RECENT MORTALITY EXPERIENCE DESCRIBED BY GOMPERTZ'S AND MAKEHAM'S LAWS - INCLUDING A GENERALIZATION

William H. Wetterstrand
Ball State University

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1. Objectives

I. Determine the extent to which Makeham's or Gompertz's Laws describe recent insurance and population mortality; investigate that Gompertz's Law holds quite closely between ages 30-90 [4].

II. Analyze how the constants in the two laws vary with respect to the factors:

A. Sex
B. Time
C. Geographical Area

III. Propose a simple extension to Makeham's Law to account for recent departures from the laws, related to difference in sex.

2. Background

Force of Mortality is defined to be, $\mu_x = \frac{-d \log l_x}{dx} = \frac{-\log s(x)}{dx}$, where l_x is the number of lives in a closed group, attaining age x, given an arbitrary number of births, l_0 , where $s(x) = l_x / l_0$ is the probability that a new birth survives to age x. We will use a first order approximation to the force of mortality $\mu_{x+\frac{1}{2}} = -\log p_x$ where $p_x = l_{x+1} / l_x$ is the probability that a life aged x survives one year to age x + 1. $\mu_{x+\frac{1}{2}} \doteq q_x$ for small $q_x = 1 - p_x$, the probability that a life aged x dies before attaining age x + 1. We will deal with $1000\mu_x$ and $1000q_x$. Makeham's law is defined to be: $\mu_x = A + Bc^x$. Gompertz's law, which preceded Makeham's, has A = 0 , i.e. $\mu_x = Bc^x$.

In words, this means that the propensity to die grows geometrically with age, like compound interest at a rate of 7½ - 10% per year. The growth rate, A, was introduced by Makeham to account for deaths due to random, chance causes, such as accidents.

The critical relation we use is

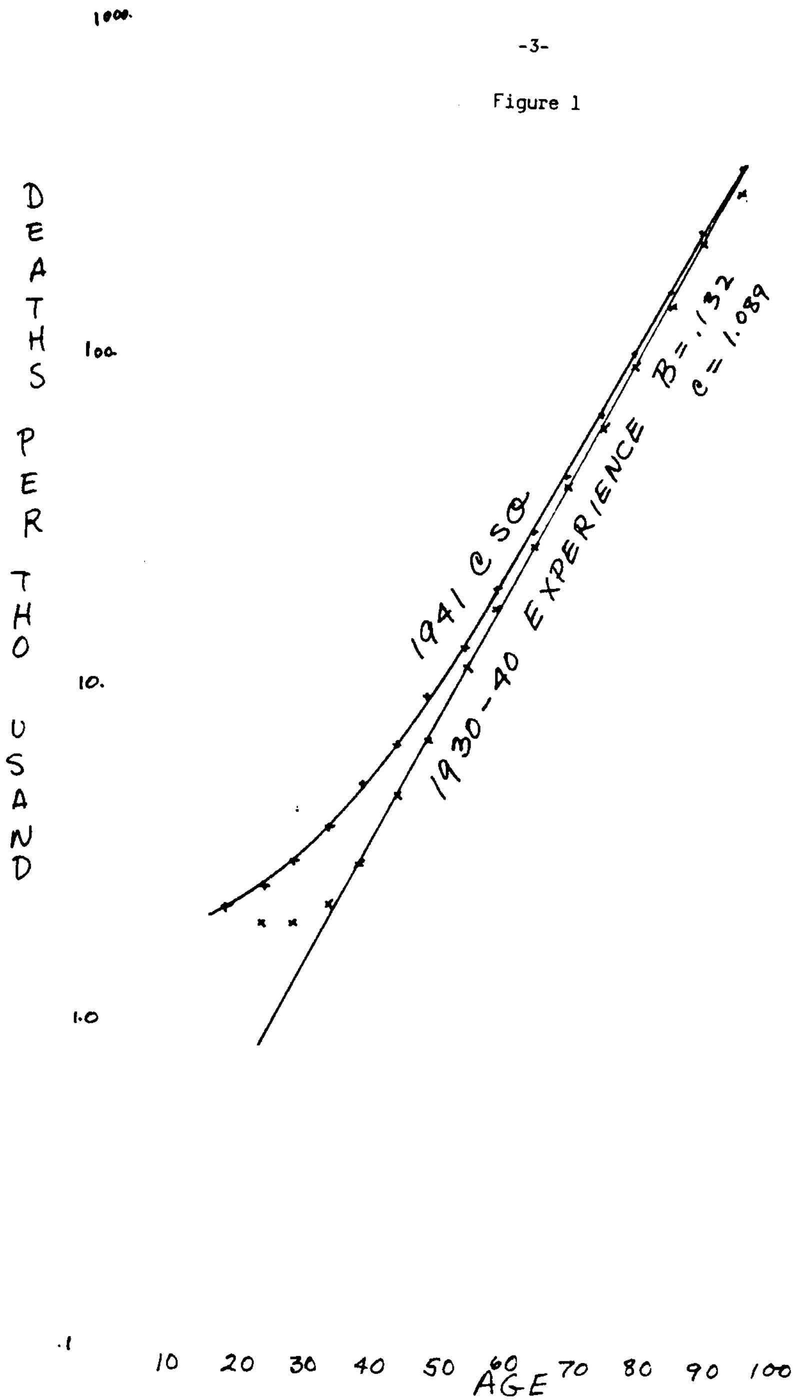
(1) $\log_b (\mu_x - A) = \log_b B + (\log_b c)x$, for any base b

This is a linear function in age x. When $\mu_x - A$ is plotted on semi-log paper, the equation (1) indicates that the result is a straight line. When μ_x is plotted on semi-log paper, for Makeham's law, the result is a curve lying above a straight line, asymptotic to a straight line as age increases, with the visual distances between the curve and the straight line diminishing with age.

