RECENTLY PUBLISHED U.K. MORTALITY TABLES

METHODS OF CONSTRUCTION AND POSSIBLE DEVELOPMENTS THEREFROM

by

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1. Introduction

Last year (1979) saw the publication of the most recent complete national life tables for England and Wales and for Scotland. By courtesy of the U.K. Government Actuary at an early date the crude data underlying these tables were made available to the present author, who with Dr J C Eilbeck carried out extensive graduation experiments. This work led ultimately to a graduation by cubic splines, which was adopted for the officially published tables. The use of splines as an actuarial tool is perhaps not all that widespread and in §3 below we describe their application to graduation. In §4 we suggest a possible improvement of the methods used for the English Life Tables No. 13 and give some illustrations of our ideas.

Earlier this year two new sets of tables, of considerable importance to the life insurance industry in the U.K., have been published by the Continuous Mortality Investigation Committee of the Institute and Faculty of Actuaries. These new tables relate to the mortality of immediate annuitants (under 'ordinary' business) and to that of pensioners (under 'group' schemes). It is expected that these new tables will be widely used for premium calculations and as valuation bases. Two particular features of these tables of special interest are the method of graduation and the procedure whereby allowance has been made for possible secular improvements in mortality. The graduation method adopted, a curve-fitting exercise based on maximum likelihood, is described in §6 below and the projection basis for improvement in mortality is considered in §7. These topics have been the subject of much recent discussion by the actuarial profession in the U.K. and it is hoped that North American actuaries will find them of interest in relation to their own work. Although these new tables (the PA(90) for pensioners and a(90) for annuitants are the responsibility of the entire Continous Mortality Investigation Committee (of which the present author is a member), particular mention must be made of Mr A D Wilkie, whose contribution to the work involved was especially large.

2. Splines

A spline is a piecewise polynomial function for which the maximum possible number of derivatives exist. More precisely, suppose that $a = x_o < x_1 < \ldots . < x_n < x_{n+1} = b$. A spline s of order k, defined on the interval $[a,b]$ with 'internal knots' $x_1, \ldots . , x_n$, is a function such that, if $o \leqslant i \leqslant n$ and $x_i \leqslant x \leqslant x_{i+1}$, then $s(x) = p_i(x)$ where $p_i(x)$ is a polynomial in x of degree k. Moreover the polynomials $p_o(x), p_1(x), \ldots . , p_n(x)$ fit together in such a manner that s is differentiable (k-1) times throughout the interval (a,b).

It is this last condition which distinguishes splines from other piecewise polynomial functions. Between knots, of course, the differentiability condition is automatically satisfied by each defining polynomial, so that the condition simply restricts the manner in which adjacent polynomials are joined together at the knots.

A simplistic (and not in practice the most efficient) way of constructing such a spline is as follows. The polynomial $p_o(x)$ of degree k on the interval $[x_o, x_1]$ is defined by (k+1) parameters. When these parameters have been fixed, the polynomial is determined. In particular the function value and the values of the first (k-1) derivatives are defined at the internal knot x_1. These must also be the value of $p_1(x)$ and its first (k-1) derivatives at x_1. If we now specify one further parameter for the polynomial $p_1(x)$ - say the value of its k^{th} derivative at x_1 - the function $p_1(x)$ is completely defined on the interval $[x_1, x_2]$. We may now repeat this argument, this time at the second internal knot x_2. By this process we are able to 'extend' our spline to the entire interval $[a,b]$. In addition to the original (k+1) parameters used to define the spline on $[x_o, x_1]$, for $1 \leqslant i \leqslant n$ we are able to choose one further parameter to extend the spline to the interval $[x_i, x_{i+1}]$. Thus, to define a spline of order k with n internal knots, a total of (n+k+1) parameters is needed.

Piecewise polynomial interpolation has been used by actuaries for many years (cf. King's method of osculatory interpolation). The application of splines to graduation and to data analysis has been studied by several authors - (references 1 and 2 are detailed works) , so that the English Life Tables No. 13 may be considered an application of earlier ideas.

