delta(t)}(-1)]^2$$

$$= \int_0^\infty M_{\Delta(t)}(-2)\ f(t)dt - \{\int_0^\infty M_{\Delta(t)}(-1)\ f(t)dt\}^2$$

If we let $\delta_2^*(t) = -\frac{d}{dt}\log M_{\Delta(t)}(-2)$, we obtain

$$V[v(t)] = \int_0^\infty exp\{-\int_0^t \delta_2^*(s)ds\}\ {}_tp_x\ \mu_{x+t}\ dt - [\int_0^\infty exp\{-\int_0^t \delta_1^*(s)ds\}\ {}_tp_x\ \mu_{x+t}\ dt]^2$$

$$= \bar{A}_x^{\delta_2^*} - \{\bar{A}_x^{\delta_1^*}\}^2$$

which can easily be determined.

For the annuity function $a(t)$ we obtain similar results.

$$V[a(t)] = E_1V_2[a(t)] + V_1E_2[a(t)]$$

$$= E_1[\int_0^t\int_0^t M_{\Delta(r)+\Delta(s)}(-1)dr\ ds] - E_1[\int_0^t M_{\Delta(s)}(-1)ds]^2$$

$$= \int_0^\infty\int_0^t\int_0^t M_{\Delta(r)+\Delta(s)}(-1)\ {}_tp_x\ \mu_{x+t}\ dr\ ds\ dt - \{\int_0^\infty\int_0^t M_{\Delta(s)}(-1)\ {}_tp_x\ \mu_{x+t}\ ds\ dt\}^2$$

$$= \int_0^\infty\int_0^s M_{\Delta(r)+\Delta(s)}(-1)\ {}_sp_x\ dr\ ds - \{\int_0^\infty M_{\Delta(s)}(-1)\ {}_sp_x\ ds\}^2$$

$$+ \int_0^\infty\int_s^\infty M_{\Delta(r)+\Delta(s)}(-1)\ {}_rp_x\ dr\ ds\ .$$

3. THE NORMAL CLASS OF PROCESSES

Suppose that, for fixed t, the stochastic processes $\delta(t)$ and δ_t are Normal processes with mean $E[\delta(t)] = \mu(t)$ and variance - covariance function $C[\delta(s),\delta(t)] = \gamma(t,s)$. Then $\Delta(t) = \int_0^t \delta(s)ds$ is a stochastic process with mean $E[\Delta(t)] = \int_0^t \mu(s)ds$, variance $V[\Delta(t)] = \int_0^t\int_0^t \gamma(r,s)dr\,ds$ and moment generating function (at $u = -1$)

$$M_{\Delta(t)}(-1) = exp\{-\int_0^t \mu(s)ds + \frac{1}{2}\int_0^t\int_0^t \gamma(r,s)dr\,ds\}.$$

Similarly $\Delta(t) = \sum_{s=1}^{t} \delta_s$ is a stochastic process with mean $\sum_{s=1}^{t} \mu(s)$, variance $\sum_{r=1}^{t}\sum_{s=1}^{t} \gamma(r,s)$ and moment generating function (at $u = -1$)

$$exp\{-\sum_{s=1}^{t} \mu(s) + \frac{1}{2}\sum_{r=1}^{t}\sum_{s=1}^{t} \gamma(r,s)\}.$$

The case considered by Boyle (1976) is a special case of Normal Process with

$$\gamma(r,s) = \sigma^2, \qquad r = s;$$
$$= 0, \qquad r \neq s.$$

In this situation the moment generating function of $\Delta(t)$, at $u = -1$, is

$$M_{\Delta(t)}(-1) = exp\{-t(\mu - \sigma^2/2)\}$$

which is the discount factor for t years at a constant force of interest $\mu - \sigma^2/2$. Similarly, at $u = -2$,

$$M_{\Delta(t)}(-2) = exp\{-t(2\mu - 2\sigma^2)\}$$

the discount factor for t years at a constant force of interest $2u - 2\sigma^2$.