Figure 1 displays two such curves. The lower curve represents the large company experience for 1930-1940, anniversaries which was used as the basis for the 1941 CSO table [6]. Note that this experience is very closely a Gompertz or linear fit between the ages 30 and 90. The final Makeham rates for the 1941 CSO table are shown in the upper curve (with typical Makeham shape). Margins in 1941 CSO were added to the basic rates, which, when expressed as a percentage of the rates, decrease with age. (A constant percentage margin would be parallel to the experience rates on semi-log paper).

Two methods were used in solving (1): graphical [2]; and least squares [1]. In the graphical method, B is read from the y intercept of the graph of $(\mu_x - A)$ on semi-log paper. The value of $\log_{10} c$ is determined as $\frac{1}{x_1 - x_2}$ where x_1 and x_2 are chosen so that $\log_{10}(\mu_{x_1} - A) = 10 \log_{10}(\mu_{x_2} - A)$. In least squares the dependent variable is $y_x = \log_{10}(\mu_x - A) = \theta_0 + \theta_1 x + e_x$. $\theta_0 + \theta_1 x$ is the predicted value of

Figure 1


1000.
100
10.
1.0
.1
DEATHS PER THOUSAND
1941 CSO
1930-40 EXPERIENCE
B = .132
c = 1.089
10 20 30 40 50 60 70 80 90 100
AGE

the dependent variable and e_x is the error or discrepency. The least squares estimates of θ_0 and θ_1 are

(2) $\hat{\theta}_1 = \sum_{x=\alpha}^{\omega} (y_x - \overline{y})(x - \overline{x}) / \sum_{x=\alpha}^{\omega}(x - \overline{x})^2$

(3) $\hat{\theta}_0 = \overline{y} - \hat{\theta}_1\overline{x}$, where $\overline{x}$ and $\overline{y}$ are sample means.

The residuals are defined to be $\hat{e}_x = y_x - \hat{\theta}_0 - \hat{\theta}_1 x$, and the residual sum of squares, ESS $= \sum_{x=\alpha}^{\omega} \hat{e}_x$, is minimized with respect to θ_0 and θ_1 by using (2) and (3). For any value of A we get a different ESS. We can vary A systematically and choose the one with minimal ESS as our least squares estimate of A, namely $\hat{A}$. By reference to (1) we se that $\hat{B} = 10^{\hat{\theta}_0}$ and $\hat{c} = 10^{\hat{\theta}_1}$.

The results presented below are partially graphical and partially least squares on a computer (note is made of which).

3. Departures from Gompertz's Law based on sex

We saw in Figure 1 that the 1930-40 experience followed Gompertz's Law quite closely between ages 30 and 90. However, when it came time to construct a new table for valuation purposes, based on 1950-54 experience, a significant departure from Gompertz's Law became evident. This can be seen in Figure 2. Curvature exists in the graph of the raw rates [5] centered around age 62. The rates from ages 50-80 are above a straight line. Similar curvature was retained in the 1958 CSO table, based on 1950-54 experience, and that table was not Makehamized because of the curvature. This can be seen in Figure 3.

Since insurance company experience is based, predominantly, upon white male mortality, the question arises whether this departure is characteristic of male mortality. Figure 4 shows graphs of white male and female mortality from the 1969-71 U.S. Life tables [7]. The male mortality exhibits the characteristic curvature above a straight line, centered at age 62, whereas the female mortality exhibits an opposing curvature below a straight line centered at age 67. When male and female mortality is combined as in Figure 5, Gompertz's Law holds except at age 50. Male and female mortality is seen to complement each other, with respect to Gompertz's Law.

Figure 6 shows the graphs of mortality rates for the 1965-70 Ultimate Tables, which are recent tables of insurance experience. Again male mortality has the characteristic curvature above the straight line and female mortality has the chacteristic curvature below the straight line, both at ages 50-75. However, there is additional curvature below age 50 as well. Since the combined table, shown in Figure 7, has male and female experience

