APPORTIONABLE PREMIUMS

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This is a sequel to my earlier paper on Installment Premiums. There it was shown that if the net premiums $P_x^{[m]}$ and $P^{[m]}(\bar{A}_x)$, for end-of-year and moment-of-death benefit payment respectively, were appropriately defined, and if the premium deduction at death was appropriately calculated, then policies using these schemes were identical to the corresponding annual premium policies with net premiums P_x and $P(\bar{A}_x)$. Consequently the terminal reserves for such corresponding policies are exactly equal: ${}_tV_x^{[m]} = {}_tV_x$ and ${}_tV^{[m]}(\bar{A}_x) = {}_tV(\bar{A}_x)$.

This paper explores the parallel case of apportionable premiums, another situation in which a special adjustment payment at death causes an otherwise somewhat complex policy to be identical with a more simple and familiar one. As with installment premiums, this will result in terminal reserves for the complex policy being exactly equal to those for its corresponding familiar policy.

This problem has been explored by Schar [2], without identifying it as the apportionable premium policy. It is repeated here to stress its parallelism with the installment premium policy, to derive the general m-thly premium case, to present a more straightforward algebraic derivation, and to include the end-of-year benefit case.

As with the installment premium case, if the net premiums

$P_x^{\{m\}}$ and $P^{\{m\}}(\bar{A}_x)$, for end-of-year and moment-of-death benefit payment respectively, are appropriately defined, and if the premium refund is appropriately calculated, then policies using these schemes are identical to the corresponding continuous premium policies with net premiums $\bar{P}_x$ and $\bar{P}(\bar{A}_x)$. Consequently the terminal reserves for such corresponding policies are exactly equal: ${}_tV_x^{\{m\}} = {}_t\bar{V}_x$ and ${}_tV^{\{m\}}(\bar{A}_x) = {}_t\bar{V}(\bar{A}_x)$.

As in the installment premium case, Jordan [1] fails to take interest into account when dealing with the premium refund. Thus he has different definitions for the net premiums, and does not reach the result for reserves stated in the preceding paragraph. The beauty of the result for reserves, of course, is that awkward approximations for apportionable premium policy reserves can be avoided. Reserve factors ${}_t\bar{V}_x$ and ${}_t\bar{V}(\bar{A}_x)$ are readily available from continuous and discrete commutation functions.

II. MOMENT-OF-DEATH BENEFIT CASE

Consider first the moment-of-death benefit case with m=1.

Continuous payment of premium, at annual rate $\bar{P}(\bar{A}_x)$, is physically impossible. Its beginning-of-year lump-sum equivalent is $\bar{P}(\bar{A}_x) \cdot \bar{a}_{\overline{1|}}$, which is the proper definition of $P^{\{1\}}(\bar{A}_x)$, referred to by Scher as the discounted continuous annual premium. Thus the two premium schemes, $\bar{P}(\bar{A}_x)$ and $P^{\{1\}}(\bar{A}_x)$ are equivalent for years the insured lives through. They are not equivalent in the year of death, since we have collected a whole year's premium in $P^{\{1\}}(\bar{A}_x)$, but would have collected continuously up to the point of death only under $\bar{P}(\bar{A}_x)$.

To make the two schemes equivalent in the year of death, there should be a refund at death equal to the value then of what was collected, but would not have been under the $\bar{P}(\bar{A}_x)$ scheme. This amount, the appropriate refund, is $\bar{P}(\bar{A}_x)\cdot\bar{a}_{\overline{1-s}|}$ for death at duration s within that year. If this is done, we should be able to see intuitively that the apportionable scheme and the fully continuous scheme are identical situations, and thus call for identical reserves, namely ${}_t\bar{V}(\bar{A}_x)$.

