DUALITY BETWEEN UNIFORM DEATHS AND BALDUCCI ASSUMPTIONS

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This paper is motivated by [4] and [2] in which exposure formulas based upon the assumption of the uniform distribution of deaths (U.D.D.) are analysed.

The U.D.D. assumption states that the function ${}_tp_x$ is linear in t for $0 \leq t \leq 1$, whereas the function ${}_{1-t}p_{x+t}$ is assumed to be linear in t under the Balducci hypothesis. Since $p_x = {}_tp_x \; {}_{1-t}p_{x+t}$, we expect a close "duality" relationship exists between these two assumptions. We are surprised by some of the conclusions drawn in [4]: An observed death has an exposure of unity, and an unobserved death has an exposure of zero. (The terminology is due to H. Gershenson; see question 6.5c on p. 185 of [3].) This treatment of deaths is very much different from that using the Balducci assumption which gives to an observed death an exposure equal to the time from the exact age at entry to the end of the year of age, and for an unobserved death there is no special exposure adjustment. The purpose of this paper is to reinterpret the exposure formulas illustrating the duality between the U.D.D. and Balducci assumptions.

The basic problem in the measurement of mortality is the following:

$$\text{Given } \ell_{x+s} \text{ and } \ell_{x+t}, \qquad 0 \leq s < t \leq 1,$$
$$\text{find } q_x.$$

To determine q_x, we first find the values of ℓ_x and ℓ_{x+1}. Thus we assume that certain (invertible) function of $\ell_{x+\tau}$ is linear in τ and extrapolate such a function for the values at $\tau = 0$ and $\tau = 1$. In the U.D.D., Balducci and constant force of mortality assumptions, we hypothesize that the function $\ell_{x+\tau}$ itself is linear, its reciprocal is linear and its logarithm is linear, respectively.

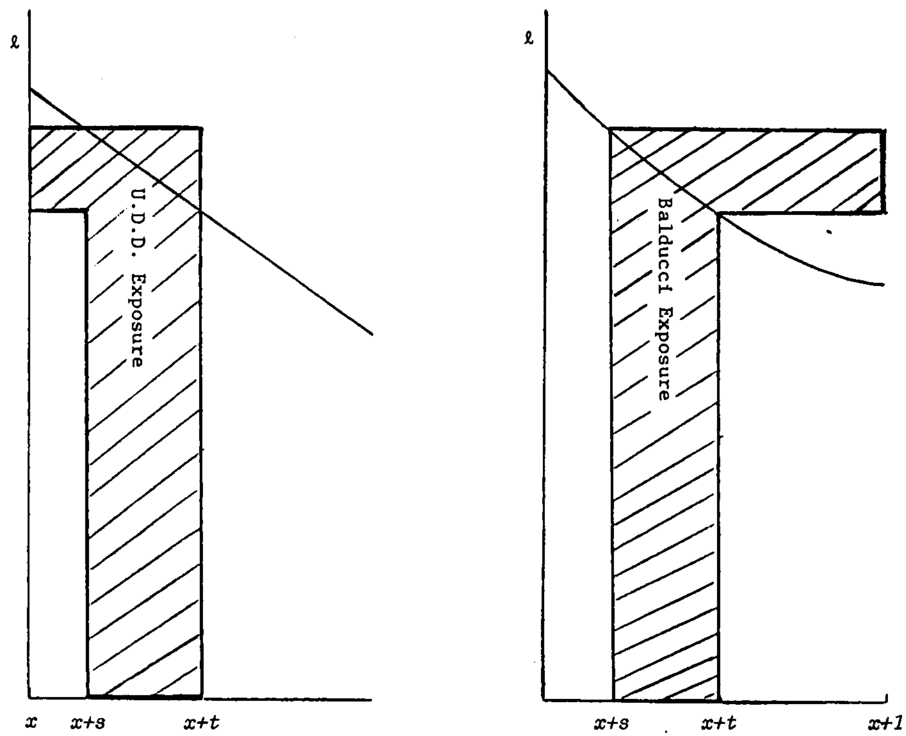
In the Appendix, we shall prove that, under the U.D.D. assumption,