Figure 2

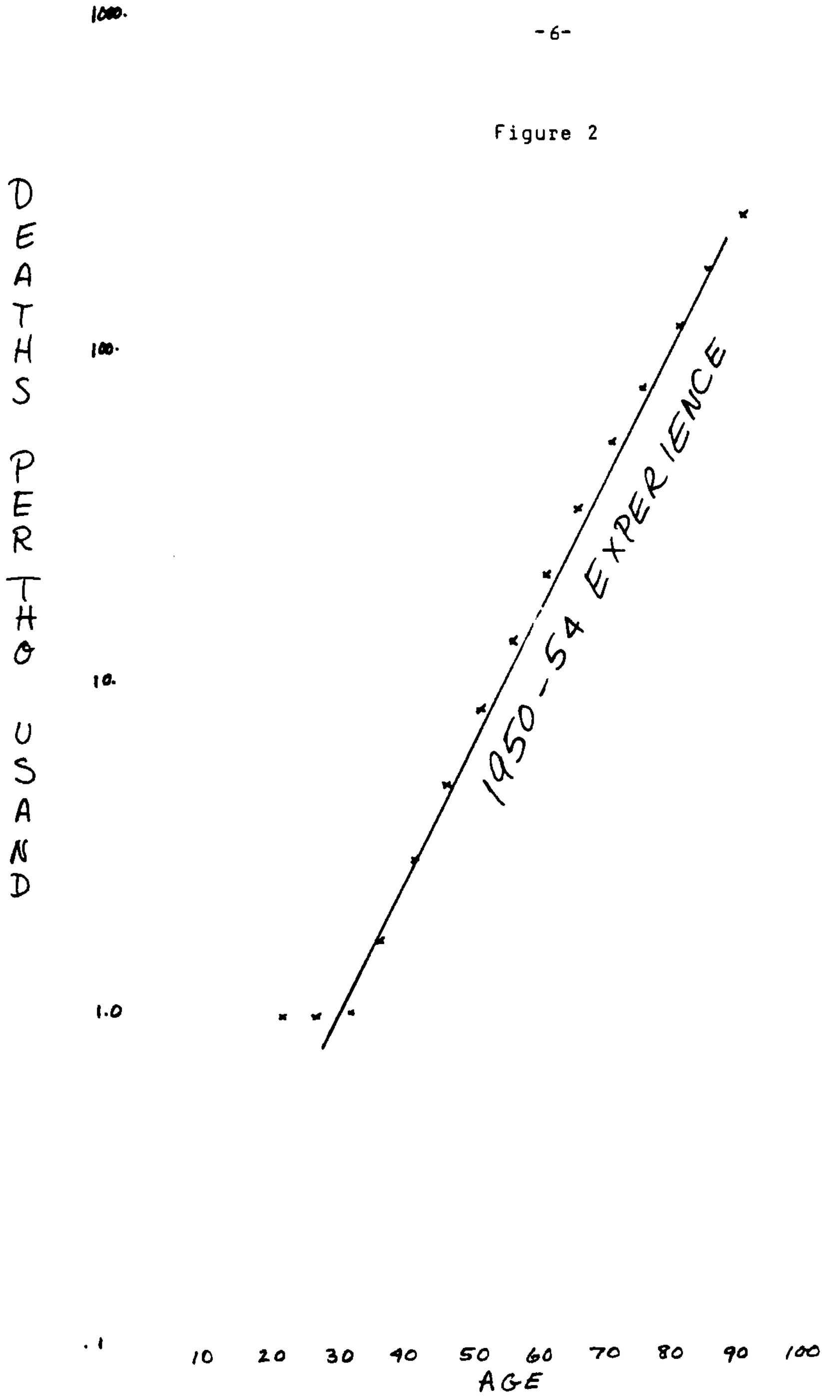

DEATHS PER THOUSAND
1000.
100.
10.
1.0
.1
10
20
30
40
50
60
70
80
90
