SOME FURTHER COMMENTS ON THE UNIFORM DEATHS ASSUMPTION

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Introduction

In his paper "Exposure Formulas Based on the Assumption of Uniform Distribution of Deaths," T N E Greville* has developed a mathematical interpretation of the uniform distribution of deaths (UDD) assumption in mortality studies A simple explanation of the mathematical consequences of this familiar assumption has traditionally proven to be rather elusive because it involves the concepts of "observed" and "unobserved"[1] deaths (as defined by Harry Gershenson in Chapter Six of Measurement of Mortality[2]) Unobserved deaths, as pointed out by Greville, pose obvious problems to the actuary because the number of deaths which occur among persons after they have left a defined group of lives is difficult, if not impossible, to determine from the data gathered for a typical mortality study

The entire subject of the logical or diagrammatic interpretation of UDD formulas was deliberately omitted from the text Mortality Table Construction[3] because such an analysis has traditionally proven confusing to students, while adding little to their overall understanding of the principles of table construction

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[1]An unobserved death is defined to be one which occurs after a life leaves the observed group between ages x and x+1 but before attainment of age x+1

[2]Gershenson, Harry, Measurement of Mortality Chicago, Illinois: The Society of Actuaries, 1961

[3]Batten, Robert W Mortality Table Construction Englewood Cliffs, New Jersey: Prentice-Hall, Inc , 1978

*Editor's note: Mr Greville's Conference paper had the title quoted in the text. However, the paper was expanded for publication in the Conference Proceedings and re-titled "Estimation of the Rate of Mortality in the Presence of In-and-Out Movement".

Yet, Greville is quite correct that the uniform deaths assumption poses intriguing questions and provides surprisingly satisfying answers to the theoretician.

Analysis of the Assumption

The application of the UDD assumption to the derivation of exposure formulas should be examined from two perspectives, first with a seriatim approach and then with the more familiar valuation schedule method. Some results which appear to be evident when a seriatim technique is used are not always substantiated when valuation schedule formulas are analyzed.

Greville's Formula (U), while basically identical to that at the bottom of page 183 of the Gershenson text, has been rearranged into a form which is more open to ready analysis It is rather easy to conclude therefrom that the UDD assumption produces exposures different from those generated under the Balducci hypothesis only in the cases of death. If the death is "observed," the life is credited with one full unit of exposure; if the death is "unobserved," no exposure is credited These two conclusions are independent of the length of time between ages x and x+1 during which the individual life was actually "on the risk." Clearly, then, observed deaths are credited with a greater exposure (on the average) under the uniform deaths assumption than under the Balducci hypothesis; the opposite is true for unobserved deaths Further, the treatment of deaths seems less esthetically satisfying under the uniform deaths assumption than under the Balducci hypothesis

Another interesting consequence of the application of the UDD assumption comes to light when one considers a person who dies in the unit interval $[x, x+1]$ after becoming an ender in that same unit interval The mathematical derivations presented by Gershenson and Greville result in the treatment of enders at exact age $x+t$, $0 \le t \le 1$, just as if they were withdrawals Thus, for example, we have the rather implausible but mathematically correct result that a life which enters a study at $x+\frac{1}{4}$, becomes an ender at $x+\frac{3}{4}$, and dies at $x+\frac{7}{8}$ is treated under UDD as an unobserved death and therefore contributes no exposure, rather than the seemingly logical one-half year. We must then conclude that the UDD assumption mandates a temporary monitoring of subsequent mortality experience not only for all withdrawn lives but also for enders in date-to-date studies

By depending upon these interpretations, the exposure contribution of any individual life is quite easy to determine, as Greville has indicated. A seriatim method, therefore, may be applied in the presence of the uniform deaths assumption as readily as with the Balducci hypothesis For the reason mentioned earlier, the seriatim method was illustrated in Mortality Table Construction only in conjunction with the Balducci hypothesis [4]

As pointed out by Greville, since each observed death is credited with exposure of exactly one year by the seriatim method based on uniform deaths, the numerator of a mortality rate can never exceed its denominator as may happen in rare instances when the Balducci hypothesis is applied However, it will be seen shortly that this conclusion is not necessarily valid when valuation schedule formulas of the uniform death genre are analyzed.

