ACTUARIAL FUNCTIONS AS RANDOM VARIABLES

by

Aaron Tenenbein
New York University

Talk given at the thirteenth annual Actuarial Research Conference held at Ball State University, Muncie, Indiana, September 1, 1978.

1. Introduction

The classical approach to life contingencies has treated actuarial functions as fixed quantities. As examples, consider the following actuarial functions (Jordan (1967)):

$$a_{x:\overline{n}|} = \sum_{t=1}^{n} v^t \, {}_tp_x$$

$$e_x = \sum_{t=1}^{\infty} {}_tp_x$$

$$A_x = \sum_{t=0}^{\infty} v^{t+1} \, {}_{t|}q_x$$

In point of fact, these functions really represent the mean or expected values of random variables which are functions of the length of time to death of a life aged x. By considering only the mean of these random variables, information is lost on other aspects of the probability distribution of actual benefits received on various life insurance contracts, such as the variability of actual benefits paid out.

The work of Menge (1937), Pollard and Pollard (1969), Pollard (1976) and Taylor (1976) discuss various aspects of this problem. Fibiger and Kellison (1971) review some of this literature and Boyle (1974) and Hickman (1964) discuss applications arising from this approach.

In this paper, we will present some examples of this approach. In Section 2, we consider the length of time to death as a discrete random variable and compute the mean and variance. In Section 3, we consider the case of a whole life policy and life annuity with annual payments. In Section 4, we consider the length of time to death as a continuous random variable and compute its mean and variance. In Section 5, we consider the Endowment Insurance Policy and the term annuity with continuous payments.

2. The Number of Years to Death as a Discrete Random Variable

Let T be defined as the number of full years to death of a randomly selected person aged x. Let f(t) be the probability function. It follows that:

$$f(t) = \Pr[T=t] = {}_{t|}q_x = d_{x+t}/\ell_x \quad \text{for } t = 0,1,2,\ldots \tag{2.1}$$

It is easy to show that $\sum_{t=0}^{\infty} f(t) = 1$ and $f(t) \geq 0$ for all t. By definition:

$$E(T) = \sum_{t=0}^{\infty} t\, f(t)$$

Thus:

$$E(T) = \sum_{t=0}^{\infty} t \cdot {}_{t|}q_x \tag{2.2}$$

$$= \sum_{t=0}^{\infty} t\, d_{x+t}/\ell_x$$

or

$$E(T) = \sum_{t=1}^{\infty} \ell_{x+t}/\ell_x = e_x$$

Consequently, the mean of this random variable is e_x, as is to be expected. Now:

$$\sigma_T^2 = V(T) = E(T^2) - [E(T)]^2$$

or

$$\sigma_T^2 = V(T) = \sum_{t=0}^{\infty} t^2 \cdot {}_{t|}q_x - e_x^2 \tag{2.3}$$

To get an idea of how variable the distribution is, let us assume DeMoirre's Law (Jordan (1967)) with $\omega = 100$. Thus:

$$q_x = {}_{t|}q_x = \mu_x = 1/(100 - x).$$

Hence, from (2.2) and (2.3):

$$E(T) = e_x = (99 - x)/2$$

$$\sigma_T^2 = (99 - x)(101 - x)/12.$$

Thus the coefficient of variation σ_T/μ_T is approximately $1/\sqrt{3} = .577$, except for ages near the terminal age of 100. This implies that the standard deviation is relatively high, approximately 58% of the mean. As an example for life aged 50, the value of e_x is 24.5 years but the standard deviation about the mean is 14.43 years.

3. Whole Life Insurance and Immediate Life Annuity with Annual Installments

Suppose we consider an whole life policy on a life aged x where benefits are paid at the end of the year of death. The benefit paid on this policy is a random variable, denoted as L, which depends upon the random variable T of the previous section.

If a life aged x has exactly T full years to death, he will die between ages X + T and X + T + 1. Hence, the present value of the benefits received is:

$$L = v^{T+1} \tag{3.1}$$

The mean of this random is, from (2.1):

$$E(L) = E[v^{T+1}] = \sum_{t=0}^{\infty} v^{t+1} \; {}_{t|}q_x = A_x \tag{3.2}$$

which is as expected. The variance of x is $E(L^2) - [E(L)]^2 = E[v^{2T+2}] - A_x^2$.