We shall consider some examples in which interest rates at different points in time are correlated. It is not unreasonable to assume that the relationship between interest rates at two different points in time depends only on the length of the time interval between the two points in time. We then write the covariance as $\gamma(r,s) = \gamma(|r-s|)$ which is a function of a single variable, the distance $|r-s|$. Boyle's example is the special case where $\gamma(0) = \sigma^2$ and $\gamma(x) = 0$ for $x \neq 0$.

4. SOME EXAMPLES - DISCRETE TIME

First, assume that the covariance between the forces of interest at two different points in time depends only on the length of the interval between those two points in time. Then the covariance can be written as

$$C[\delta_t, \delta_s] = \gamma(|t - s|).$$

Suppose now that $\{\delta_t;\ t = 1,2,\ldots\}$ is a discrete parameter stochastic process such that, for fixed t, δ_t is Normally distributed with mean μ and variance σ^2. Let $\rho(|t - s|) = \gamma(|t - s|)/\sigma^2$ denote the correlation coefficient between δ_t and δ_s. Then $\Delta(t) = \sum_{s=1}^{t} \delta_s$ is Normally distributed with mean μt and variance

$$\sigma^2 \sum_{r=1}^{t} \sum_{s=1}^{t} \rho(|r - s|) = \sigma^2 \rho(0)t + 2\sigma^2 \sum_{r=1}^{t-1} (t - r)\rho(r)$$
$$= \sigma^2\{t + 2 \sum_{r=1}^{t-1} (t - r)\rho(r)\}$$

since $\rho(0) = 1$.

Let $G(x) = \frac{x}{2} + \sum_{r=1}^{x-1} (x - r)\rho(r)$. We then have immediately that

$$E[v(t)] = M_{\Delta(t)}(-1) = exp\{-t\mu + \sigma^2\, G(t)\}$$

and

$$E[v(t)^2] = M_{\Delta(t)}(-2) = exp\{-2t\mu + 4\sigma^2\, G(t)\}.$$

We now obtain the cross product moments

$$E[v(s)v(t)] = M_{\Delta(s)+\Delta(t)}(-1).$$

Since $E[\delta_t] = \mu$, we have $E[\Delta(t) + \Delta(s)] = (t + s)\mu$. We will use $V[\delta_t] = \sigma^2$ and $C[\delta_t, \delta_s] = \sigma^2\rho(|x - y|)$ in computing $V[\Delta(t) + \Delta(s)]$. Suppose $s \leq t$. Then

$$\Delta(t) + \Delta(s) = 2\Delta(s) + \Delta(t) - \Delta(s),$$

so that

$$\begin{aligned} V[\Delta(t) + \Delta(s)] &= 4V[\Delta(s)] + 4C[\Delta(s), \Delta(t) - \Delta(s)] \\ &\quad + V[\Delta(t) - \Delta(s)] \\ &= \sigma^2\{4 \sum_{r=1}^{s} \sum_{w=1}^{s} \rho(|r - w|) + 4 \sum_{r=1}^{s} \sum_{w=s+1}^{t} \rho(|r - w|) \\ &\quad + \sum_{r=s+1}^{t} \sum_{=s+1}^{t} \rho(|r' - w|)\} \\ &= \sigma^2\{4 \sum_{r=1}^{s} \sum_{w=1}^{t} \rho(|r - w|) + \sum_{r=s+1}^{t} \sum_{w=s+1}^{t} \rho(|r - w|)\}. \end{aligned}$$

Now the second double sum can be written as

$$\begin{aligned} \sum_{r=s+1}^{t} \sum_{w=s+1}^{t} \rho(|r - w|) &= (t - s)\cdot\rho(0) + 2 \sum_{r=1}^{t-s-1} (t - s - r)\rho(r) \\ &= t - s + 2 \sum_{r=1}^{t-s-1} (t - s - r)\rho(r). \end{aligned}$$

The first double sum is similarly

$$s + \sum_{r=1}^{s-1} (s - r)\rho(r) + s \sum_{r=1}^{t-s} \rho(r) + \sum_{r=t-s+1}^{t-1} (t - r)\rho(r).$$

