Definitions for Compound and Simple Interest

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While we agree with the conclusions expressed by Silver and Hedges ("The Classical Definition of Compound Interest is Adequate" by Murray Silver and Bob Hedges, ARCH 1981.2), we do not believe the theorem follows from the stated Axiom. We present an alternative proof which uses only the accumulation function and not the force of interest.

The result that compound interest demands $a(t) = (1 + i)^t$ for all t must depend on some statement with regard to the behavior of $a(t)$ for noninteger t. We suggest the following definition of compound interest.

Definition 1: Interest is said to be compounded at annual rate $i$ if (1) $a(1) = 1 + i$ and (2) $a(t + s) = a(t)a(s)$ for all real s and t.

The second statement may be explained as follows: A $1 investment accumulates to $a(t + s)$ after $t + s$ years. If however the accumulated value is withdrawn after just t years and immediately reinvested, the investment will grow to $a(t)a(s)$ after s additional years. The definition requires that the final accumulated value be unaffected by the intermediate transaction. Clearly compounding is occurring since interest earned during the first t years, earns interest during the final s years. The appropriate theorem is:

Theorem 1: If interest is compounded at rate $i$ and $a(t)$ is differentiable for all t, then $a(t) = (1 + i)^t$.

Proof: $a'(t) = \lim_{s\to 0} \frac{a(t+s)-a(t)}{s} = \lim_{s\to 0} \frac{a(t)(a(s)-1)}{s} = a(t)a'(0)$

Therefore $\frac{a'(t)}{a(t)} = a'(0)$ and so $\frac{d}{dt} \ln a(t) = a'(0)$,

which implies $\ln a(t) = a'(0)t + c$.

But $\ln a(0) = \ln(1) = 0 = c$.

and $\ln a(1) = \ln(1+i) = a'(0)$.

Therefore $\ln a(t) = t \ln(1+i)$,

and finally $a(t) = (1+i)^t$.

Our proof is similar to that given by Silver and Hedges, but it more directly uses the basic assumption that reinvestment and additional investments accumulate in the same manner as continuing deposits.

We also note that simple interest may be developed in a similar manner.

Definition 2: Interest is said to be simple at annual rate i if (1) $a(1) = 1+i$ and (2) $a(t+s) = a(t)+a(s)-1$ for all real s and t.

The motivation for (2) is provided by the same reinvestment example. The value after t years is $a(t) = 1 + [a(t) - 1]$, which has been separated into principal and interest components. Since we want only the principal to earn interest, the final value is $a(s) + [a(t) - 1]$.

Theorem 2: If interest is simple at rate i and $a(t)$ is differentiable for all t, then $a(t) = 1 + it$.

The proof is analogous to that of Theorem 1. In this case $a(t) = a'(0)t + c$ and the constants are determined from $a(0) = 1$ and $a(1) = 1 + i$.