$$q_x = (\ell_{x+s} - \ell_{x+t})/(t\ell_{x+s} - s\ell_{x+t}) \quad (1),$$

and under the Balducci assumption,

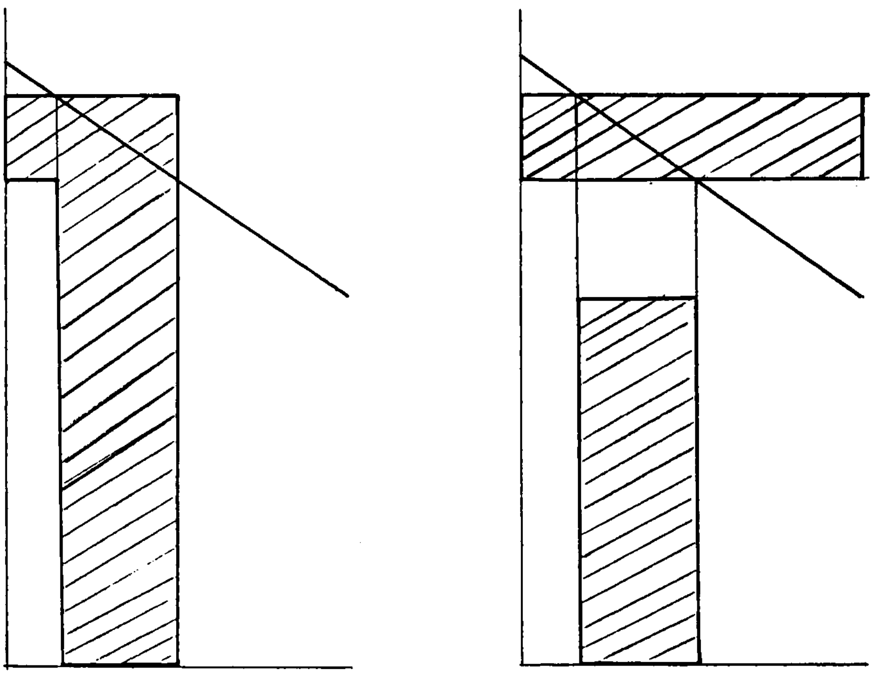
$$q_x = (\ell_{x+s} - \ell_{x+t})/((1-s)\ell_{x+s} - (1-t)\ell_{x+t}) \quad (2).$$

We can interpret the exposures, i.e. the denominators in the formulas, as the shaded areas in the following diagrams.



The duality between these two assumptions is now obvious. The exposure of an observed death is equal to the time from the age at entry to his next birthday under the Balducci assumption, but under the U.D.D. assumption it is the time from his last birthday to the end of the observation period or the next migration point. In this interpretation both assumptions give the same exposure for an unobserved death.

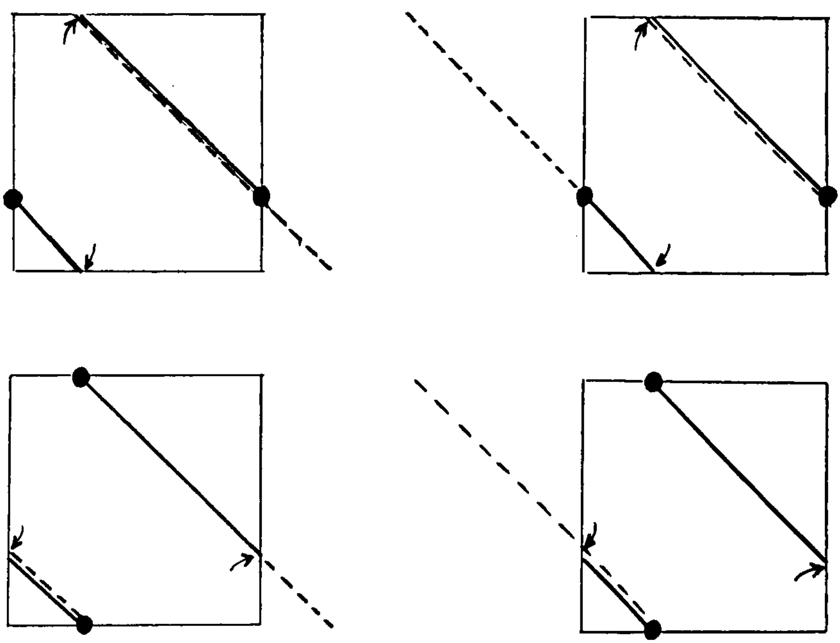
We can reconcile our interpretation with the one given in [3] and [4]. Observe that the following two diagrams have equal shaded areas.