Although in theory it is possible to work with splines of any order, for many practical applications cubic splines (i.e. splines of order 3) are an excellent tool. Such functions are twice-differentiable and, if suitably defined, have minimum-curvature properties. A cubic spline over the interval $[a,b]$ with n internal knots is defined by (n+4) parameters. In practice the spline may be defined by its values at the end points and at the internal knots (i.e. n+2 points in all) together with the values at the end points of either its first or second derivatives. If the second derivative is zero at each the end point, the function is called a 'natural' spline.

3. The Use of Splines to Graduate Mortality Rates

Suppose that $b = a+m-1$ and that we have m 'data' points $(a,y(a))$, $(a+1,y(a+1))$, ..., $(b,y(b))$. Suppose further that it is required to fit a cubic spline as closely as possible to the data. The spline is to have n specified internal knots x_1, ..., x_n. (We discuss how the knots may be chosen in §4 below.) As we have remarked above, the spline is defined by n+4 suitably chosen parameters. We write $s(x;\lambda_1,\lambda_2,....,\lambda_{n+4})$ to denote the value at x of the spline determined by the particular parameter set $\lambda_1,\lambda_2,....,\lambda_{n+4}$. (The parameters $\lambda_1,....,\lambda_{n+4}$ need not be those described in §2 above. For example, in practice they will probably be the coefficients in a B-spline representation of our function. (See references 1 and 2.) The important point to note is that the value of $s(x;\lambda_1,\lambda_2,....,\lambda_{n+4})$, although cubic in x, is a <u>linear</u> expression in $\lambda_1,\lambda_2,....,\lambda_{n+4}$. If $w(a),....,w(b)$ are a given set of weights, we define the best-fitting spline to be that for which

$$\sum_{x=a}^{x=b} w(x) . [y(x)-s(x;\lambda_1,....,\lambda_{n+4})]^2 \tag{3.1}$$

has a minimum value. Thus our best-fit spline is that which has a minimum weighted least square error. Since $s(x;\lambda_1,....,\lambda_{n+4})$ is linear in $\lambda_1,....,\lambda_{n+4}$, the expression (3.1) is a quadratic form in these parameters and it is a relatively simple task to obtain the parameter set corresponding to the minimum value.

Thus, given the crude data points on the interval $[a,b]$, the weights $w(a),\ldots,w(b)$, and a particular choice of internal knots, we are able to determine a unique cubic spline which yields the best fit according to the above definition.

In mortality analysis the available data often consist of exposures to risk and observed numbers of deaths, from which crude death rates can be found at each age. Suppose that for $x = a,a+1,\ldots,b$ the exposure to risk is denoted by E_x and the number of deaths by θ_x. The quotient θ_x/E_x is the crude death rate at age x, say q'_x. The 'true' death rate at age x is denoted by q_x. The number of deaths arising from an exposure of E_x is regarded as a binomially distributed variable with mean $E_x q_x$ and variance $E_x q_x(1-q_x)$. If the exposures and number of deaths are sufficiently large, we may regard the corresponding standardised variable as having a normal distribution. On this basis, if we have independence at successive ages, the distribution of

$$\chi^2 = \sum_a^b \left[\frac{\theta_x - E_x q_x}{\sqrt{E_x q_x(1-q_x)}}\right]^2$$

$$= \sum_a^b \frac{(\theta_x - E_x q_x)^2}{E_x q_x(1-q_x)} \tag{3.2}$$

will be closely approximated by that of a χ^2 variable.

This is well known and is repeated here solely for completeness. Several graduation methods are based on the minimisation of the above expression. We may write

$$\chi^2 = \sum_a^b \frac{E_x}{q_x(1-q_x)} \cdot \left(\frac{\theta_x}{E_x} - q_x\right)^2$$

$$= \sum_a^b w(x) . \quad (q'_x - q_x)^2 \quad , \tag{3.3}$$

where

$$w(x) = \frac{E_x}{q_x(1-q_x)} \quad . \tag{3.4}$$

Equation 3.3 shows that χ^2 may be regarded as a weighted sum of the squares of the differences between the crude and the true mortality rates. From this viewpoint, however, the weights $w(x)$ depend on the true rates.

