A REVISION OF THE MINIMUM R_z THEOREM

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In this note, we rearrange the statement and the proof of a theorem in Greville [1,p.13,pp.65-70]. The presentation appears to be more compact; the materials, however, are not new.

THEOREM: Among moving weighted average formulas of the form

$$u_x = \sum_{s=-n}^{n} a_s \, u''_{x+s} \tag{1}$$

and being exact for quadratics, there is a unique formula with minimum R_z. Its weights $\{ \hat{a}_s \}$ are given by

$$\begin{aligned} \hat{a}_s &= [(n+1)^2-s^2][(n+2)^2-s^2]\cdots[(n+z)^2-s^2](a+bs^2) \qquad (2) \\ &= q(s)(a+bs^2) \end{aligned}$$

where a and b are determined uniquely by

$$\sum_{s=-n}^{n} q(s)(a+bs^2) = 1 \quad , \tag{3}$$

$$\sum_{s=-n}^{n} q(s)(a+bs^2)\, s^2 = 0 \, . \tag{4}$$

Since $\{ \hat{a}_s \}$ is symmetric, the formula is exact for cubics. It has minimum R_z among exact for quadratics formulas and consequently among exact for cubics formulas.

Proof:

The system (3) and (4) fails to have a unique solution if and only if the left hand sides of (3) and (4) are linearly dependent. That is, for some $\alpha\neq0$, $\beta\neq0$,

$$0 = \alpha \sum_{s=-n}^{n} q(s)(a+bs^2) + \beta \sum_{s=-n}^{n} q(s)(a+bs^2)\, s^2$$

$$= \sum_{s=-n}^{n} q(s)(a+bs^2)(\alpha+\beta s^2)$$

for all a, b. With $a=\alpha$, $b=\beta$,

$$\sum_{s=-n}^{n} q(s)(\alpha+\beta s^2)^2 = 0$$

implies $\alpha = \beta = 0$ since $q(s) > 0$ for $|s| \leq n$. Thus the system (3) and (4) has a unique solution.

We adopt the convention that $a_s=0$ for $|s| > n$ and change the limits of summation to $-\infty$ and ∞. For any formula of the form (1) and being exact for quadratics, consider

$$\Sigma\, (\Delta^z a_s)^2 = \Sigma\, [\, \Delta^z \hat{a}_s + \Delta^z(a_s-\hat{a}_s)\,]^2$$
$$= \Sigma\, (\Delta^z \hat{a}_s)^2 + \Sigma\, (\Delta^z(a_s-\hat{a}_s))^2 + 2\, \Sigma\, \Delta^z a_s \cdot \Delta^z(a_s-\hat{a}_s)\ .$$

Apply summation by parts [2,(6.14)] z times to obtain

$$\sum_{s=-\infty}^{\infty} \Delta^z \hat{a}_s\ \Delta^z(a_s-\hat{a}_s)$$

$$= [\, \Delta^z \hat{a}_{s-1}\ \Delta^{z-1}(a_s-\hat{a}_s)\,]_{-\infty}^{\infty} - \sum_{s=-\infty}^{\infty} \Delta^{z+1}\hat{a}_{s-1}\ \Delta^{z-1}(a_s-\hat{a}_s)$$

$$= (-1)^z \sum_{s=-\infty}^{\infty} (\Delta^{2z}\hat{a}_{s-z})\ (a_s-\hat{a}_s)\ .$$

This last quantity vanishes because $\deg(\Delta^{2z}\hat{a}_{s-z}) = 2z+2-2z = 2$ and both $\{ a_s \}$ and $\{ \hat{a}_s \}$ come from exact for quadratics formulas. Thus for $\{ a_s \}$ to have minimum R_z, we must have $\Delta^z(a_s-\hat{a}_s) = 0$ for all s. When z=0, this implies $a_s = \hat{a}_s$ for all s. When $z > 0$, since $a_s=\hat{a}_s=0$ on $\pm(n+1)$, $\pm(n+2)$, ··· $\pm(n\pm z)$, $a_s-\hat{a}_s$, a polynomial in s of degree at most z-1, must be identically zero. This shows that the minimum R_z formula is unique. Q.E.D.

REFERENCES

1. Greville, T.N.E., *Graduation*, Society of Actuaries, Chicago, 1973.

2. Kellison, S.G., *Fundamentals of Numerical Analysis*, Irwin, Homewood, Illinois, 1974.