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DEFINING THE YIELD RATE

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In discussing yield rates in the Theory of Interest class at the University of Nebraska, I have been encouraging students to try to approach yield rate problems by starting with the definition of yield rate found on page 117 of Steve Kellison's text:

> "The yield rate is that effective rate of interest at which the present value of his expenditures is equal to the present value of his returns."

This may be contrasted to the more intuitive starting point illustrated in Example 5.10, found on page 120.

> Example 5.10. An investor buys an n-year annuity with a present value of \$1000 at 5% at a price which will allow him to accumulate a sinking fund to replace his capital at 4% and will produce an overall yield rate of 6%. Find the purchase price of the annuity.

The implicit assumption of a level net return each year results in the relationship:

$$R = \frac{1000}{a_{\overline{n}|.05}} - .06P \qquad (1)$$

where P is the purchase price and R is the sinking fund deposit.

In this particular case, and I believe in all cases where the returns - in this case, the annuity payments - and the expenditures (beyond the initial purchase price, P) - in this case, the sinking fund deposits - are both level, the definitional starting point gives the same result, since the relationship (1) is only a simplification of the definitional relationship:

$$\underbrace{P + R \cdot a_{\overline{n}|.06}}_{\text{Present Value of expenditures at desired yield rate}} = \underbrace{\frac{1000}{a_{\overline{n}|.05}} \cdot a_{\overline{n}|.06} + P v^n_{.06}}_{\text{Present Value of returns at desired yield rate}} \qquad (2)$$

This is simply illustrated as follows:

$$(2) \quad P + R \cdot a_{\overline{n}|.06} = \frac{1000}{a_{\overline{n}|.05}} \cdot a_{\overline{n}|.06} + P v^n_{.06}$$

$$\Longrightarrow \quad P(1 - v^n_{.06}) + R \cdot a_{\overline{n}|.06} = \frac{1000}{a_{\overline{n}|.05}} \cdot a_{\overline{n}|.06}$$

$$\Longrightarrow \quad .06 \frac{P(1 - v^n_{.06})}{.06} + R \cdot a_{\overline{n}|.06} = \frac{1000}{a_{\overline{n}|.05}} \cdot a_{\overline{n}|.06}$$

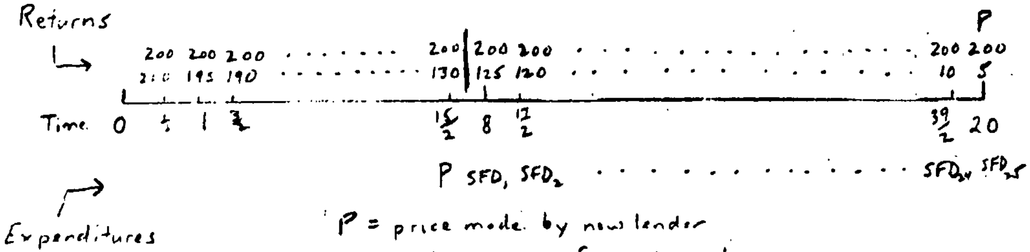
$$\Longrightarrow \quad .06P + R = \frac{1000}{a_{\overline{n}|.05}}$$

$$\text{or} \quad R = \frac{1000}{a_{\overline{n}|.05}} - .06P \qquad (1)$$

In solving problem 36 in Chapter 5 of the text, the definitional starting point leads to some difficulties.

> 36. A mortgage of \$8000 is repayable in 20 years by semiannual installments of \$200 each plus interest on the unpaid balance at 5%. Just after the 15th payment the lender sells the mortgage at a price which yields the new lender 6% and allows him to accumulate a sinking fund to replace his capital at 4%. Assuming that all interest rates are convertible semiannually, show that the price is
>
> $$\frac{75\, s_{\overline{25}|.02} + 6250}{1 + .03\, s_{\overline{25}|.02}}$$

A diagram of the situation presented by the problem is as follows:



P = price made by new lender

SFD_i = i^{th} sinking fund deposit

For the new lender, the returns are the level \$200 he receives every 6 months for the next 12½ years, the decreasing interest payment he receives every 6 months for the next 12½ years, and the return of his capital, P, from the sinking fund at the end of the 12½ years. The expenditures are the price, P, he pays to the initial investor, and the payment he makes to the sinking fund at the end of every 6 months for the next 12½ years.

The definitional starting point implies that one should equate the present value at 6% convertible semiannually of the expenditures and the present value at 6% convertible semiannually of the returns. The equation suggested is:

$$\underbrace{P + PV\ SFD_i\ (\text{at } i^{(2)} = .06)}_{\text{Present Value of expenditures at desired yield rate}} = \underbrace{200\, a_{\overline{25}|.03} + 5(Da)_{\overline{25}|.03} + P\, v^{25}_{.03}}_{\text{Present Value of returns at desired yield rate}} \qquad (3)$$

But what is the present value of the sinking fund deposits? If one makes the logical assumption that the best way to assure a 6% convertible semiannually return on this transaction is to deduct .03P from the returns at the end of every 6 months and place the balance in the sinking fund, the following equation for the sinking fund results:

$$(200 - .03P)\, s_{\overline{25}|.02} + 5\,(Ds)_{\overline{25}|.02} = P \qquad (4)$$