For a given set of internal knots the graduated rates q_x are taken to be the values of that cubic spline which leads to the smallest possible value for χ^2. In its simplest form - with the n internal knot positions not determined directly by the data (see below) - a deduction of $(n+4)$ must be made from the number of degrees of freedom in certain statistical tests, to allow for the number of parameters which have been fitted.

One complication compared with our earlier discussion is to be found in the fact that the weights $w(x)$ in 3.3 above themselves depend on the spline. We allow for this by an iterative method. An initial set of weights $w^{(o)}(x)$ is chosen somewhat arbitrarily and the best-fitting spline is found by minimising 3.3 above with $w(x) = w^{(o)}(x)$. The values of q_x from this spline are then substituted in 3.4 to obtain a new set of weights, say $w^{(1)}(x)$. These new weights are then used in 3.3 (i.e. we put $w(x) = w^{(1)}(x)$) to find a revised best-fitting spline, from which yet another set of weights can be found by 3.4. This iterative process may be continued until the values of successive splines are equal at every age (to within any desired degree of accuracy), at which point the latest spline values are taken as the graduated rates. In practice only a few iterations (between 3 and 6 in most cases) are needed .

4. A Possible Criterion for the Knot Positions

Two important aspects of the spline graduation method are (i) the number and (ii) the position of the internal knots. At present we have no definite way of determining the "best" number of knots to use. Practical experience shows that in some situations, if there are too few knots, then there is no possibility of obtaining an acceptable graduation. On the other hand, if the number of knots is excessive, the spline will adhere too closely to the crude rates and in fact there will be little graduation. This matter is perhaps worth further study. (See below for a possible line of attack.) In the construction of the English Life Tables the knots were chosen by dividing the entire age-range (2-95) into two subintervals (2-17 and 17-95 for males; 2-23 and 23-95 for females) - to allow for the pattern of mortality rates in the late teens and early twenties - and then by inserting equally-spaced knots over each of these subintervals (cf. references 3 and 8). This was a somewhat arbitrary procedure. In view of our remarks above, the following method of determining the knot positions may be considered a better (and, with hindsight, rather obvious!) approach.

Suppose that $n \geqslant 1$ and that $a < x_1 < .... < x_n < b$. Let $z(x_1,....,x_n)$ denote the minimum value of the expression 3.2 above, when the rates q_x are given by a cubic spline on $[a,b]$ with knots $x_1,....,x_n$. We now consider $z(x_1,....,x_n)$ as a function of the knot positions. The criterion which we suggest is that the knot positions themselves be chosen to minimise $z(x_1,....,x_n)$. Thus, the number of knots having been fixed as n, we define the graduated rates by the minimum-χ^2 n-knot cubic spline on $[a,b]$. Theoretically it is not obvious that the criterion does in fact lead to a unique spline, but in practice this seems always to be the case - although in unusual circumstances it may be necessary to allow two or more of the knots to become coincident (in which case there is a reduction in the number of derivatives which exist). The choice of the knot positions thus becomes a problem of constrained minimisation - the constraints arising since the knots must lie within the interval $[a,b]$. Since, for a given set of knots the evaluation of $z(x_1,....,x_n)$ is relatively simple, with a modern computer the solution of such a problem is not too formidable a task even when the number of knots is quite large. (The fact that the knots must be in ascending order causes little difficulty.) It should be noted that with this criterion we have (2n+4) parameters available to fit our curve - first we have n parameters to choose for the knot positions and then a further (n+4) parameters to determine the spline based on these knots.

It is worth pointing out that a sensible choice of the knot positions can lead to a striking reduction in the number of knots required. The method described above for the English Life Table had 20 knots. Yet, for example, a smaller χ^2 can be obtained with only 12 knots, suitably chosen.

As the number of knots increases, the minimum χ^2 decreases and this may provide some clue as to how many knots should be chosen. We may ask whether or not the addition of a further knot leads to a 'significant' reduction in the minimum χ^2. However this is something which we have not yet had time to consider.