Thus, $\sigma^2 = \Sigma(v^2)^{t+1} \; {}_{t|}q_x - A_x^2$ or

$$\sigma_L^2 = A'_x - (A_x)^2 \tag{3.3}$$

where A'_x is calculated at interest rate $i' = (1+i)^2 - 1$.

Suppose we consider an immediate life annuity with annual payments. Let the random variable Y be the present value of all payments actually made for a life annuity on a randomly selected life aged x. Hence:

$$Y = a_{\overline{T}|} = \frac{1-v^T}{i} \tag{3.4}$$

From (3.1), it is evident that:

$$Y = \frac{1-(1+i)L}{i} \tag{3.5}$$

Thus, from (3.2):

$$E(Y) = \frac{1}{i} - \frac{1+i}{i} E(L)$$

$$= \frac{1}{i} - \frac{1+i}{i} A_x$$

$$= \frac{1}{i} - \frac{1+i}{i} (1-d\ddot{a}_x)$$

$$= \frac{1}{i} [i\ \ddot{a}_x - i]$$

or $E(Y) = a_x$.

From (3.3) and (3.5):

$$\sigma_Y^2 = \sigma_x^2/d^2 = \frac{A_x' - (A_x)^2}{d^2} \tag{3.6}$$

If we assume DeMoirre's Law with $i = .03$, $i' = .0609$, we have that:

$$A_x = a_{\overline{100-x}|} / (100-x) \tag{3.7}$$

$$a_x = [1 - (1+i)A_x]/i \tag{3.8}$$

From (3.3):

$$\sigma_L^2 = a^{i'}_{\overline{100-x}|} / (100-x) - A_x^2 \tag{3.9}$$

and

$$\sigma_Y^2 = \sigma_L^2/d^2 \tag{3.10}$$

Using equations (3.7), (3.8), (3.9) and (3.10), the values of A_x, a_x and the corresponding standard deviations were computed and tabulated in Table 1 for various ages x. As can be seen, the coefficient of variation increases for the life annuity random variable and decreases for the life insurance random variable. We do not imply that this happens under all mortality experience.

4. The Number of Years to Death as a Continuous Random Variable

Suppose we define T as the number of years to death of a randomly life aged x. It follows that:

$$F(t) = \Pr[T \le t] = 1 - {}_tp_x \tag{4.1}$$

Thus, the probability density function of T is:

$$f(t) = F'(t) = {}_tp_x\,\mu_{x+t} \qquad \text{for } t \ge 0 \tag{4.2}$$

If we compute the mean, we obtain:

$$E(T) = \int_0^\infty t \cdot {}_tp_x\,\mu_{x+t}\,dt \tag{4.3}$$

Using integration by parts, we can show that:

$$E(T) = \int_0^\infty {}_tp_x\,dt = e_x \tag{4.4}$$

$$\sigma_T^2 = E(T^2) - e_x^2$$

$$= \int_0^\infty t^2 \cdot {}_tp_x\,\mu_{x+t}\,dt - e_x^2$$

Thus,

$$\sigma_T^2 = \frac{2}{\ell_x}\int_0^\infty t\,\ell_{x+t}\,dt - e_x^2 \tag{4.5}$$

or

$$\sigma_T^2 = \frac{2Y_x}{\ell_x} - e_x^2 \tag{4.6}$$

where Y_x is defined by Jordan (1967).

If we assume DeMoirre's Law with:

$$\ell_{x+t} = 100 - x - t \qquad \text{for } 0 \le t \le 100\text{-}x$$

$$_tp_x = 1 - \frac{1}{100\text{-}x} \qquad \text{for } 0 \le t \quad 100\text{-}x$$

in equations (4.4) and (4.5), we obtain:

$$\mu_T = E(T) = e_x = (100\text{-}x)/2$$

$$\sigma_T^2 = (100\text{-}x)^2/12$$

Hence, the coefficient of variation σ_T/μ_T is exactly equal to $1/\sqrt{3}$.