As a result, we have

$$V[\Delta(t) + \Delta(s)] = \sigma^2\{4s + 4\sum_{r=1}^{s-1}(s - r)\rho(r) + 4s\sum_{r=1}^{t-s}\rho(r)$$

$$+ 4\sum_{r=t-s+1}^{t-1}(t - r)\rho(r) + (t - s) + 2\sum_{r=1}^{t-s-1}(t - s - r)\rho(r)\}$$

$$= 2\sigma^2\{2G(s) + 2G(t) - G(t - s)\}.$$

Similarly for $s > t$, we have

$$V[\Delta(t) + \Delta(s)] = 2\sigma^2\{2G(s) + 2G(t) - G(s - t)\}$$

Therefore, for $s \neq t$, we have

$$V[\Delta(t) + \Delta(s)] = 2\sigma^2\{2G(s) + 2G(t) - G(|t - s|)\}.$$

Finally, we can evaluate the product moment

$$E[v(t)^2] = exp\{-2t\mu + V[2\Delta(t)]/2\},$$

where $V[2\Delta(t)] = 8\sigma^2 G(t) = 4\sigma^2 t + 8\sigma^2 \sum_{r=1}^{t}(t - r)\rho(r)$ which is the result obtained by evaluating $M_{\Delta(t)}(-2)$. We now apply these results to examples using autoregressive models. The reader should refer to Box and Jenkins (1970) for the details of the following models.

a) ***Autoregressive Process of Order One - AR(1)***

Assume that the force of interest in year t can be modelled as

$$\delta_t = \mu + \phi(\delta_{t-1} - \mu) + \varepsilon_t$$

where $\{\varepsilon_t;\ t = 1,2,\ldots\}$ are independently Normally distributed with mean 0 and variance γ^2. Then, according to Box and Jenkins (1970)

$$E[\delta_t] = \mu\ ,$$

$$V[\delta_t] = \gamma^2/(1 - \phi^2) = \sigma^2\ ,$$

and

$$C[\delta_t, \delta_s] = \sigma^2\ \phi^{|t-s|}\ ;$$

i.e. $\rho(r) = \phi^r$ for $r > 0$.

The model suggests that interest rates in any year depend on 1) the level of interest rates the previous year and 2) some constant level determined from the force of interest . For the process to be stationary, we require that $-1 < \phi < 1$. Note that the model of Boyle (1976) is this one with $\phi = 0$ and $\phi^0 = 1$. We now evaluate $E[v(t)]$, $E[v(t)^2]$ and $E[v(s)v(t)]$ for this model. First,

$$E[v(t)] = M_{\Delta(t)}(-1) = exp\{-t\mu + \sigma^2 G(t)\}$$

where

$$G(t) = \frac{t}{2} + \sum_{r=1}^{t-1} (t - r)\rho(r) = \frac{t}{2} + \sum_{r=1}^{t-1} (t - r)\ \phi^r$$

$$= \frac{t}{2} \cdot \frac{1 + \phi}{1 - \phi} - \phi \cdot \frac{1 - \phi^t}{(1 - \phi)^2} .$$

Similarly,

$$E[v(t)^2] = exp\{-2t\mu + 4\sigma^2 G(t)\}$$

and for s ≠ t

$$E[v(s)v(t)] = exp\{-(s + t)\mu + \sigma^2[2G(s) + 2G(t) - G(|t - s|)]\}.$$

b) *Autoregressive Process of Order Two - AR(2)*

Assume that the force of interest can be modelled as

$$\delta_t = \mu + \phi_1(\delta_{t-1} - \mu) + \phi_2(\delta_{t-2} - \mu) + \varepsilon_t$$

where $\{\varepsilon_t;\ t = 1, 2, \ldots\}$ are independently Normally distributed with mean 0 and variance γ^2. Then, according to Box and Jenkins (1970)

$$E[\delta_t] = \mu,$$

$$V[\delta_t] = \frac{1 - \phi_2}{1 + \phi_2} \cdot \frac{\gamma^2}{(1 - \phi_2)^2 - \phi_1^2} = \sigma^2,$$

and

$$C[\delta_t, \delta_s] = \sigma^2\{\lambda\,\psi_1^{|t-s|} + (1 - \lambda)\,\psi_2^{|t-s|}\}$$

i.e. $\rho(r) = \lambda\,\psi_1^r + (1 - \lambda)\,\psi_2^r$.