As an illustration of some of the above points, we consider the crude data underlying the English Life Table (No. 13) Males over the age range 2 to 30 inclusive. (We have chosen this range of ages simply for purposes of illustration.) In figure 1 we show the crude death rates q'_x and the best cubic spline fits with (i) 3 and (ii) 6 knots. The 3-knot spline has χ^2=92.95 with a <u>repeated</u> knot at x =16.62 and a further knot at x = 23.06. The 6-knot spline has χ^2 = 20.95 with knots at x = 5.67, 15.36, 15.38, 16.38, 16.50 and 23.98. It is obvious from these figures that over the age range in question no satisfactory graduation can be obtained with only 3 knots.

5. Curve-fitting by formula

As an alternative to the approach using splines, we may instead try to fit a specific mathematical formula to the crude mortality rates - or to some associated function. (The celebrated laws of Makeham and Gompertz are simple illustrations of this.) Much more complicated formulae may be used. For example, in the English Life Tables 11 and 12 the values of m_x, the central death rate at age x, were graduated by a seven-parameter formula which combined logistic and normal curves in the form

$$m_x = \alpha_1 + \alpha_2 [1+\exp\{-\alpha_3(x-\alpha_4)\}]^{-1} + \alpha_5 \exp\{-\alpha_6(x-\alpha_7)^2\}. \tag{5.1}$$

This somewhat unusual expression was chosen after close scrutiny of the pattern of the crude rates, the values of the coefficient $\alpha_1, \ldots, \alpha_7$ being determined by a trial-and-error method (cf. reference 10).

More generally we may suppose that the true death rates q_x are given by an equation of the form

$$q_x = q(x;\underline{\alpha}) \quad , \tag{5.2}$$

where $\underline{\alpha}$ is the vector of formula coefficients. Two criteria then come to mind as obvious possible ways of finding the 'best' choice for $\underline{\alpha}$. These are the methods of minimum χ^2 and maximum likelihood. The former approach has been discussed in §3 above. For our present purpose we regard equation 3.2 as defining

$$\chi^2 = \chi^2(\underline{\alpha}) \tag{5.3}$$

and choose $\underline{\alpha}$ to minimise this expression. In practice for a particular formula under consideration it is usually simple to calculate the partial derivatives $\frac{\partial q_x}{\partial \alpha_i}$ ($1 \leqslant i \leqslant n$, where n is the number of formula coefficients), so that using 3.2 above we can find not only $\chi^2(\underline{\alpha})$ but also $\frac{\partial \chi^2}{\partial \alpha_i}$ for each value of i. With these partial derivatives and the power of a modern computer the minimisation problem can be solved rapidly in most circumstances.

The maximum likelihood approach is particularly suited to the determination of formula coefficients when exposures and deaths are available as described in §3 above. The likelihood function to be maximised is

$$L = \prod_{x=a}^{x=b} \binom{E_x}{\theta_x} \cdot (q_x)^{\theta_x} \cdot (1-q_x)^{E_x-\theta_x} . \tag{5.4}$$

Determining the maximum of L is equivalent to finding that of log L. Since

$$\log L = \sum_a^b \left[\log \left\{ \binom{E_x}{\theta_x} \right\} + \theta_x \log q_x + (E_x-\theta_x)\log(1-q_x) \right]$$

and first term in this last expression does not depend on q_x, the function to be maximised may be taken as

$$L^*(\underline{\alpha}) = \sum_a^b [\, \theta_x \log q_x + (E_x-\theta_x)\log(1-q_x)] \quad . \tag{5.5}$$

Again with any particular formula for q_x it is generally easy to find the values of $L^*(\underline{\alpha})$ and its partial derivatives with respect to the formula coefficients. This means that the vector $\underline{\alpha}$ which maximises L^* can be found fairly quickly.

Although the two methods are theoretically quite distinct, in practical terms the minimum χ^2 and maximum likelihood approaches ususally produce very similar graduations. Obviously one vital aspect of either method is the choice of the particular formula to be fitted to the crude rates. The age range over which the curve is being fitted is often an important consideration here and the pattern of the crude rates must be studied carefully.

The maximum likelihood method was used for the annuitant and pensioner mortality tables referred to in §1 above. In our next section we discuss the application of this method to these tables, with reference to the particular formula used and some interesting consequences thereof.

