A GENERAL MODEL FOR LIFE CONTINGENCIES

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1. Formulation of the Model and Net Reserves

A policy is defined by two sequences of random variables:

$$X_0,\ X_1,\ X_2,\ \ldots$$
$$S_0,\ S_1,\ S_2,\ \ldots$$

Here, X_t is the balance of benefits paid plus expenses incured minus premiums received at time t . S_t (possibly a random vector) is the "state of the world" or the additional information obtained, at time t . Let

$$H_t = (S_0,\ S_1,\ \ldots,\ S_t)$$

denote the total information available at time t . It is assumed that X_t is a function of H_t , i.e. at time t the outcome of X_t is known.

The net reserve at time t is the expected present value of future outgo:

$$V_t = E[\sum_{k=1}^{\infty} v^k X_{t+k} \mid H_t] \ .$$

By splitting off the first term in this sum, and conditioning on H_{t+1} , it can be seen that

$$V_t = v\, E[X_{t+1} \mid H_t] + v\, E[V_{t+1} \mid H_t]$$

This recursive formula for reserves has an obvious interpretation.

Note that reserves were defined prospectively (in the general model they cannot be defined retrospectively). Also, observe that these reserves are "initial" reserves. Since no formal distinction is made between premiums and benefits, "terminal" reserves cannot be defined.

For some of the analysis it is convenient to introduce the present values of X_t and V_t . Let

$$Y_t = v^t X_t \text{ , } W_t = v^t V_t \text{ .}$$

Then

$$L = Y_0 + Y_1 + Y_2 + \ldots$$

is the present value of the total loss. Its predicted value at time t can be shown to be

$$E[L \mid H_t] = Y_0 + \ldots + Y_t + W_t \text{ .}$$

Let

$$L_t = Y_t + W_t - W_{t-1} \quad \text{for} \quad t = 1, 2, \ldots$$

and $L_0 = Y_0 + W_0$. Thus L_t is the present value of the loss incured at time t . It is easy to show that

$$L = L_0 + L_1 + L_2 + \dots$$

and that

$$E[L \mid H_t] = L_0 + L_1 + \dots + L_t \ .$$

Let $s < t$. From the recursive formula for reserves it can be seen that

$$E[L_t \mid H_s] = 0 \ .$$

Thus $E[L_t] = 0$ for $t = 1, 2, 3, \dots$. Furthermore,

$$E[L_s \, L_t \mid H_s] = L_s \, E[L_t \mid H_s] = 0 \ .$$

Thus $E[L_s \, L_t] = 0$, which means that L_s and L_t are uncorrelated. (Since they are not independent, this is not obvious a priori). From this it follows that

$$\mathrm{Var}[L] = \mathrm{Var}[\sum_{t=0}^{\infty} L_t] = \sum_{t=0}^{\infty} \mathrm{Var}[L_t] \ ,$$

which is Hattendorf's Theorem.

2. Contingency Reserves

In the following, let $\tilde{V}_t$ denote the contingency reserve (containing a loading) at time t, and let $\tilde{W}_t$ denote its present value. One possibility is to define $\tilde{V}_t$ as a biased expected value (which involves "deltas") of the present value of future outgo. Alternatively, it has been suggested to set

$$\tilde{W}_t = W_t + \alpha \cdot \mathrm{Var}[Y_{t+1} + Y_{t+2} + \dots \mid H_t] ,$$

where α is a positive parameter, or to set

$$\tilde{W}_t = \frac{1}{a} \log E[\exp(a(Y_{t+1} + Y_{t+2} + \dots)) \mid H_t] ,$$

where $a > 0$ is the parameter. This last possibility ("exponential reserves") is considered more in detail in the following.

First, exponential reserves satisfy a recurrence relation,

$$\tilde{W}_t = \frac{1}{a} \log E[\exp(a(Y_{t+1} + \tilde{W}_{t+1})) \mid H_t] ,$$

which is not the case for variance reserves. Let $\tilde{L}_0 = Y_0 + \tilde{W}_0$, and

$$\tilde{L}_t = Y_t + \tilde{W}_t - \tilde{W}_{t-1}$$

denote the present value of the loss (induced by the exponential reserves) incured at time t. It is still true that

$$L = \tilde{L}_0 + \tilde{L}_1 + \tilde{L}_2 + \ldots ,$$

but the allocation of loss is different now: For $s < t$ it can be shown that

$$E[\tilde{L}_t \mid H_s] \geq 0 ,$$

with strict inequality holding in any nontrivial case, from which it follows that $E[\tilde{L}_t] \geq 0$ for $t \geq 1$.

There is an interesting connection between exponential reserves and risk theory. If u is an amount that the insurer is willing to risk in connection with the policy, we speak of "ruin", if

$$\tilde{L}_0 + \tilde{L}_1 + \ldots + \tilde{L}_t > u$$

for some t . It can be shown that a plays the role of the adjustment coefficient, and that

$$\exp(-a(u - \tilde{L}_0))$$

is an upper bound for the probability of ruin.

References

"A probabilistic model for (life) contingencies and a delta-free approach to contingency reserves", TSA, 28, 127-148, and references quoted therein.

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