where

$$\lambda = \psi_1(1 - \psi_2^2)/\ \{(\psi_1 - \psi_2)(1 + \psi_1\psi_2)\}$$

and ψ_1 and ψ_2 are the reciprocals of the roots of the characteristic equation

$$\phi(r) = 1 - \phi_1 r - \phi_2 r^2 = 0 .$$

The model suggests that interest rates depend on 1) the level of interest rates in the two previous years and 2) some constant level. For the process to be stationary, we require that 1) $\phi_1 + \phi_2 < 1$, 2) $\phi_2 - \phi_1 < 1$ and 3) $-1 < \phi_2 < 1$. Note that the model of Pollard (1971) is this one with $\phi_1 = 2k$ and $\phi_2 = -k$ and $0 < k < 1$.

The expected values $E[v(t)]$, $E[v(t)^2]$ and $E[v(t)v(s)]$ can be computed in the same manner as in the $AR(1)$ case using the following value of $G(x)$:

$$G(x) = \frac{x}{2} + \sum_{r=1}^{x-1} (x - r)\rho(r)$$

$$= \frac{x}{2} + \sum_{r=1}^{x-1} (x - r)\ (\lambda_1\psi_1^r + \lambda_2\psi_2^r) = \lambda\ G_1(x) + (1 - \lambda)G_2(x)$$

where $G_i(x) = \frac{x}{2} \cdot \frac{1 + \psi_i}{1 - \psi_i} - \psi_i \frac{1 - \psi_i^x}{(1 - \psi_i)^2}$ $, i = 1,2.$

The roots of the characteristic function are either both real or both complex. When the roots are complex, $\rho(r)$ is a real-valued function. Thus, in either case, $G(x)$ is real-valued and so, in turn, are the expected values. When the roots are real, the autocorrelation function $\rho(r)$ consists of a mixture of damped exponentials. When the roots are complex, $\rho(r)$ is a damped sine wave. These results are given in Box and Jenkins (1970). Note that the example of Pollard (1971) admits only complex roots since $\phi_1 = 2k$, $\phi_2 = -k$ and $0 < k < 1$.

5. SOME EXAMPLES - CONTINUOUS TIME

Again, as in the previous section, assume that the covariance between the forces of interest at two different points in time depends only on the length of the interval between those two points in time. Then the covariance can be written as

$$C[\delta(t), \delta(s)] = \gamma(|t - s|).$$

Suppose now that $\{\delta(t);\ t \geq 0\}$ is a continuous parameter stochastic process such that, for fixed t, $\delta(t)$ is Normally distributed with mean μ and variance σ^2. Again, let $\rho(|t - s|) = \gamma(|t - s|)/\sigma^2$ denote the correlation coefficient between $\delta(t)$ and $\delta(s)$. Then $\Delta(t) = \int_0^t \delta(s)ds$ is Normally distributed with mean μt and variance

$$\sigma^2 \int_0^t \int_0^t \rho(|r - s|)dr\, ds = 2\sigma^2 \int_0^t \int_s^t \rho(r - s)dr\, ds$$

$$= 2\sigma^2 \int_0^t \int_0^{t-s} \rho(z)\,dz\,ds$$

$$= 2\sigma^2 \int_0^t \int_0^{t-z} \rho(z)\,ds\,dz$$

$$= 2\sigma^2 \int_0^t (t - z)\rho(z)dz \ .$$

Letting $G(t) = \int_0^t (t - r)\rho(r)dr$, we can write the variance of $\Delta(t)$ as $2\sigma^2\, G(t)$. As a result we have

$$E[v(t)] = M_{\Delta(t)}(-1) = exp\{-t\mu + \sigma^2\, G(t)\}$$

and

$$E[v(t)^2] = M_{\Delta(t)}(-2) = exp\{-2t\mu + 4\sigma^2\, G(t)\}.$$

Now examining $\Delta(t) + \Delta(s)$, we find that

$$V[\Delta(t) + \Delta(s)] = 2\sigma^2\ \{2G(s) + 2G(t) - G(|t - s|)\}$$

as in the discrete case. $E[v(s)v(t)]$ is then evaluated as in discrete case.