6. The basic formula of the PA(90) and a(90) tables

These tables were based on the experience of life office pensioners under group schemes and that of immediate annuitants under ordinary business. For pensioners the data were available on both a 'lives' and 'amounts' basis, further subdivided according to whether or not retirement took place early (i.e. before normal retirement age).

Although in this paper we are not concerned with the detailed construction of these tables, it is perhaps worth pointing out that after much discussion the annuitants' table was produced with a one-year select period while that for the pensioners was an ultimate table based on the 'amounts' data for normal retirements. (The interested reader should refer to references 4, 5, 6, and 7 for a very detailed account of these and other matters.)

In view of the relatively short age range covered by the available data, it was felt that a simple formula might be found as a basis for the graduation. Consideration was given to the pattern of the crude rates and to various related functions. We show in figure 2 the values of $\log\left(\frac{q'_x}{1-q'_x}\right)$ according to the 'lives' data for male pensioners retiring at or after the normal retirement age. (As before, q'_x denotes the crude mortality rate at exact age x. Because of the manner in which data are collected, the values of x are 50½, 51½,...., 99½. An asterisk is used to denote an age at which there were no deaths.)

The points on the graph of figure 2 lie, roughly speaking, close to a straight line. Accordingly the graduation formula was based on the equation

$$\log\left(\frac{q_x}{1-q_x}\right) = \text{pol}(x) \quad , \tag{6.1}$$

where pol(x) is a low order (and, in the final event, linear) polynomial in x. This last equation may be rewritten as

$$q_x = \frac{e^{pol(x)}}{1+e^{pol(x)}} \tag{6.2}$$

$$= 1 - \frac{1}{1+e^{pol(x)}} \tag{6.3}$$

One convenient feature of the formula 6.2 is that, whatever the values of the coefficients occurring in pol(x), the graduated rates must necessarily lie between 0 and 1. For a more general formula this last condition may impose constraints upon the region in which the optimal vector $\underline{\alpha}$ must lie and in certain circumstances such constraints might make the maximisation problem more difficult.

Using the particular formula under consideration together with equations 6.2 and 6.3 above, we may substitute in equation 5.5 to obtain in this case

$$L^*(\underline{\alpha}) = \sum_a^b [\,\theta_x(pol(x) - \log\{1+e^{pol(x)}\}) - (E_x - \theta_x)\log\{1+e^{pol(x)}\}]$$

$$= \sum_a^b [\,\theta_x pol(x) - E_x \log\{1+e^{pol(x)}\}] \quad , \tag{6.4}$$

from which it follows that

$$\frac{\partial L^*}{\partial \alpha_i} = \sum_a^b \left[\theta_x - \frac{E_x \cdot e^{pol(x)}}{1+e^{pol(x)}}\right] \frac{\partial pol(x)}{\partial \alpha_i}$$

$$= \sum_a^b [\,\theta_x - E_x q_x] \frac{\partial pol(x)}{\partial \alpha_i} \quad . \tag{6.5}$$

If our formula has n coefficients, we may write

$$pol(x) = \sum_{i=1}^{n} \alpha_i x^{i-1} \quad , \tag{6.6}$$

in which case

$$\frac{\partial pol(x)}{\partial \alpha_i} = x^{i-1} \qquad (1 \leqslant i \leqslant n). \tag{6.7}$$

Since the maximum likelihood solution must have each of the partial derivatives $\frac{\partial L^*}{\partial \alpha_i}$ equal to zero, by combining equations 6.5 and 6.7 we see that for this solution

$$\sum_a^b \theta_x . x^{i-1} = \sum_a^b E_x . q_x . x^{i-1} \qquad (1 \leqslant i \leqslant n). \tag{6.8}$$

This last equation (when i=1) shows that our solution will produce the same totals for the 'actual' and the 'expected' deaths. In addition each of the first (n-1) moments about the origin of the actual deaths will equal the corresponding moment of the expected deaths. This equality of the moments is, of course, a consequence of the particular formula being used and does <u>not</u> hold in general for the maximum likelihood method.