100
AGE
1950-54 EXPERIENCE

Figure 3

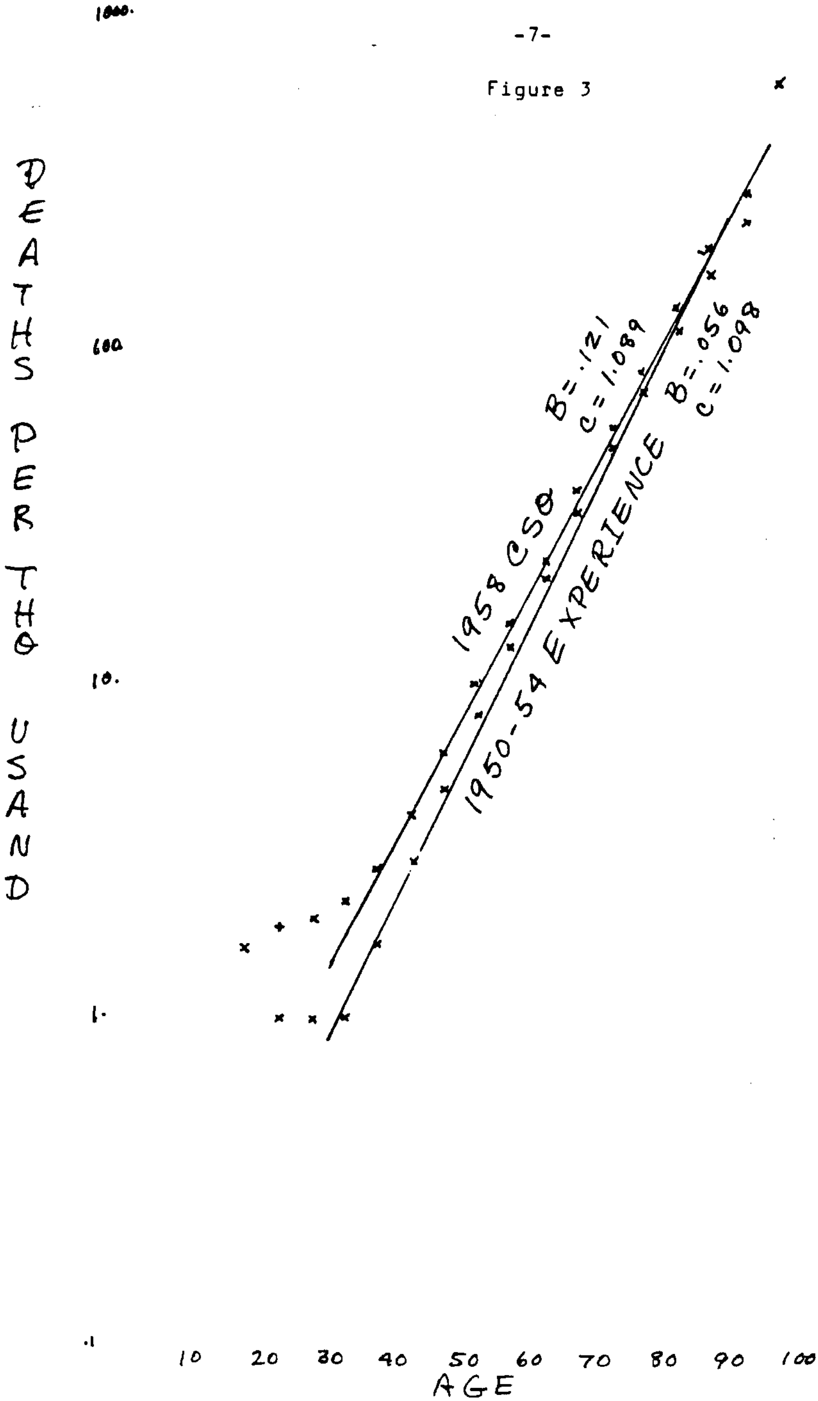
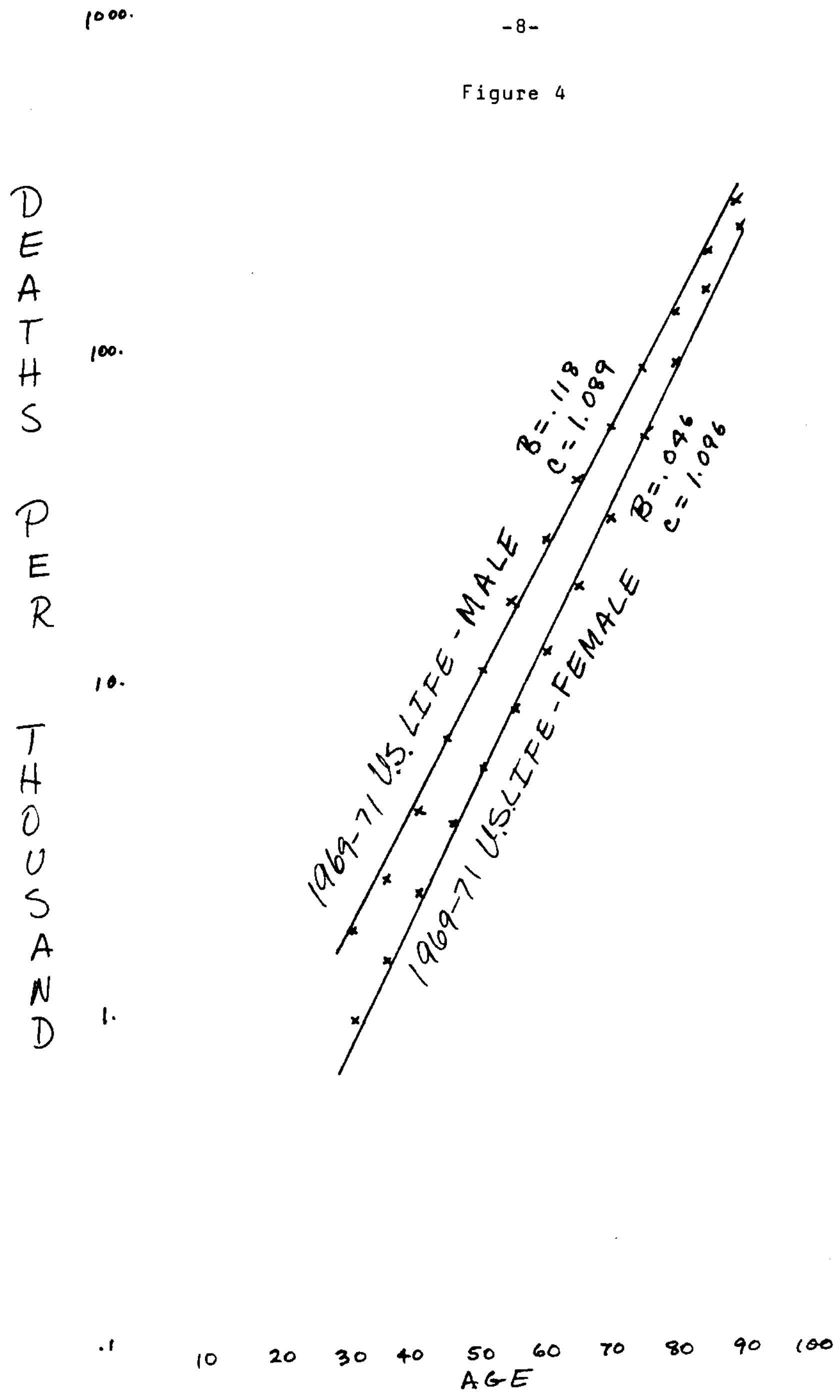