This equation can be solved for P to yield the expression required in problem 36:

$$(200 - .03P)\, s_{\overline{25|}.02} + 5\,(Ds)_{\overline{25|}.02} = P$$

$$\Rightarrow P = \frac{200\, s_{\overline{25|}.02} + 5\,(Ds)_{\overline{25|}.02}}{1 + .03\, s_{\overline{25|}.02}} = \frac{200\, s_{\overline{25|}.02} + 5\left(\dfrac{25(1.02)^{25} - s_{\overline{25|}.02}}{.02}\right)}{1 + .03\, s_{\overline{25|}.02}}$$

$$\Rightarrow P = \frac{200\, s_{\overline{25|}.02} + 5\left[25\left(\dfrac{(1.02)^{25} - 1}{.02}\right) + \dfrac{25 - s_{\overline{25|}.02}}{.02}\right]}{1 + .03\, s_{\overline{25|}.02}}$$

$$\Rightarrow P = \frac{200\, s_{\overline{25|}.02} + 125\, s_{\overline{25|}.02} + 250\,(25 - s_{\overline{25|}.02})}{1 + .03\, s_{\overline{25|}.02}}$$

$$\Rightarrow P = \frac{75\, s_{\overline{25|}.02} + 6250}{1 + .03\, s_{\overline{25|}.02}}$$ (when evaluated, $P = \$4412.38$)

Thus, the problem can be solved without directly appealing to the basic equation ((3) above) which results from the definition of yield rate, as stated on page 117 in the text.

But what if a <u>level sinking fund deposit</u>, rather than a <u>level net return (.03P)</u>, is assumed? This is equivalent to a non-level net return (returns - expenditures) every 6 months. However, the level sinking fund deposit assumption does not affect the solvability of equation (3). That is, equation (3) can still be solved for a P such that the yield rate over the investment period is 6% convertible semiannually. This is illustrated below:

Let SFD = the level sinking fund deposit every 6 months, then equation (3) becomes:

$$P + SFD\, a_{\overline{25|}.03} = 200\, a_{\overline{25|}.03} + 5\,(Da)_{\overline{25|}.03} + P v^{25}_{.03}$$

Also, by the nature of the sinking fund:

$$SFD\, s_{\overline{25|}.02} = P$$

$$\Rightarrow SFD = \frac{P}{s_{\overline{25|}.02}}$$

Substituting for SFD,

$$P + \frac{P}{s_{\overline{25|}.02}} \cdot a_{\overline{25|}.03} = 200\, a_{\overline{25|}.03} + 5\,(Da)_{\overline{25|}.03} + P\, v^{25}_{.03}$$

$$\Rightarrow \quad P(1 - v^{25}_{.03}) + \frac{P}{s_{\overline{25|}.02}} \cdot a_{\overline{25|}.03} = 200\, a_{\overline{25|}.03} + \frac{5(25 - a_{\overline{25|}.03})}{.03}$$

$$\Rightarrow \quad .03P\, a_{\overline{25|}.03} + \frac{P}{s_{\overline{25|}.02}}\, a_{\overline{25|}.03} = 200\, a_{\overline{25|}.03} + \frac{5(25 - a_{\overline{25|}.03})}{.03}$$

$$\Rightarrow \quad .03P + \frac{P}{s_{\overline{25|}.02}} = \frac{200\, a_{\overline{25|}.03}}{a_{\overline{25|}.03}} + \frac{5(25 - a_{\overline{25|}.03})}{.03\, a_{\overline{25|}.03}}$$

$$\Rightarrow \quad P\left(.03 + \frac{1}{s_{\overline{25|}.02}}\right) = \frac{.03(200)\, a_{\overline{25|}.03} + 125 - 5\, a_{\overline{25|}.03}}{.03\, a_{\overline{25|}.03}}$$

$$\Rightarrow \quad P\left(\frac{.03\, s_{\overline{25|}.02} + 1}{s_{\overline{25|}.02}}\right) = \frac{a_{\overline{25|}.03} + 125}{.03\, a_{\overline{25|}.03}}$$

$$\Rightarrow \quad P = \frac{(a_{\overline{25|}.03} + 125)(s_{\overline{25|}.02})}{.03\, a_{\overline{25|}.03}\,(1 + .03\, s_{\overline{25|}.02})} \qquad \left(\text{when evaluated, } P \doteq \$4453.04\right)$$

Of course, as you might expect, the price is not the same under this latter assumption (level sinking fund deposit) as it is under the level net return assumption. Under the level net return assumption the price is $4412.38, while under the level sinking fund deposit assumption the price is $4453.04.

It is interesting to reason why the second price is higher than the first. Obviously, it should only be higher - assuming the same yield rate - if one "gets more". How does one "get more" with the level sinking fund deposit? The net return is greater in the earlier years under the level sinking fund deposit assumption. Of course, the net return is smaller in the later years under the level sinking fund assumption. However, the present value of the net returns is greater under the level sinking fund deposit assumption.

But what does this all mean? I believe the two main points to remember are:

1. The definition of yield rate does not imply any specific net return pattern. Thus, in my mind, either price determined above may be considered a correct solution to problem 36 as stated.

2. The definitional approach has to be applied carefully, recognising what assumptions one is making when it is applied.

Finally, the statement of problem 36 could perhaps be improved by adding either the phrase "assuming a level net return every 6 months", or the phrase "assuming a level sinking fund deposit every 6 months", at the end of the second to the last sentence. Or, perhaps better yet, it could be developed into an "(a),(b),(c)" problem:

(a) assuming a level net return every 6 months
(b) assuming a level sinking fund deposit every 6 months
(c) explain why the price in (b) is greater than the price in (a)