In fact for the graduations in question the polynomial pol(x) - in 6.1 above - was expressed in terms of a transformed variable

$$t = \frac{x-70}{50}$$

and the Chebyshev polynomials $C_i(t)$ - ($C_0(t)=1$, $C_1(t)=t$, $C_2(t)=2t^2-1$, etc.) - in the form

$$\text{pol}(x) = \sum_{i=1}^{n} \alpha_i C_{i-1}(t) \quad . \tag{6.9}$$

The advantages of using such an expression lie in the fact that the values of the coefficients $\{\alpha_i\}$ are a convenient order of magnitude and change only to a moderate extent as the degree of the polynomial is increased.

As an illustration of the above ideas we consider the best maximum likelihood quadratic fit to the 'lives' data for male pensioners. In this example the formula used is

$$\log\left(\frac{q_x}{1-q_x}\right) = \alpha_1 + \alpha_2\left(\frac{x-70}{50}\right) + \alpha_3\left[2\left(\frac{x-70}{50}\right)^2 - 1\right] \tag{6.10}$$

The values of the coefficients obtained by the maximum likelihood method described above are $\alpha_1 = -3.156928$, $\alpha_2 = 4.286465$, and $\alpha_3 = -0.187533$. The graph of the curve given by the right hand side of equation 6.10 with these particular coefficients is shown in figure 2.

It is perhaps of interest for purposes of comparison to consider the minimum χ^2 graduation by equation 6.10 above. For this method the values of the coefficients are $\alpha_1 = -3.101088$, $\alpha_2 = 4.264721$ and $\alpha_3 = -0.131656$. The rates determined by these coefficients are, of course, very similar to those arising from the first graduation. This is borne out by the fact that while the minimum value of χ^2 is 77.32 the χ^2 value from the maximum likelihood graduation is only 77.51.

Experiments were carried out not only with a quadratic expression for pol(x) but also with a linear and a cubic form. For each of the sets of 'normal' (i.e. not 'early') retirement data the linear formula produced graduations not significantly worse than the quadratic and for only one set of such data did the cubic lead to a significant improvement. In the final event the maximum likelihood method with a linear polynomial was used for all graduations (cf. references 2 and 3). This choice of formula has interesting consequences when allowance is made for possible improvements in mortality. This is discussed in the following section, where we give a brief description of the construction of the tables actually published.

7. Allowance for possible improvement in mortality

The data to which the above discussion relates are obtained from the experience of U.K. life offices over the four years 1967 to 1970. The basic graduated rates at age x refer on average to a life attaining age x in 1968.

For tables likely to be widely used for premium and valuation purposes relating to annuity benefits it is obviously prudent to make some allowance for possible improvement in mortality with the passage of time. One well-known method, commonly used for projection of mortality rates, is to assume that the rates decrease geometrically with time, so that

$$q_{x,t} = (r_x)^t \cdot q_{x,o} \qquad (0 < r_x < 1) \qquad (7.1)$$

where $q_{x,t}$ denotes the mortality rate at age x for a life attaining this age in calendar year t (measured from some suitable base year) and r_x is the 'mortality improvement factor' at age x. For many practical purposes it is often assumed that the improvement factors do not vary with age, in which case

$$q_{x,t} = r^t . q_{x,o} \quad . \tag{7.2}$$

In view of the graduation method used, for the pensioner and annuitant tables it was decided to allow for mortality improvement by the equation

$$\frac{q_{x,t}}{1-q_{x,t}} = r^t . \frac{q_{x,o}}{1-q_{x,o}} \quad . \tag{7.3}$$

(It should be noted that in general, for a given value of r, the projected rates $q_{x,t}$ arising from equation 7.3 above will be very close to those arising from equation 7.2, The difference in the two methods of projection will be noticeable only at advanced ages, say where $q_{x,o}$ is ½ or more.)

