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SOCIETY OF ACTUARIES
Part 3 Examination Committee

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To Readers of "The Actuary":

In connection with our work on the Part 3 Committee we would like to solicit opinions on the following problem which seems to crop up each time an examination is to be set.

The symbols

$$(ax + b)_h^{(n)} \quad \text{and its generalization} \quad (f(x))_h^{(n)}$$

are not explicitly defined in the references for Numerical Analysis. Two definitions have been proposed:

Proposed Definition I:

$$(ax + b)_h^{(n)} = (ax + b)\ (ax + b - h)\ .\ .\ .\ (ax + b - (n - 1)h)$$

or

$$(f(x))_h^{(n)} = (f(x))\ (f(x) - h)\ .\ .\ .\ (f(x) - (n - 1)h)$$

Proposed Definition II:

$$(ax + b)_h^{(n)} = (ax + b)\ (a(x - h) + b)\ .\ .\ .\ (a(x - (n - 1)h) + b)$$

or

$$(f(x))_h^{(n)} = f(x)\ f(x - h)\ .\ .\ .\ f(x - (n - 1)h)$$

Proposed Definition I is implicitly used in some of the exercises in Kellison, Fundamentals of Numerical Analysis (cf., Exercise 17, page 57, and Exercise 51, page 59). However, the definition is never explicitly stated, and no conclusive evidence is given on its behalf.

Proposed Definition II does appear to have a clear cut advantage: it permits the statement of a finite difference "chain rule" analogous to the chain rule of differential calculus. To see this, recall that the operators Δ, ∇,

δ and μ are always assumed to apply to the variable "x" unless explicitly noted. Thus,

$$\underset{h}{\Delta} f(x) = f(x + h) - f(x)$$

If Proposed Definition I is used, the expression closest in form to a "finite difference chain rule" is

$$\underset{h}{\Delta} (ax + b)_{ha}^{(n)} = han(ax + b)_{ha}^{(n-1)}$$

However, this formula applies only to factorial polynomials "built" on a linear function such as ax + b, since the coefficient "a" of the variable "x" must appear in the final subscript. In other words, if f(x) is a function other than a linear function, there isn't a consistant result for the expression

$$\underset{h}{\triangle} (f(x))_h^{(n)}$$

There is no analogy to the chain rule in this case.

Utilizing Proposed Definition II, we have

$$\underset{h}{\triangle} (ax + b)_h^{(n)} = han(ax + b)_h^{(n-1)}$$

which generalizes to

$$\underset{h}{\triangle} (f(x))_h^{(n)} = \left[\frac{1}{n} \underset{nh}{\triangle} f(x - [n - 1] \cdot h)\right] \cdot \left[n(f(x))_h^{(n-1)}\right]$$

This result is an analogy to the chain rule.

From another point of view, the result obtained above (using Definition II) simply reflects the fact that whenever a finite difference operator is applied to a function, it operates on the argument of that function. Proposed Definition II may be looked at as defining a new operator which, similarly, acts on arguments.

On this basis it appears logical to make the following definitions for Part 3 purposes:

<u>DEFINITION</u>

$$(f(x))_h^{(n)} = f(x)\ f(x - h)\ .\ .\ .\ f(x - (n - 1)h)$$

and

$$(f(x))_h^{(-m)} = \frac{1}{(f(x + mh))_h^{(m)}}$$

provided none of the factors in the denominator is equal to zero.

The Part 3 Committee intends to recommend a study note on this subject. Therefore, we would appreciate receiving comments at this time from other actuaries and mathematicians. Comments should be sent to:
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