Proof: For death between x+r and x+r+1, at time s within that year, the value of the refund benefit at issue is

$$\bar{P}(\bar{A}_x)\cdot\sum_{r=0}^{\infty} v^r\,{}_rp_x \int_0^1 v^s\,{}_sp_{x+r}\,\mu_{x+r+s}\cdot\bar{a}_{\overline{1-s}|}\,ds$$

or $$\frac{\bar{P}(\bar{A}_x)}{\delta}\sum_{r=0}^{\infty} v^r\,{}_rp_x \int_0^1 v^s\,{}_sp_{x+r}\,\mu_{x+r+s}(1-v^{1-s})\,ds$$

or $$\frac{\bar{P}(\bar{A}_x)}{\delta}\sum_{r=0}^{\infty} v^r\,{}_rp_x\,[\bar{A}^{\,1}_{x+r:\overline{1}|} - \underbrace{\int_0^1 v\cdot{}_sp_{x+r}\,\mu_{x+r+s}\,ds}_{A^{\,1}_{x+r:\overline{1}|}}]$$

or $$\frac{\bar{P}(\bar{A}_x)}{\delta}\,[\bar{A}_x - A_x].$$

The value at issue of all benefits is $\bar{A}_x + \frac{\bar{P}(\bar{A}_x)}{\delta}[\bar{A}_x - A_x]$, and the value at issue of future net premiums is $P^{\{1\}}(\bar{A}_x)\cdot\ddot{a}_x$.

Then $$P^{\{1\}}(\bar{A}_x)\cdot\ddot{a}_x = \bar{A}_x + \frac{\bar{P}(\bar{A}_x)}{\delta}[\bar{A}_x - A_x]$$

$$= \bar{A}_x + \bar{P}(\bar{A}_x)\cdot\frac{1}{\delta}[1 - \delta\cdot\bar{a}_x - 1 + d\cdot\ddot{a}_x]$$

$$= \underbrace{\bar{A}_x - \bar{P}(\bar{A}_x)\cdot\bar{a}_x}_{0} + \frac{d}{\delta}\cdot\ddot{a}_x\cdot\bar{P}(\bar{A}_x)$$

Thus $P^{\{1\}}(\bar{A}_x) = \bar{P}(\bar{A}_x)\cdot\frac{d}{\delta} = \bar{P}(\bar{A}_x)\cdot\bar{a}_{\overline{1}|}$, which verifies our earlier definition.

The present value of future benefits at duration t is

$$\bar{A}_{x+t} + \bar{P}(\bar{A}_x)\cdot\frac{1}{\delta}\,[\bar{A}_{x+t} - A_{x+t}],$$

and the present value of future premiums at duration t is

$$P^{\{1\}}(\bar{A}_x)\cdot\ddot{a}_{x+t} = \bar{P}(\bar{A}_x)\cdot\bar{a}_{\overline{1}|}\cdot\ddot{a}_{x+t}.$$

Finally the reserve is PVFB - PVFP

$$= \bar{A}_{x+t} + \frac{1}{\delta}\bar{P}(\bar{A}_x)\,[1 - \delta\cdot\bar{a}_{x+t} - 1 + d\cdot\ddot{a}_{x+t}] - \frac{d}{\delta}\bar{P}(\bar{A}_x)\cdot\ddot{a}_{x+t}$$

$$= \bar{A}_{x+t} - \bar{P}(\bar{A}_x)\cdot\bar{a}_{x+t} = {}_t\bar{V}(\bar{A}_x). \qquad \underline{\text{q.e.d.}}$$

The case for m in general is similar but more algebraically intricate. The net premium collected at the beginning of each m^{th} is $\frac{1}{m}\cdot P^{\{m\}}(\bar{A}_x) = \bar{P}(\bar{A}_x)\cdot\bar{a}_{\overline{1/m}|}$. The refund for death at time s within the m^{th} of death is $\bar{P}(\bar{A}_x)\cdot\bar{a}_{\overline{1/m - s}|}$, $0 \le s \le \frac{1}{m}$.

We again wish to show that the reserve is ${}_t\bar{V}(\bar{A}_x)$.

<u>Proof</u>: For death at age $x+r+\frac{k}{m}+s$, the value of the refund benefit at issue is

$$\sum_{r=0}^{\infty} v^r\,{}_rp_x \sum_{k=0}^{m-1} v^{k/m}\,{}_{k/m}p_{x+r}\cdot\bar{P}(\bar{A}_x)\int_0^{1/m} v^s\,{}_sp_{x+r+k/m}\,\mu_{x+r+k/m+s}\cdot\bar{a}_{\overline{1/m - s}|}\,ds.$$

Parallel to the m=1 case, this simplifies to $\frac{1}{\delta}\cdot\bar{P}(\bar{A}_x)\,[\bar{A}_x - A_x^{(m)}]$.