We shall examine the continuous analogue of the "Box-Jenkins" autoregressive models discussed in the discrete case. The reader is referred to Koopmans (1974). Let $y(t) = \delta(t) - \mu$ and let D^k denote the k-th order derivative with respect to t.

Then the n-th order continuous time process can be described as

$$y(t) = a_1 D[y(t)] + a_2 D^2[y(t)] + \ldots + a_n D^n[y(t)] + \varepsilon(t)$$

where $\{\varepsilon(t);\ t \geq 0\}$ are, for fixed t, independently Normally distributed with mean 0 and variance γ^2. The mean of the process $y(t)$ is 0. Inverting the process we obtain

$$y(t) = \int_{-\infty}^{\infty} h(r)\ \varepsilon(t - r)dr$$

where $h(r)$ is a function to be defined by the order of the process.

The covariance function is given by

$$E[y(t)y(s)] = E \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(r)h(w)\ \varepsilon(t - r)\ \varepsilon(s - w)dr\,dw\ .$$

Since

$$E[\varepsilon(t - r)\varepsilon(s - w)] = \gamma^2 \quad \text{when } t-r = s-w,$$
$$= 0 \quad \text{otherwise},$$

we write the covariance as

$$E[y(t)y(s)] = \gamma^2 \int_{-\infty}^{\infty} h(t - s + r)\ h(r)dr\ .$$

The characteristic equation of this process is

$$1 - a_1 r - a_2 r^2 - \ldots - a_n r^n = 0$$

Its roots $r_1, r_2, \ldots, r_n$ may be real or complex, with complex roots in pairs. We shall assume that the process depends only on the present and past of the $\delta(t)$ process, as we did implicitly in the discrete case. This assumption requires that the real part of the roots $r_1, r_2, \ldots, r_n$ be negative [see Koopmans (1974, pp. 104-111)]. Then, for special cases, we need examine only the case of roots with negative real parts.

a) *Autoregressive Process of Order One - AR(1)*

Assume that the force of interest at time t can be modelled as

$$\delta(t) = \mu + a_1 \, D\delta(t) + \varepsilon(t)$$

or equivalently

$$y(t) = a_1 \, Dy(t) + \varepsilon(t) \, .$$

The characteristic equation is

$$1 - a_1 r = 0$$

with root $\alpha = 1/a_1$. Then by the arguments given in Koopmans [1974, p.110], we can show, for $\alpha < 0$ (as assumed)

$$h(r) = exp\{\alpha r\} \qquad \text{for } r > 0$$
$$= 0 \qquad \text{for } r \leq 0$$

To derive the covariance function, $\gamma^2 \int_{-\infty}^{\infty} h(t - s + r)h(r)dr$, we distinguish two cases:

Case 1: $\alpha < 0$, $t - s > 0$

In this case the product of the h-functions is non zero when $r > 0$ and $t - s + r > 0$. Since $t - s > 0$ we need only $r > 0$. Hence

$$\int_{-\infty}^{\infty} h(t - s - r)h(r)dr = \int_{0}^{\infty} exp\{\alpha(t - s + 2r)\}dr$$
$$= -\frac{1}{2\alpha}\, exp\{\alpha(t - s)\}.$$

Case 2: $\alpha < 0$, $t - s \leq 0$

In this case the product is non-zero when $r > 0$ and $t - s + r > 0$. Thus, we need $r > s - t$. Hence

$$\int_{\infty}^{\infty} h(t - s + r)h(r)dr = \int_{s-t}^{\infty} exp\{\alpha(t - s + 2r)\}dr$$
$$= -\frac{1}{2\alpha}\, exp\{\alpha(s - t)\}.$$

Combining the two cases, we find that the covariance between $\delta(t)$ and $\delta(s)$ can be written as

$$C[\delta(t), \delta(s)] = C[y(t), y(s)] = -\frac{\gamma^2}{2\alpha} \exp\{\alpha \cdot |t - s|\}.$$

Note that the covariance is a function of only the distance $|t - s|$.