1000.
100
10.
1.
.1
DEATHS PER THOUSAND
1958 CSO
B=.121
c=1.089
1950-54 EXPERIENCE
B=.056
c=1.098
10
20
30
40
50
60
70
80
90
100
AGE

Figure 4



1000.
100.
10.
1.
.1
DEATHS PER THOUSAND
B=.118
c=1.089
1969-71 U.S. LIFE-MALE
B=.046
c=1.096
1969-71 U.S. LIFE-FEMALE
10
20
30
40
50
60
70
80
90
100
AGE

Figure 5

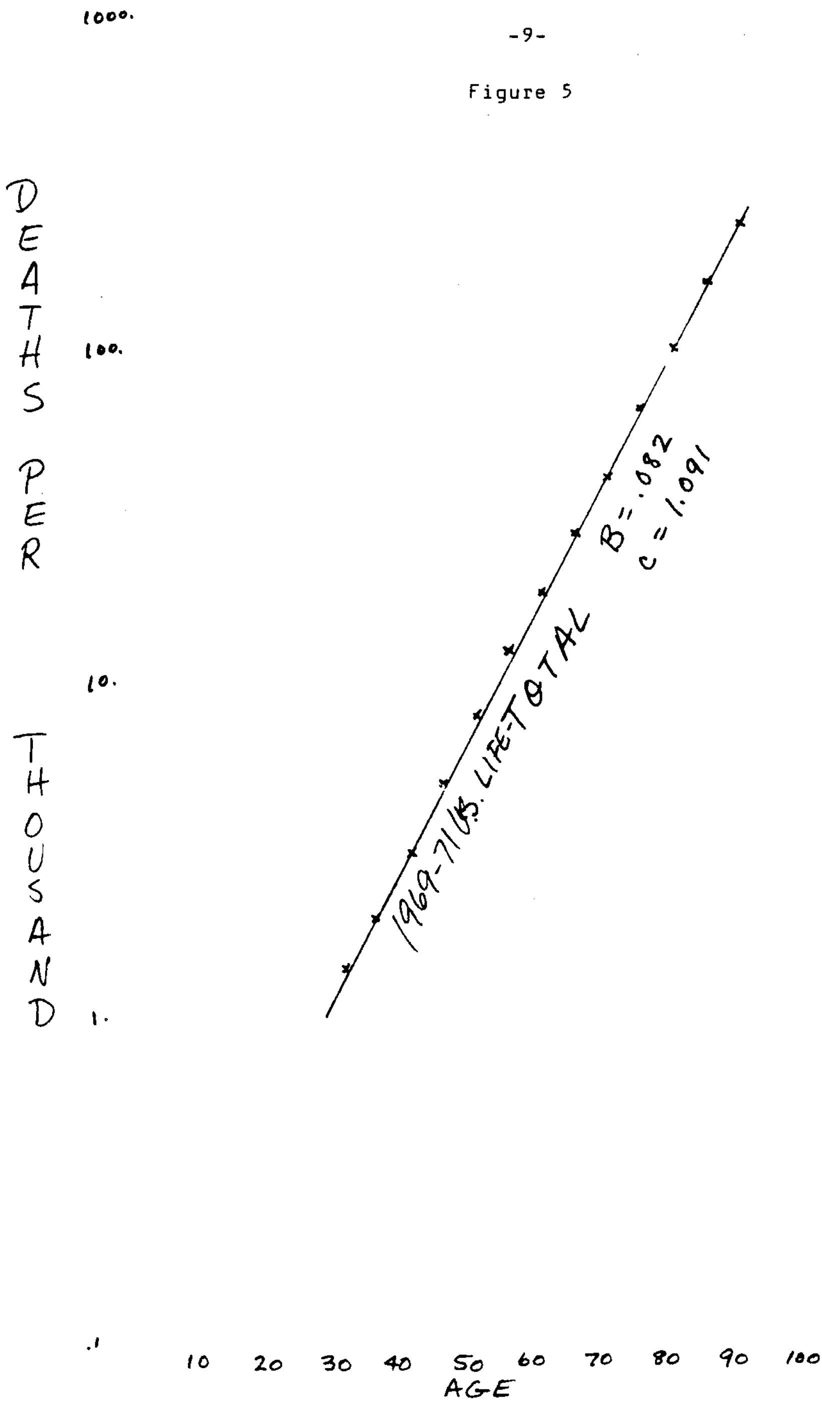

1000.
100.
10.
1.
.1
DEATHS PER THOUSAND
1969-71 U.S. LIFE-TOTAL
B = .082
c = 1.091
10
20
30
40
50
60
70
80
90
100
AGE

Figure 6

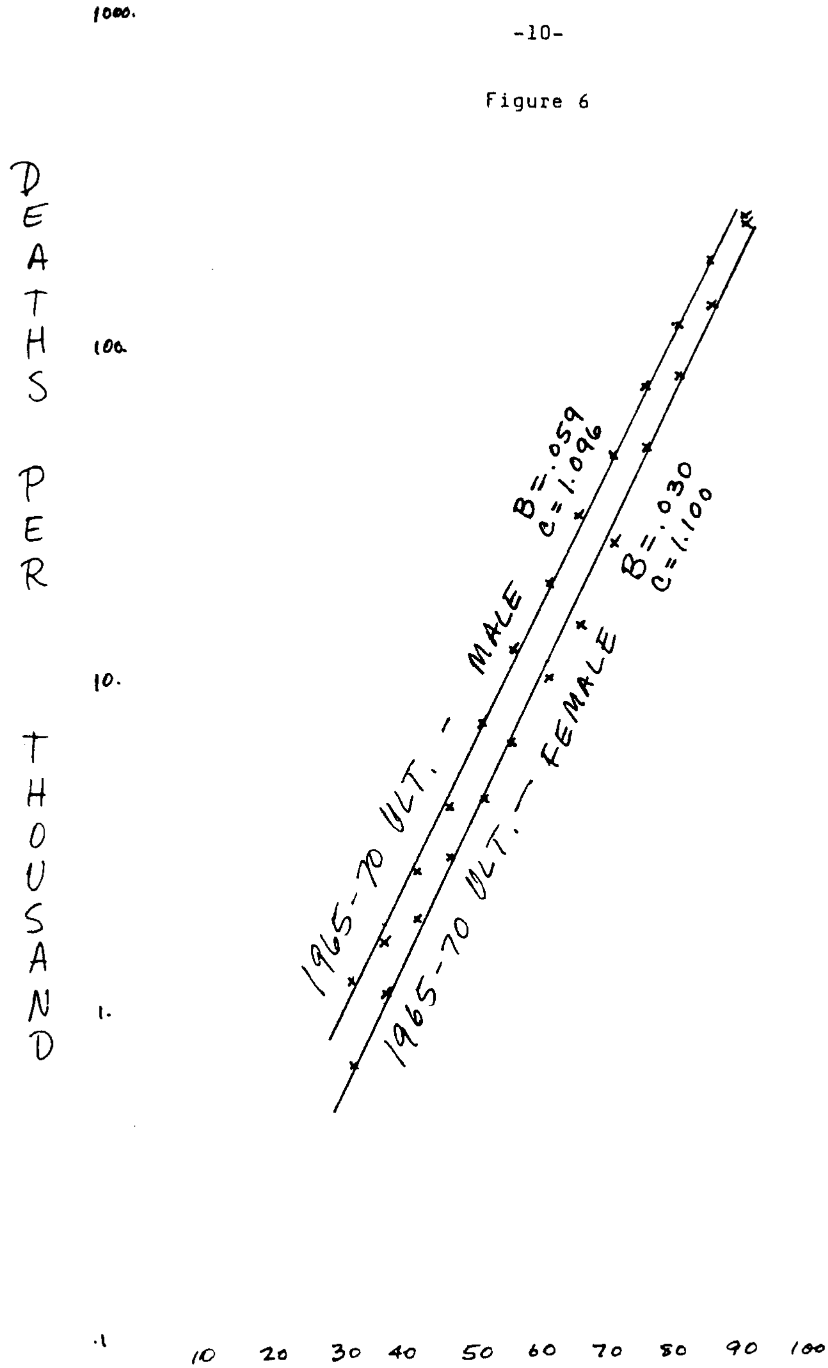
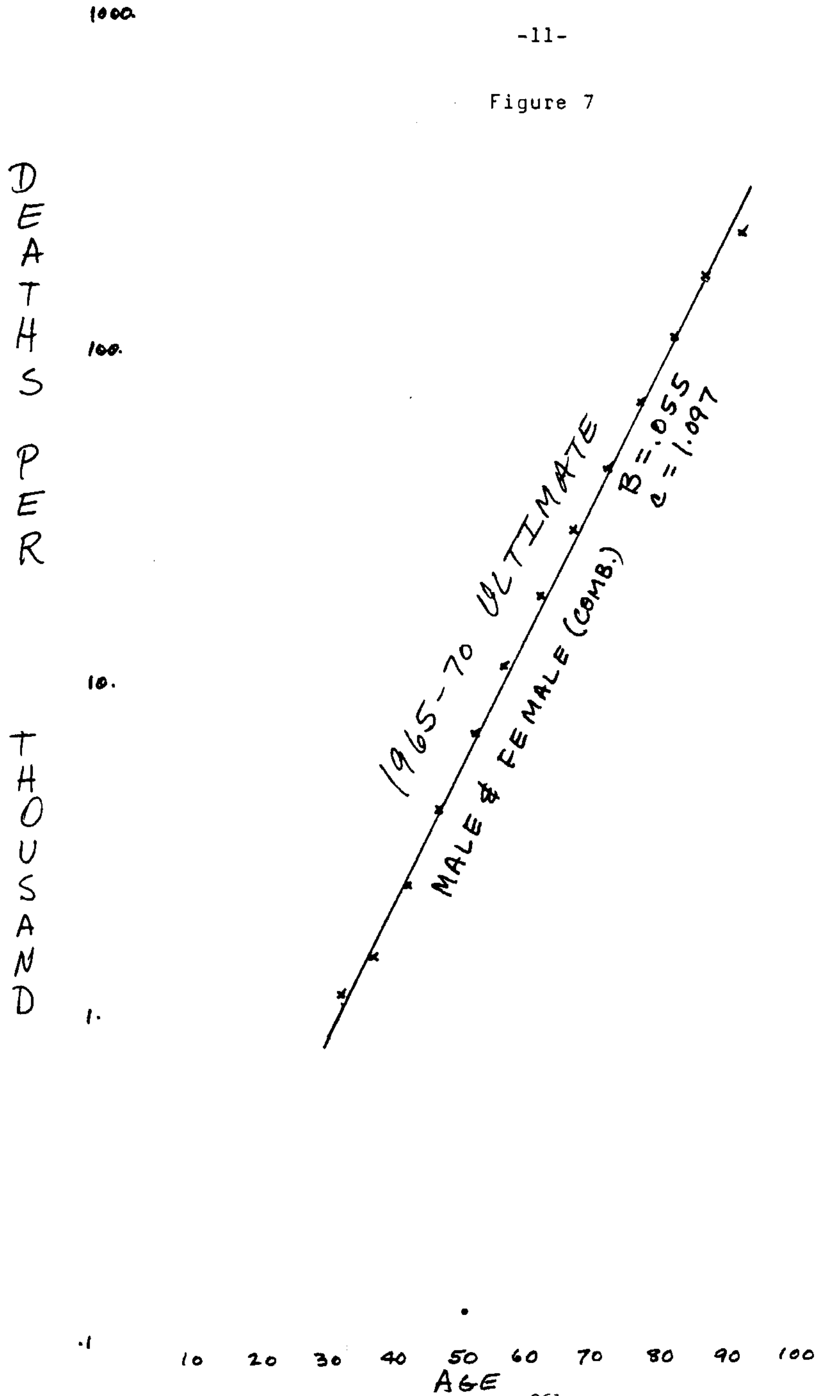

1000.
100.
10.
1.
.1
DEATHS PER THOUSAND
1965-70 ULT. - MALE
B=.059
c=1.096
1965-70 ULT. — FEMALE
B=.030
c=1.100
10
20
30
40
50
60
70
80
90
100

Figure 7



1000.
100.
10.
1.
.1
DEATHS PER THOUSAND
10 20 30 40 50 60 70 80 90 100
AGE
1965-70 ULTIMATE
MALE & FEMALE (COMB.)
B=.055
c=1.097

combined in proportion to their respective amounts of experience, white male experience predominates, and the Gompertz fit is not so close as in the combined 1969-71 U.S. Life Table.

Table 1 summarizes the determination of Gompertz constants for the data in Figures 1 - 7 by computer program. Table 2 compares determination of Makeham constants by computer program with actual, published results [6]. The results agree to three significant figures, which offers some confirmation of the computer algorithm used here. Similar results were obtained in [3], where a similar algorithm was used.

Table 1

Determination of Gompertz Constants

Mortality Table	B	c
1930-40 Experience	.13260	1.08934
1950-54 Experience	.05685	1.09788
1958 CSO	.12065	1.08879
1965-70 Ult. M & F	.05532	1.09671
1965-70 Ult. M	.05902	1.09628
1965-70 Ult. F	.03000	1.10000