Equating the value at issue of all net premiums to the value of all benefits

$$P^{\{m\}}(\bar{A}_x)\cdot\ddot{a}_x^{(m)} = \bar{A}_x + \frac{1}{\delta}\cdot\bar{P}(\bar{A}_x)[\bar{A}_x - A_x^{(m)}].$$

Making use of the identity $A_x^{(m)} = 1 - d^{(m)}\cdot\ddot{a}_x^{(m)}$, this equation verifies our earlier definition of $P^{\{m\}}(\bar{A}_x)$.

Finally the reserve at duration t is PVFB - PVFP

$$= \bar{A}_{x+t} + \frac{1}{\delta}\cdot\bar{P}(\bar{A}_x)[\bar{A}_{x+t} - A_{x+t}^{(m)}] - \frac{d^{(m)}}{\delta}\cdot\bar{P}(\bar{A}_x)\cdot\ddot{a}_{x+t}^{(m)}.$$

Again using $\bar{A}_{x+t} = 1 - \delta\cdot\bar{a}_{x+t}$ and $A_{x+t}^{(m)} = 1 - d^{(m)}\cdot\ddot{a}_{x+t}^{(m)}$, the reserve expression easily reduces to ${}_t\bar{V}(\bar{A}_x)$. q.e.d.

III. END-OF-YEAR BENEFIT CASE

For the end-of-year benefit case, not discussed by Scher, the derivations are quite analogous to those in Section II. As a result, they will be somewhat abbreviated here.

For m=1, the net premium $P_x^{\{1\}}$ is defined as $\bar{P}_x\cdot\bar{a}_{\overline{1|}}$, and the appropriate refund for death at time s, but paid with the basic benefit at year-end, is $\bar{P}_x\cdot\bar{s}_{\overline{1-s|}}$. We wish to show that ${}_tV_x^{\{1\}} = {}_t\bar{V}_x$.

Proof: The value of the refund benefit at issue is

$\bar{P}_x \sum_{r=0}^{\infty} v^r {}_rp_x \int_0^1 v\cdot {}_sp_{x+r}\mu_{x+r+s}\cdot\bar{s}_{\overline{1-s|}}\,ds$, which reduces to

$\frac{1}{\delta}\cdot\bar{P}_x[\bar{A}_x - A_x]$.

Equating the values of net premiums and all benefits at issue verifies our earlier definition of $P_x^{\{1\}}$. Taking PVFB - PVFP at duration t verifies that the reserve is ${}_t\bar{V}_x$. q.e.d.

For m in general, the net premium collected at the beginning of each m^{th} is $\frac{1}{m} \cdot P_x^{\{m\}} = \bar{P}_x \cdot \bar{a}_{\overline{1/m}|}$, and the refund for death at time s within the m^{th} of death, paid at year-end, is $\bar{P}_x \cdot \bar{s}_{\overline{1/m - s}|}$, further accumulated at interest to year-end. Thus for death at age $x+r+\frac{k}{m}+s$, the refund is $\bar{P}_x \cdot \bar{s}_{\overline{1/m - s}|} \cdot (1+i)^{\frac{m-k-1}{m}}$. We again wish to show that the reserve is ${}_t\bar{V}_x$.

Proof: The value of the refund benefit at issue is

$$\bar{P}_x \sum_{r=0}^{\infty} v^r {}_rp_x \sum_{k=0}^{m-1} v^{\cdot k/m} {}_{k/m}p_{x+r} \int_0^{1/m} {}_sp_{x+r+k/m} \mu_{x+r+k/m+s} \cdot \bar{s}_{\overline{1/m - s}|} \cdot (1+i)^{\frac{m-k-1}{m}} ds,$$

which reduces to $\frac{1}{\delta} \cdot \bar{P}_x [\bar{A}_x - A_x^{(m)}]$, as expected.

Equating the values at issue of net premiums and all benefits verifies our definition of $P_x^{\{m\}}$. Taking PVFB - PVFP at duration t verifies that the reserve is ${}_t\bar{V}_x$. q.e.d.

REFERENCES

1. Jordan, C.W. Life Contingencies, Society of Actuaries: Chicago, 1967.

2. Scher, E. "Relationships Among the Fully Continuous, the Discounted Continuous, and the Semi-Continuous Reserve Bases for Ordinary Life Insurance", TSA, XXVI(1974), 597.