By substituting

$$\sigma^2 = -\frac{\gamma^2}{2\alpha} \quad \text{and} \quad \phi = \exp\{\alpha\}$$

we obtain

$$C[\delta(t), \delta(s)] = \sigma^2 \cdot \phi^{|t-s|},$$

which is the result in the corresponding discrete case. We now obtain the function $G(t)$:

$$G(t) = \int_0^t (t - r)\rho(r)dr$$

$$= \int_0^t (t - r) \quad \phi^r \, dr$$

$$= -\frac{t}{\ln \phi} - \frac{1 - \phi^t}{(\ln \phi)^2}$$

$$= -\frac{t}{\alpha} - \frac{1 - \exp\{\alpha t\}}{\alpha^2}$$

The moments $E[v(t)]$, $E[v(t)^2]$ and $E[v(s)v(t)]$ can now be evaluated using this function.

b) *Autoregressive Process of Order Two - AR(2)*

The process is given by

$$y(t) = a_1 D\, y(t) + a_2 D^2\, y(t) + \varepsilon(t)$$

when $y(t) = \delta(t) - \mu$. It has characteristic equation

$$a_2 r^2 + a_1 r - 1 = 0$$

with roots

$$r_1 = \frac{-a_1 + \sqrt{a_1^2 + 4a_2}}{2a_2} = \alpha_1 + i\beta_1$$

and

$$r_2 = \frac{-a_1 - \sqrt{a_1^2 + 4a_2}}{2a_2} = \alpha_2 + i\beta_2$$

where $i = \sqrt{-1}$

According to Koopmans (1974, p.110), in the *AR(2)* case we can write $h(x)$ as

$$h(x) = \frac{h_1(x) - h_2(x)}{r_1 - r_2}$$

where

$$h_j(x) = exp\{r_j x\}, \qquad x > 0$$
$$= 0 \qquad , \qquad x \le 0$$

for $r_j < 0$, $j = 1,2$.

Although $h_1(x)$ and $h_2(x)$ may be complex-valued, $h(x)$ is always real. As in the $AR(1)$ case we consider two cases:

<u>Case 1:</u> $r_1 < 0$, $r_2 < 0$, $t - s \ge 0$

$$\int_{-\infty}^{\infty} h(t - s + r)h(r)dr$$

$$= \frac{1}{(r_1 - r_2)^2}\int_0^{\infty} [exp\{r_1(t - s + r) + r_1 r\} - exp\{r_2(t - s + r) + r_1 r\}$$

$$- exp\{r_1(t - s + r) + r_2 r\} + exp\{r_2(t - s + r) + r_2 r\}]dr$$

$$= \frac{1}{2(r_1^2 - r_2^2)}\left[\frac{exp\{r_1(t - s)\}}{r_1} - \frac{exp\{r_2(t - s)\}}{r_2}\right]$$

after some simplification.

<u>*Case 2:*</u> $r_1 < 0,\ r_2 < 0,\ t - s < 0$

$$\int_{-\infty}^{\infty} h(t - s + r)h(r)dr$$

$$= \int_{s-t}^{\infty} [exp\{r_1(t - s + r) + r_1 r\} - exp\{r_2(t - s + r) + r_1 r\}$$

$$- exp\{r_1(t - s + r) + r_2 r\} + exp\{r_2(t - s + r) + r_2 r\}]dr$$

$$= \frac{1}{2(r_1^2 - r_2^2)}\left[\frac{exp\{r_1(s - t)\}}{r_1} - \frac{exp\{r_2(s - t)\}}{r_2}\right]$$