As described in §6 above the graduated rates in the base year (1968) were obtained by the formula

$$\log\left(\frac{q_{x,o}}{1-q_{x,o}}\right) = \alpha_1 + \alpha_2 \left(\frac{x-70}{50}\right) . \tag{7.4}$$

Note that equation 7.3 above implies

$$\log\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = t.\log r + \log\left(\frac{q_{x,o}}{1-q_{x,o}}\right) . \tag{7.5}$$

By combining our last two equations we obtain

$$\log\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = t.\log r + \alpha_1 + \alpha_2 \left(\frac{x-70}{50}\right)$$

$$= \alpha_1 + \alpha_2 \left[\frac{x- \frac{50}{\alpha_2} .\log\{\frac{1}{r}\}.t-70}{50}\right]$$

$$= \alpha_1 + \alpha_2 \left(\frac{x-\lambda t-70}{50}\right) , \tag{7.6}$$

where

$$\lambda = \frac{50}{\alpha_2} \log \{\frac{1}{r}\} \quad . \tag{7.7}$$

The right-hand side of equation 7.6 is of course the value of

$$\log \left\{ \frac{q_{x-\lambda t,o}}{1-q_{x-\lambda t,o}} \right\} .$$

Hence it is a consequence of the projection method used (i.e. equation 7.3 above) and the graduation formula that

$$q_{x,t} = q_{x-\lambda t,o} \quad , \tag{7.8}$$

where λ is defined by equation 7.7 above.

Thus the mortality rate at age x in year t equals the rate at age $(x-\lambda t)$ in the base year. Loosely speaking, therefore, we may say that each year's advance in time corresponds to a deduction of λ from the age. Equivalently we may say that an advance of $\frac{1}{\lambda}$ years in time corresponds to a deduction of 1 year from the age. (Of course it would be quite possible to replace r^t in equation 7.3 by r_x^t, in which case λ - in equations 7.7 and 7.8 - would be changed to λ_x. On practical grounds, however, it may be difficult to justify this refinement.) If the mortality rates of the base year are given by a mathematical formula or spline, the rates applicable to any future year can easily be found from equation 7.8.

When the allowance for mortality improvement to be incorporated in the published tables was under consideration, careful attention was paid to past trends. Strictly speaking it follows from equation 7.7 that λ depends on both α_2 and r. Since, however, any method of projection can be at best a realistic estimate of future trends, it is wrong to infer a greater degree of accuracy than in reality exists. As many factors as possible having been considered (cf. reference 5), it was decided to take $\lambda = \frac{1}{20}$ for both male and female pensioners and annuitants. This choice of the same value of λ for all the tables does in fact imply a slightly greater rate of improvement in mortality for females than for males. Roughly speaking, however, this value of λ implies that mortality rates will decrease by about 10% every 20 years.

In determining the mortality rates to be used for the new tables, the Continuous Mortality Investigation Committee considered (among other methods) using the mortality rates of one particular generation or those applicable in some specified future year. As far as the former method is concerned the following simple argument is of some interest.

Let $q_x^{(n)}$ denote the mortality rate at age x for a life born in calendar year n (where the base year - in our case 1968 - is taken as year 0). Since such a life will attain age x in calendar year n+x, using our earlier notation, we have

$$q_x^{(n)} = q_{x,x+n} \quad .$$

Now suppose that the calendar year of birth N cohort is taken as a basis. Then

$$q_x^{(N+t)} = q_{x,x+N+t}$$

$$= q_{x-\lambda(x+N+t),o} \qquad (7.9)$$

(by 7.8) above).

Similarly it follows that

$$q_{x-\beta}^{(N)} = q_{x-\beta,x-\beta+N}$$

$$= q_{x-\beta-\lambda(x-\beta+N),o} \quad . \qquad (7.10)$$

The right-hand sides of equations 7.9 and 7.10 are equal if

$$x-\lambda(x+N+t) = x-\beta-\lambda(x-\beta+N)$$

which is equivalent to

$$\beta = \frac{\lambda}{1-\lambda} \cdot t \qquad (7.11)$$

Combining this last equation with equations 7.9 and 7.10 above we see that

$$q_x^{(N+t)} = q_{x-\mu t}^{(N)} \tag{7.12}$$

where

$$\mu = \frac{\lambda}{1-\lambda} \quad . \tag{7.13}$$

(Note that, if $\lambda = \frac{1}{n}$, then $\mu = \frac{1}{n-1}$.)