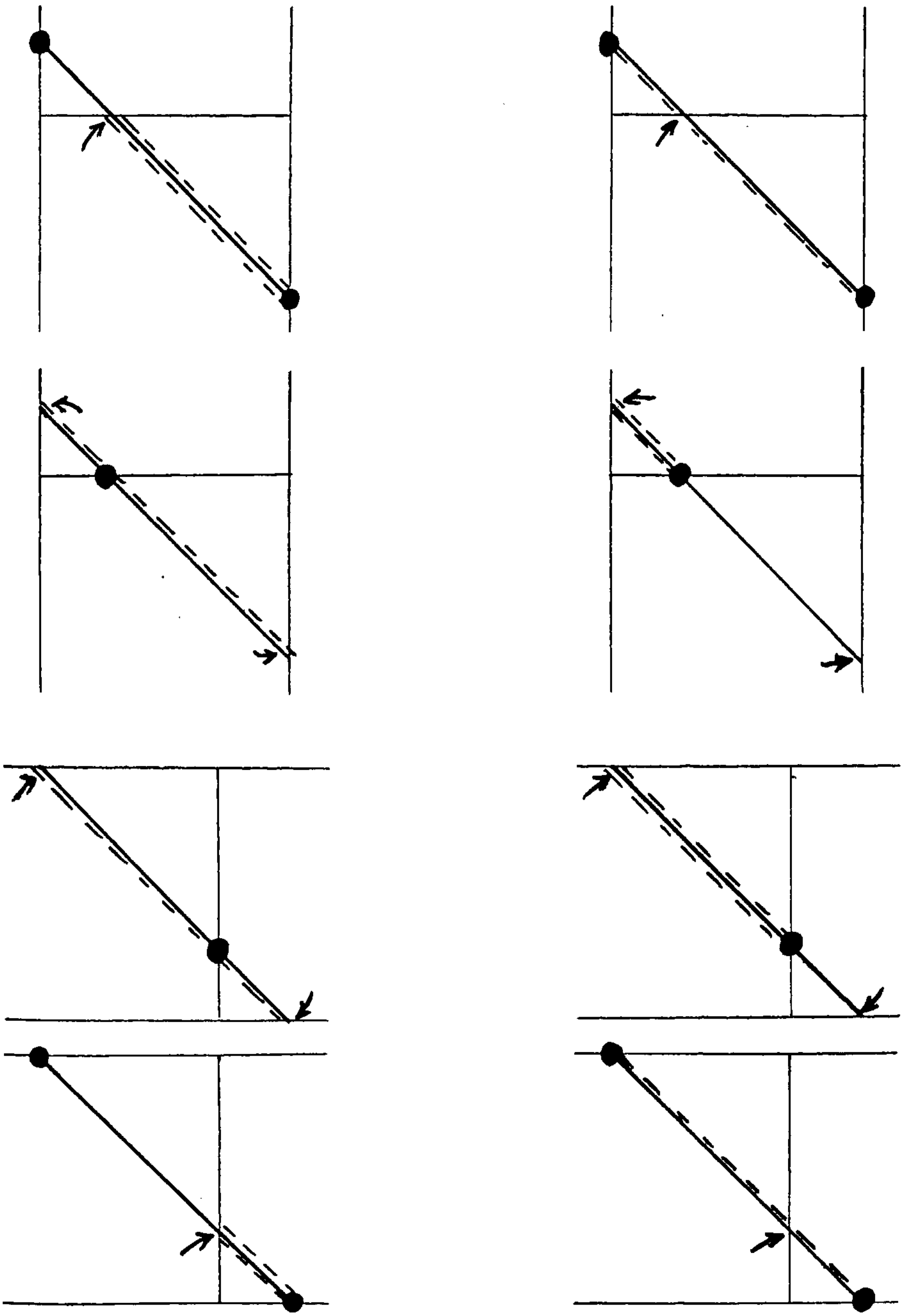
Valuation Schedule Formulas

In the case of the basic valuation schedule exposure formulas, the duality relationship between the Balducci and U.D.D. assumptions has already been observed by Professor R.W. Batten. (See the last paragraph on page 158 of [1].) The following diagrams illustrate the four exposure formulas given in Table 5-6 on page 159 of [1]. The arrows indicate the migration points, the round dots the census points and the dotted lines the exposure adjustments on the deaths.

BALDUCCI U.D.D.



The duality relationship between these two assumptions is clearly seen in the above diagrams. Similar diagrams for the exposure formulas in Tables 5-7 and 5-8 are sketched on the next page.