1969-71 U.S. Life White lot	.08217	1.09064
1969-71 U.S. Life White M	.11836	1.08885
1969-71 U.S. Life White F	.04595	1.09558

Note: Above results determined by computer program

Table 2

Determination of Makeham Constants

1941 CSO	A	B	c
Actual	1.6050	.14471	1.08913
Estimated	1.6100	.14405	1.08920

Note: Above results determined by computer program

4. Changes in Gompertz Constants Across Time

Assuming that Gompertz Law describes mortality data reasonably well between the ages of 30 and 90, the next question to answer is how do the constants change across time. In order to answer this question it is necessary to analyze a homogenous body of data at various times. An ideal body of such data would be the large company experience for life insurance collected by the Society of Actuaries on an ongoing basis. I chose the latter data because of its easy availability. The results of this analysis are shown in Table 3. The data are separated by duration from issue and separated into three categories:

(i) Policy Years 1 to 5
(ii) Policy Years 6 to 15
(iii) Policy Years 16 and over [9]

In interpretting this data it is important to note that B and c do not change independently of each other. For example, if B increases from one time period to another, then c will probably decrease, and vice versa. In analyzing the first pair of columns, you will note that the changes are U - shaped. For B there is a decrease towards the middle with pronounced high points towards the ends. For c, the opposite direction is evident: low points on the ends with high points toward the middle. In particular the values for 1970 to 75 seem especially suspicious, with pronounced changes from previous time periods.