Application of the Assumption

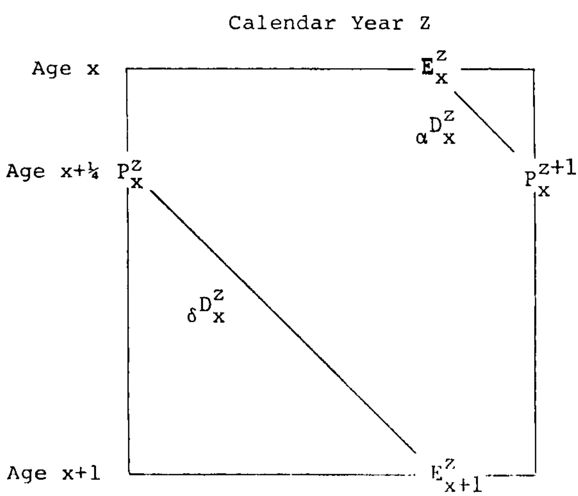
Greville has used Figure 5-13 from Mortality Table Construction, along with its underlying assumptions, as an illustration of how his Formula (U) can be applied to derive valuation schedule formulas for double-diagonal situations Similar application to a single-diagonal formula appears to be somewhat more difficult. The following technique, essentially equivalent

[4]Some actuaries have suggested that a seriatim approach be used in a pure sense, that is, without any accompanying mortality assumption In such a case, every observed life would be credited with the amount of exposure corresponding to its own true period of observation, resulting in a mortality measure more closely associated with a force of mortality μ than with a rate of mortality q Such a procedure may well come into greater favor with the increasing availability and capability of high-speed data processing equiement.

to that of Greville, works equally well for either type of formula under the uniform deaths assumption and may, at least in some instances, be easier to apply

The duration from the beginning of the unit age interval to the point at which the second demographic in-force symbol appears should be considered independent of the remainder of the diagram. The appropriate exposure for this interval is simply the number of lives present at the beginning of the interval multiplied by its length This is essentially stated by Greville as his Formula (U') [5] The second segment of the valuation schedule diagram can be handled exactly as Greville suggests, i e , by application of Formula (U)

As an illustration of each type of formula, let us consider the derivations of Formulas (5-20) and (5-23) in <u>Mortality Table Construction</u>, ignoring migration The appropriate diagram for Formula (5-20) is as follows:



[5]Greville's Formula (U'), i e , $E_x = A \cdot t$ is correct under the uniform deaths assumption It does not, however, hold under the Balducci hypothesis The Balducci form would be $E_x = A \cdot t + D(1-t)$ since deaths must be exposed to age x+1 even though the endpoint of the period of observation intervenes before the end of the unit age interval

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The exposure along the top diagonal is

$$\frac{1}{4} E_x^z \ ;$$

along the bottom diagonal, we have

$$\frac{3}{4} \left(P_x^z - {}_{\delta}D_x^z\right) + {}_{\delta}D_x^z = \frac{3}{4} P_x^z + \frac{1}{4} \, {}_{\delta}D_x^z$$

by Greville's Formula (U) Adding, we have

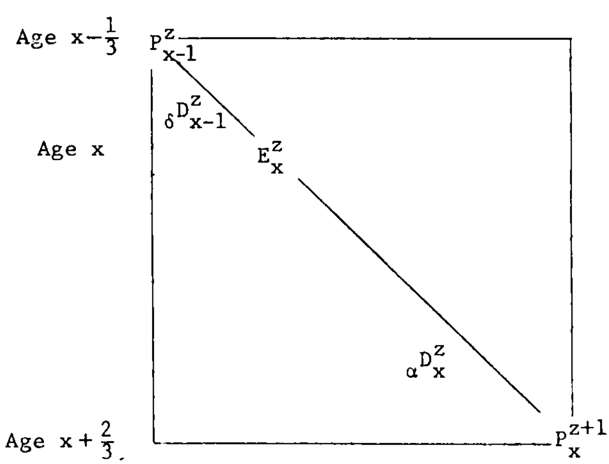
$$E_x\Big]_x^{x+1} = \frac{1}{4} E_x^z + \frac{3}{4} P_x^z + \frac{1}{4} \, {}_{\delta}D_x^z$$

$$= \frac{1}{4} P_x^{z+1} + \frac{3}{4} P_x^z + \frac{1}{4} D_x^z$$

$$= \frac{1}{4} E_x^z + \frac{3}{4} E_{x+1}^z + {}_{\delta}D_x^z$$

The single-diagonal diagram for Formula (5-23) is Figure 5-14 of <u>Mortality Table Construction</u>, shown below with the migration terms omitted

Calendar Year Z



From the upper third of the diagonal, we obtain the exposure

$$\frac{1}{3} P^z_{x-1}$$

From the remainder, we have

$$(E^z_x - {}_\alpha D^z_x)\frac{2}{3} + {}_\alpha D^z_x = \frac{2}{3} E^z_x + \frac{1}{3} {}_\alpha D^z_x$$