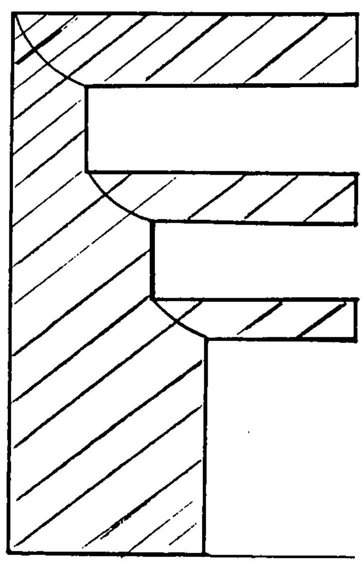
To derive a valuation schedule exposure formula, we first partition the diagonal(s) into segments. Each segment corresponds to a "closed group", i.e., it begins from a migration point or census point and ends at the next migration point or census point. For each of these segments the exposure is given by our method immediately. Then the required exposure is the sum of the segment exposures since we have the following simple

PROPOSITION. If $q = \frac{DEATHS_i}{EXPOSURE_i}$, $i = 1,2,3,\ldots,$

$$\text{then } q = \frac{\Sigma_i\ DEATHS_i}{\Sigma_i\ EXPOSURE_i} .$$

Proof. If $\frac{a}{b} = \frac{c}{d}$ and $a \neq -c$,

$$\text{then } \frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d} .$$



Appendix

To simplify notation, we shall write ℓ for ℓ_x, ℓ_t for ℓ_{x+t}, etc. We wish to prove two results:

$$\begin{aligned} &\text{If} \quad \ell_s = (1-s)\ell + s\ell_1 \\ &\text{and} \quad \ell_t = (1-t)\ell + t\ell_1 \quad , \\ &\text{then} \quad q = (\ell_s - \ell_t)/(t\ell_s - s\ell_t) \end{aligned} \tag{1}$$

$$\begin{aligned} &\text{If} \quad 1/\ell_s = (1-s)/\ell + s/\ell_1 \\ &\text{and} \quad 1/\ell_t = (1-t)/\ell + t/\ell_1 \quad , \\ &\text{then} \quad q = (\ell_s - \ell_t)/((1-s)\ell_s - (1-t)\ell_t) \end{aligned} \tag{2}$$

These two equations can be derived by high school algebra. However, we observe that to compute q, we do not need to know the individual values of ℓ and ℓ_1; what we need to know is merely the value of the ratio ℓ_1/ℓ. This leads us to consider the ℓ's as "homogeneous coordinates".

LEMMA. Suppose $\frac{u}{v} = \frac{aw + bx}{cw + dx}$ and $\frac{w}{x} = \frac{ey + fz}{gy + hz}$.

Then (i) $\frac{u}{v} = \frac{iy + jz}{ky + lz}$, where $\begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$;

(ii) $\frac{w}{x} = \frac{du - bv}{-cu + av}$, if $ad \neq bc$.

The proof of (i) is straightforward. Conclusion (ii) follows from (i) immediately if we note that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \Big/ (ad - bc) \quad .$$

PROOF of (1)

Under the U.D.D. assumption, we have

$$\frac{\ell_s}{\ell_t} = \frac{(1-s)\ell + s\ell_1}{(1-t)\ell + t\ell_1} \quad .$$

$$\text{Since} \quad q = \frac{\ell - \ell_1}{\ell - 0\ell_1} \quad \text{and}$$

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t & -s \\ -(1-t) & 1-s \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ t & -s \end{pmatrix} ,$$

$$\text{we have} \quad q = \frac{\ell_s - \ell_t}{t\ell_s - s\ell_t} \quad .$$

PROOF of (2)

Under the Balducci assumption, we have

$$\frac{1/\ell_s}{1/\ell_t} = \frac{(1-s)/\ell + s/\ell_1}{(1-t)/\ell + t/\ell_1} \quad ,$$

$$\text{or} \quad \frac{\ell_t}{\ell_s} = \frac{s\ell + (1-s)\ell_1}{t\ell + (1-t)\ell_1} \quad .$$

$$\text{Since} \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1-t & -(1-s) \\ -t & s \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1-t & -(1-s) \end{pmatrix} ,$$

$$\text{we have} \quad q = \frac{\ell_t - \ell_s}{(1-t)\ell_t - (1-s)\ell_s} \quad .$$

REFERENCES

[1] R.W. Batten, Mortality Table Construction. Prentice-Hall, Inc., 1978.

[2] ___________, Some further comments on the uniform deaths assumption, ARCH, 1978.2 issue, 57-64.

[3] H. Gershenson, Measurement of Mortality. Society of Actuaries, 1961.

[4] T.N.E. Greville, Estimation of the rate of mortality in the presence of in-and-out movement, ARCH, 1978.2 issue, 41-56.