Combining the two cases, we find that

$$C[\delta(s), \delta(t)] = E[y(s)y(t)]$$

$$= \frac{\gamma^2}{2(r_1^2 - r_2^2)}\left[\frac{exp\{r_1|t - s|\}}{r_1} - \frac{exp\{r_2|t - s|\}}{r_2}\right]$$

By letting $s = t$, we see that

$$V[\delta(t)] = -\frac{\gamma^2}{2r_1r_2(r_1 + r_2)} = \sigma^2.$$

The correlation between $\delta(s)$ and $\delta(t)$ is then expressed as

$$\rho(|t - s|) = \frac{r_2}{r_2 - r_1} exp\{r_1 \cdot |t - s|\} - \frac{r_1}{r_2 - r_1} exp\{r_2 \cdot |t - s|\}$$

If r_1 and r_2 are real-valued, the correlation is a linear combination of decreasing exponentials. If the roots are complex-valued, the correlation is an exponentially damped sine wave.

The function $G(x) = \int_0^x (x - r)\rho(r)dr$ can now easily be computed. It is

$$\frac{r_2}{r_2 - r_1}\left[\frac{x}{r_1} + \frac{exp\{r_1t\} - 1}{r_1^2}\right] - \frac{r_1}{r_2 - r_1}\left[\frac{x}{r_2} + \frac{exp\{r_2t\} - 1}{r_2^2}\right]$$

and can then be used to compute $E[v(t)]$, $E[v(t)^2]$ and $E[v(s)v(t)]$.

6. CONCLUSIONS

To apply the results of this paper, the following procedure is followed:

1. An interest rate model is selected.
2. The function $G(x)$ is determined for the model selected.
3. The values of $E[v(t)]$, $E[v(t)^2]$ and $E[v(s)v(t)]$ are computed using the function $G(x)$, where the expectation is over interest only.
4. These values can then be applied in life or other contingency settings as in section 2.

The methods in this paper generalize the results of Boyle (1976) and Pollard (1971) by using more realistic models. The use of the moment generating function greatly simplifies the analysis. Since the Normal distribution is characterized by the mean and covariance structure, the exact values of the expected values in 3. above can be analytically expressed in simple form. The annuity values can be calculated by integration (or summation) over the insurance values.

We hope that the results will prove to be useful in a variety of applications.

7. REFERENCES

1. Box, G.E.P., and Jenkins, G.M. (1970), *Time Series Analysis*, San Francisco: Holden-Day.

2. Boyle, P.P. (1973), "Note on the Variance of a Widow's Pension," *Journal of the Institute of Actuaries*, 99, 103-107.

3. Boyle, P.P. (1976), "Rates of Return as Random Variables," *Journal of Risk and Insurance*, XLIII, 693-713.

4. Hickman, J.C. (1964), "A Statistical Approach to Premiums and Reserves in Multiple Decrement Theory," *Transactions of the Society of Actuaries*, XVI, 1-16.

5. Hickman, J.C., and Gayda, J.J. (1971), "Discussion of the paper 'Actuarial Functions as Expected Values'," *Transactions of the Society of Actuaries*, XXIII, 143-149.

6. Koopmans, L.H. (1974), *The Spectral Analysis of Time Series*, New York: Academic Press.

7. Panjer, H.H. (1978), "Stochastic Foundation of Life Contingencies," presented at Canadian Institute of Actuaries, London, Canada.

8. Pollard, A.H., and Pollard, J.H. (1969), "A Stochastic Approach to Actuarial Functions," *Journal of the Institute of Actuaries*, 95, 79-113.

9. Pollard, J.H. (1971), "On Fluctuating Interest Rates," *Bulletin de l'Association Royal des Actuaries Belges*, 66, 68-97.

10. Pollard, J.H. (1976), "Premium Loadings for Non-Participating Business," *Journal of the Institute of Actuaries*, 103, 205-212.

11. Taylor, G.C. (1972), "On Calculation of Moments of the Value of a Life Assurance Liability," *Journal of the Institute of Actuaries*, 98, 157-164.