Adding, we have

$$E_x\Big]_{x-1/3}^{x+2/3} = \frac{2}{3} E^z_x + \frac{1}{3} P^z_{x-1} + \frac{1}{3} {}_\alpha D^z_x$$

$$= \frac{2}{3} P^{z+1}_x + \frac{1}{3} P^z_{x-1} + {}_\alpha D^z_x$$

$$= E^z_x + \frac{1}{3} D^z_x \ . \qquad \text{(A)}$$

In connection with Formula (A), consider an extreme example in which $P^z_{x-1} = 3$ and ${}_\alpha D^z_x = 3$ The exposure is easily seen to be four units, a quantity which is not an integral number of years for each death nor is it consistent with the earlier supposition that no life may be credited with more than one unit of exposure in a given unit age interval. Suppose next that the three deaths had been of the δ- rather than the α- variety Formula (A) would then produce an exposure of one unit and a corresponding mortality rate of 3.

These examples reflect the rather illusory nature of valuation schedule formulas Although Formula (A) is theoretically correct, the numerical values used in the example were inconsistent with the UDD requirement that ${}_\alpha D^z_x = 2 \cdot {}_\delta D^z_{x-1}$ If we had observed $P^z_{x-1} = 3$, ${}_\delta D^z_{x-1} = 1$, and ${}_\alpha D^z_x = 2$, we would have obtained a total exposure of three units and a mortality rate of unity, a much more satisfying result These observations should not be surprising, as only the seriatim method can accurately reflect the exposure contribution of each life independently Any valuation schedule formula will produce the theoretically desirable results, i.e., reflecting those of the seriatim method, only if the grouped data exactly reflect the underlying mortality assumption. It thus cannot be stated that valuation schedule exposure formulas under the uniform deaths assumption possess the same simple and logical characteristics that are suggested when a seriatim approach is used.

It should also be noted that the consideration of <u>any</u> "ordered pair," as described in <u>Mortality Table Construction</u>, must result in an exposure value between 0 and 1, inclusive, for any unit age interval when an exposure formula of the Balducci variety is applied. Formulas of the UDD type, however, may often suggest an exposure contribution of more than one unit for a specific life, such as for the pair $(P^z_{x-1}, {}_{\alpha}D^z_x)$ in Formula (A). In fact, as will now be demonstrated, the familiar technique of deriving and verifying exposure formulas through the use of ordered pairs loses much of its appeal when the uniform deaths assumption is imposed.

Consider the case in which there are A starters at exact age x and D_1 deaths prior to age x+t, at which age the observation period is terminated, leaving $A - D_1$ enders. Let us further suppose that D_2 of these enders die before attaining age x+1. This produces exposure, under UDD, of

$$E = (A - D_1 - D_2)t + D_1 ,$$

or

$$E = A \cdot t + D_1(1-t) - D_2 \cdot t ,$$

suggesting that the ordered pair (A, D_1) be credited with $t + (1-t)$, or one unit of exposure. This is consistent with the conclusion that all <u>observed</u> deaths be credited, under UDD, with one exposure unit. However, the UDD assumption clearly requires that

$$D_2 = \frac{1-t}{t} \cdot D_1 ,$$

a relationship which transforms the exposure formula into

$$E = A \cdot t .$$

Now it appears that the exposure contribution for the ordered pair (A, D_1) is t instead of 1. Thus, from two formulas, each of which is mathematically sound, the same ordered pair seems to be credited with two different amounts of exposure. The discrepancy can be attributed to the necessity to subtract exposure for the D_2 deaths which occurred after the observation period ended, a unique requirement of the UDD assumption which was discussed earlier in this paper. Since D_2 is a function of D_1, the adjustment for D_2 inherently produced a reduction of (1-t) in exposure for the D_1 lives, explaining the seeming inconsistency. We may conclude from this simple example that analysis of ordered pairs under the uniform deaths assumption should be attempted only with extreme caution.

Conclusion

It is hoped that this paper, along with that of Greville, has shed some light upon the ramifications of the uniform deaths assumption. Notwithstanding this, however, the end result of both papers seems to be that the Balducci hypothesis remains the easier of the two to describe logically and to apply to the development of exposure formulas.

It should also be observed that the theoretical analysis of mortality assumptions should not obscure the fact that, from a practical standpoint, any of the three primary assumptions may be imposed with confidence. The variation in results, when reasonable amounts of data are available, is generally minimal. In fact, it would not be surprising if assumptions as to birth dates, migration points, and issue dates could be shown to be much more critical to accurate results than the choice of assumption as to inter-age mortality patterns.