5. Endowment Policy and Term Annuity with Continuous Installments

Suppose we consider an n year endowment policy where benefits are paid at the moment of death. Let L be the random variable which denotes the actual payment made on this policy issued on a randomly selected life aged x. This implies that:

$$\begin{aligned} L &= v^T & T &\le n \\ &= v^n & T &> n \end{aligned} \tag{5.1}$$

where T is defined in Section 4. By (4.2):

$$E(L) = \int_0^n v^t\, {}_tp_x\, \mu_{x+t}\, dt + \int_n^\infty v^n\, {}_tp_x\, \mu_{x+t}\, dt$$

Thus:

$$E(L) = \bar{A}^{\,1}_{x:\overline{n|}} + v^n {}_np_x = \bar{A}_{x:n} \tag{5.2}$$

and

$$\sigma_L^2 = E(L^2) - [E(L)]^2$$

$$= \int_0^n v^{2t} {}_tp_x \mu_{x+t} dt + \int_n^\infty v^{2n} {}_tp_x \mu_{x+t} dt - \bar{A}^2_{x:\overline{n|}}$$

or

$$\sigma_L^2 = {}^{i'}\bar{A}_{x:\overline{n|}} - (\bar{A}_{x:\overline{n|}})^2 \tag{5.3}$$

If we define Y to be the actual payments made on an n year term annuity with continuous installments on a randomly selected life aged x, then:

$$\begin{aligned} Y &= \bar{a}_{\overline{T|}} && \text{for } T \le n \\ &= \bar{a}_{\overline{n|}} && \text{for } T > n \end{aligned} \tag{5.4}$$

For (5.1), we have that:

$$Y = (1-L)/\delta \tag{5.5}$$

Since Y is a linear function of L, we have that:

$$E(Y) = (1-E(L))/\delta$$

$$= (1-\bar{A}_{x:\overline{n|}})/\delta$$

or

$$E(Y) = \bar{a}_{x:\overline{n|}} \tag{5.6}$$

$$\therefore \quad \sigma_Y^2 = \sigma_L^2/\delta^2 \tag{5.7}$$

7. Summary

In this paper, we have attempted to describe several basic actuarial functions as random variables. We have shown that their expected values are exactly the same expression as are given in standard actuarial texts [see Jordan (1967)]. By considering the corresponding random variables, more information can be obtained concerning such aspects as the variability of benefits which are actually payable. This information is extremely useful for some aspects of risk theory. The new actuarial textbook being jointly written by Bowers, Gerber, Hickman and Jones and Nesbitt (1978) is adopting this random variable approach.

The material presented here is not intended to represent all of the actuarial functions possible. For example, we have not considered the idea of reserves as the expected value of a loss function. We have attempted to present the flavor of this approach. For further development, the reader is referred to the bibliography.

BIBLIOGRAPHY

1. Bowers, N., Gerber, H., Hickman, J., Jones, D., and Nesbitt, C., "New Societies of Actuaries Textbook on Life Contingencies," Panel Presentation, Thirteenth Annual Actuarial Research Conference, Ball State University, Munsie, Indiana, September 1, 1978.

2. Boyle, P.P., "Note on the Variance of a Widow's Pension," JIA, Vol. CCI (1974), 103-8.

3. Fibiger, J.A., and Kellison, S.G., "Actuarial Functions as Expected Values," TSA, Vol. XXIII (1971), 1-5 (Discussion on pp. 141-50).

4. Hickman, J.C., "A Statistical Approach to Premiums and Reserves in Multiple Decrement Theory," TSA, Vol. XVI, Part I (1964), 1-16.

5. Jordan, C.W., Life Contingencies, Chicago: Society of Actuaries (1967).

6. Menge, W.O., "A Statistical Treatment of Actuarial Functions," RAIA, Vol. XXVI (1937), 65-88.

7. Pollard, A.H. and Pollard, J.H., "A Stochastic Approach to Actuarial Functions," JIA, Vol. XCV (1969), Part I, No. 400, 79-114.

8. Pollard, J.H., "Premium Loadings for Non-Participating Business," JIA, Vol. CCIII (1976), 205-212.

9. Taylor, G.C., "On Calculation of Moments of the Value of a Life Assurance Liability," JIA, Vol. XCVIII (1972), 157-64.

TABLE 1

Values of A_x, a_x and the Standard Deviations

σ_L and σ_Y under DeMoivre's Law

Age (x)	A_x	σ_y	Coefficient of Variation	a_x	σ_L	Coefficient of Variation
20	.3775	.2468	.654	20.37	8.475	.416
30	.4160	.2403	.578	19.05	8.250	.433
40	.4613	.2303	.499	17.50	7.906	.452
50	.5146	.2159	.419	15.67	7.405	.473
60	.5779	.1949	.337	13.49	6.693	.496
70	.6533	.1661	.246	10.90	5.702	.523
80	.7439	.1264	.170	7.79	4.340	.557
90	.8530	.0724	.085	4.05	2.485	.614
99	.9709	0	--	0	0	--