For the second and third pairs of columns a more easily explainable situation obtains. B decreases fairly smoothly over time while c is fairly stable. These changes can be seen as conforming to a general lowering of mortality rates in the general population over time.

Table 3

Variation in Gompertz Constants over Time

Large Company Insurance Experience

Year	Pol. Year 1-5 B	Pol. Year 1-5 c	Pol. Year 6-15 B	Pol. Year 6-15 c	Pol. Year 16 & over B	Pol. Year 16 & over c
39-43	.071	1.088	.071	1.098	.081	1.096
43-47	.048	1.093	.066	1.095	.102	1.090
47-51	.048	1.093	.066	1.093	.067	1.097
51-55	.048	1.090	.058	1.093	.054	1.098
55-59	.041	1.093	.058	1.093	.050	1.099
59-64	.057	1.086	.045	1.096	.047	1.100
64-70	.052	1.087	.058	1.091	.055	1.097
70-75	.071	1.077	.044	1.095	.041	1.101

Note: Above results determined graphically.

5. Changes in Gompertz Constants Across U.S. Metropolitan Areas

Another criterion across which variation in the Gompertz constants can be measured is geographical area. The geographical unit chosen was the Standard Metropolitan Statistical Area used in the U.S. Census. I am indebted to the class members of Math 464/564 , the Graduation of Data class, in the spring quarter of 1978, at Ball State University, for analyzing 14 such units as part of a class project. Their results, after some modification for obvious errors, are shown in Table 4.

The method for these analyses was to pool the deaths in the SMSA for the years 1969, 70, and 71, as they appear in [11]. These deaths, D_x, are available for decennial age groups, and become the numerators in age-specific mortality rates. The denominators are taken to be $3P_x + \frac{1}{2}D_x$, where P_x is the population in the age group in the 1970 census [8] . The procedure is the same one as that used in forming the 1969-71 U.S. Life Tables [10] .

The results in Table 4 are ordered by increasing B. Besides B and c, the complete expectation of life at age 30 is also listed as a single, indicative measure of mortality. Honolulu stands out with a low value of B and high expectation of life. This is consistent with Greville's findings for Hawaii in the 1969-71 U.S. Life Tables by State [10] .

The next group of metropolitan areas, starting with Evansville and ending with Erie, form a fairly consistent group of midwestern cities with similar Gompertz constant. The group from Gary/Hammond to Miami have higher levels of B and lower levels of c than the first group. New York City has a substantially higher value for B a fairly small value for c but a marked decrease in the expectation of life. In summary, data show definite patterns of mortality within geographical area as well as marked differences between areas, as described by the Gompertz constants. It is perhaps surprising that Gompertz law still holds within each area despite the differences between areas.

Table 4

Variation in Gompertz Constants
Across U.S. Metropolitan Areas

SMSA	Discrepant Ages*	$\overset{o}{e}_{30}$	B	$\log_{10}c$	c
Honolulu	40	46.7	.093	.0356	1.085
Evansville	none	44.1	.110	.0357	1.086
Muncie	none	44.7	.125	.0345	1.083
Indianapolis	none	44.0	.125	.0350	1.084
Springfield, Ohio	90	42.6	.126	.0360	1.086
Kansas City	none	44.3	.13	.0345	1.083
Erie, Penn.	80	42.6	.15	.0345	1.083
Gary/Hammond	70,80	43.9	.17	.0333	1.079
Portland, Maine	none	41.1	.17	.0345	1.083
Bakersfield, Calif.	40,50	42.6	.212	.0323	1.077
Miami	40	43.6	.22	.0315	1.075
New York City	60,70	38.3	.271	.0330	1.079
Male	none	35.7	.42	.0313	1.075
Female	60,70	41.8	.135	.0357	1.086

*Ages between 30 and 90 at which crude rates were not "close" to predicted rates.

6. An Extension to Makeham's Law

The curvature about a straight line evident in Figures 2, 3, 4, 6, and 7, which is related to differences in sex poses a problem which we would like to solve. This inspires a search for a more flexible parametric representation to account for this curvature. The extension to Makeham's law considered here is a simple one: instead of fitting a straight line to $\log_{10}(\mu_x - A)$ as in equation (1), we try fitting a higher degree curve, which was limited in the present study to the fourth degree. For any given value of A, equation (1) then becomes:

$$(4)\quad \log_{10}(\mu_x - A) = \theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3 + \theta_4 x^4 + \varepsilon_x$$

This equation can then be fitted using a linear least squares computer program, producing a vector of parameters θ_i and a vector residuals ε_x. For a given value of A the residual sum of squares, $RSS(A) = \sum_x \varepsilon_x$, is minimal with respect to θ_i. Now by choosing the value of A through trial and error, a true least squares solution for θ_i and A is obtained by choosing that value of A for which RSS(A) is minimal.* We call that minimal value, $\hat{A}$, the resulting parameters, $\hat{\theta}_i$, and the corresponding residuals, $\hat{\varepsilon}_x$. Then by substituting these values into (4) and raising both sides of (4) to a power of 10 we obtain the following:

$$B = 10^{\hat{\theta}_0}\ ;\quad \hat{c} = 10^{\hat{\theta}_1}\ ;\ d = \hat{\theta}_2 / \hat{\theta}_1\ ;$$

$$f = \hat{\theta}_3 / \hat{\theta}_1\ ;\ \hat{h} = \hat{\theta}_4 / \hat{\theta}_1\ ;\ 1 + \hat{P}_x = 10^{\hat{\varepsilon}_x}$$

Combining the results in (4) we obtain a generalization to Makeham's law.

$$(5)\quad \hat{\mu}_x = \hat{A} + \hat{B}\hat{c}^{\,x(1 + \hat{d}x + \hat{f}x^2 + \hat{h}x^3)}\,(1 + \hat{P}_x)$$

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*The results reported below use R^2, the square of the multiple correlation coefficient as the least squares criteria. I am indebted to Prof. Aaron Tenenbein for pointing out that RSS(A) is the correct criterion.

The $\hat{P}_x$ constitute a relative error model for $\mu_x - \hat{A}$ which is desirable to prevent the estimation of negative values for $\hat{A}$ and $\hat{\mu}_x$ for small x. A final relative error norm for the entire procedure is max $| \hat{P}_x |$, the maximum percentage error.

In Table 5 we see this procedure applied to quinquennial data from the 1958 CSO table. Four sets of results are presented, corresponding to linear, quadratic, cubic and quartic fits. The cubic fit is to be preferred for several reasons:

a) It is an improvement over the linear and quadratic fits, with smaller percentage errors.

b) The quartic fit is only a slight improvement over the cubic.

c) Furthermore the quartic fit produces a value for c that is less than one, and, therefore, c is no longer an aging parameter (rather, the opposite).

We feel that the cubic fit in Table 5 is adequate with an acceptable level of percentage error. The residuals, labeled DU, seem to be random, without cyclic pattern. The percentage errors might even be better. (No normality tests were applied to either of these residuals.)

Although apparently successful with 1958 CSO data, the extension to Makeham's Law was less successful with the 1965-70 Ultimate Table (not shown). One reason for this may have been the graduation of the table. Curve fitting to graduated data presents a problem because the graduation process itself may introduce cyclic patterns into the data with which the subsequent curve fitting must deal. It would seem preferable to fit a curve to ungraduated data to avoid such problems (not done with the 1965-70 Ultimate Table).

Table 5

Extension to Makeham's Law
1958 CSO Table

Linear Fit

R^2 0.99896

A 0.50000 B 0.09051 C 1.09274

X	μ	$\hat{\mu}$	Δμ	%Error
32.5	2.252	2.116	0.136	6.053
37.5	2.804	3.018	-0.214	-7.637
42.5	4.179	4.423	-0.245	-5.854
47.5	6.380	6.613	-0.232	-3.642
52.5	10.010	10.024	-0.014	-0.140
57.5	15.662	15.339	0.323	2.064
62.5	24.610	23.620	0.991	4.026
67.5	38.782	36.522	2.261	5.829
72.5	60.440	56.624	3.816	6.314
77.5	89.596	87.944	1.652	1.844
82.5	138.308	136.743	1.566	1.132
87.5	204.727	212.774	-8.047	-3.931
92.5	309.151	331.234	-22.084	-7.143

Quadratic Fit

R^2 0.99964

A 1.50000 B 0.00657 C 1.17580

D -3.08328E-03

X	μ	$\hat{\mu}$	Δμ	%Error
32.5	2.252	2.248	0.004	0.187
37.5	2.804	2.912	-0.108	-3.855
42.5	4.179	4.099	0.080	1.912
47.5	6.380	6.165	0.215	3.374
52.5	10.010	9.668	0.342	3.420
57.5	15.662	15.448	0.214	1.369
62.5	24.610	24.730	-0.120	-0.487
67.5	38.782	39.237	-0.455	-1.172
72.5	60.440	61.291	-0.851	-1.408
77.5	89.596	93.899	-4.302	-4.802
82.5	138.308	140.768	-2.459	-1.778
87.5	204.727	206.234	-1.508	-0.736
92.5	309.151	295.055	14.095	4.559

Table 5 (Continued)

Extension to Makeham's Law
1958 CSO Table

Cubic Fit (Preferred)

R^2	0.99979				
A	1.50000	B	0.00255	C	1.23765
D	-6.50552E-03			F	2.16723E-05

X	μ	$\hat{\mu}$	$\Delta\mu$	%Error
32.5	2.252	2.220	0.032	1.429
37.5	2.804	2.912	-0.108	-3.856
42.5	4.179	4.153	0.025	0.605
47.5	6.380	6.296	0.084	1.318
52.5	10.010	9.868	0.142	1.415
57.5	15.662	15.642	0.020	0.126
62.5	24.610	24.730	-0.120	-0.487
67.5	38.782	38.717	0.065	0.167
72.5	60.440	59.858	0.582	0.963
77.5	89.596	91.372	-1.776	-1.982
82.5	138.308	137.902	0.407	0.294
87.5	204.727	206.234	-1.508	-0.736
92.5	309.151	306.462	2.689	0.870

Quartic Fit

R^2	0.99992				
A	0.00000	B	65.93153	C	0.71962
D	-2.93845E-02	F	2.91771E-04	H	-1.04785E-06

X	μ	$\hat{\mu}$	$\Delta\mu$	%Error
32.5	2.252	2.218	0.035	1.543
37.5	2.804	2.902	-0.098	-3.498
42.5	4.179	4.152	0.027	0.636
47.5	6.380	6.300	0.081	1.266
52.5	10.010	9.880	0.129	1.293
57.5	15.662	15.699	-0.037	-0.233
62.5	24.610	24.888	-0.277	-1.126
67.5	38.782	38.981	-0.198	-0.511
72.5	60.440	60.029	0.411	0.680
77.5	89.596	90.935	-1.338	-1.493
82.5	138.308	136.253	2.056	1.486
87.5	204.727	204.411	0.612	0.299
92.5	309.151	310.618	-1.467	-0.475

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