Equation 7.12 shows that - with the projection method described above - the mortality rates for any particular generation can be obtained from those of some base generation by making an age deduction of μ for each year by which the year of birth of the given cohort is later than the year of birth of the base cohort. Note also that equation 7.10 (with β suitably chosen) provides the link between the mortality rates of a particular generation and those applicable in the base calendar year.

The mortality rates upon which the new tables are based are in fact the projected rates applicable to the calendar year 1990. This is the reason for the tables being given the names PA(90) and a(90) - for Pensioners (based on Amounts data) and for annuitants.

For practical reasons it was considered desirable to extend the tables to younger ages where the data are virtually non-existent. Extrapolation to the younger ages was carried out by a somewhat ad hoc method, based on the mortality of assured lives and that of the general population. Precise details are outwith the scope of the present paper and the interested reader should refer to reference 2 for a full account of the methods employed.

It is obviously essential to know reasonably accurately the financial consequences for annuity and pensions business of different rates of improvement in mortality. At the time of construction of the new tables described above extensive experiments were carried out relating to possible alternative projection bases (cf. references 4 and 6). As these experiments may be of some general interest, we give below an abridged form of certain of our earlier results.

Although the basic projection equation 7.3 does not in fact imply a constant geometric rate of decrease in mortality at all ages, for practical purposes the projected values are virtually the same as those which arise from the assumption that male mortality rates decrease (at a constant annual rate) by about 8% every 20 years and those for females by about 10% over the same period. For purposes of illustration we give below figures which arise from the base year (1968) mortality rates on alternative assumptions of constant annual rates of decrease in mortality at all ages. We have assumed instead that every 20 years mortality rates will decrease by (i) 5% or (ii) 15%. In terms of equation 7.1 above these alternative models are given by (i) $r_x^{20} = .95$ or (ii) $r_x^{20} = .85$ (for all values of x).

Given the base year mortality rates, an annuity value applicable to any individual in the future depends on the age at entry, the calendar year of entry, the mortality improvement rate, and the rate of interest. (The annuity value must be calculated using the appropriate "generation" mortality table.) We consider entry dates from 1980 to 2000 at intervals of 5 years and interest rates of 0%, 5%, 10% and 15%. For single lives (male) and joint lives (male/female) we have calculated annuity values and expressed the results as an appropriate <u>percentage</u> adjustment to the values on the published table (which, for purposes of illustration we have taken as the PA(90)). A percentage addition means that the published table understates the 'true' annuity value and a percentage subtraction means that there is some safety margin in the tabulated value. The results of our calculations are given in tables 1,2 and 3 below.

TABLE 1

PA(90) Annuity Values (Single-life male : Joint-life male and female)

Interest rate p.a.	a_{60}	a_{65}	a_{70}	$a_{65:60}$	$a_{65:65}$	$a_{\overline{65:60}}$	$a_{\overline{65:65}}$
0%	17.512	14.093	11.063	12.018	10.966	24.099	21.040
5%	10.607	9.149	7.669	8.189	7.658	13.364	12.392
10%	7.296	6.561	5.736	6.049	5.748	8.670	8.305
15%	5.463	5.039	4.528	4.734	4.545	6.247	6.089

TABLE 2

Male Pensioners

(Percentage addition to PA(90) values of specified function)

Improvement basis	Function	Entry Date 1980	1985	1990	1995	2000
$r_x^{20} = 0.95$ (all x)	a_{60}	-1.3	-0.7	-0.0	0.6	1.3
		-1.1	-0.7	-0.3	0.1	0.5
		-0.9	-0.6	-0.3	-0.1	0.2
		-0.8	-0.5	-0.3	-0.1	0.1
	a_{65}	-1.8	-1.1	-0.4	0.4	1.1
		-1.5	-1.0	-0.5	0.0	0.5
		-1.2	-0.9	-0.5	-0.1	0.2
		-1.1	-0.8	-0.5	-0.2	0.1
	a_{70}	-2.3	-1.5	-0.7	0.1	1.0
		-1.9	-1.3	-0.7	-0.1	0.5
		-1.6	-1.2	-0.7	-0.2	0.2
		-1.4	-1.0	-0.7	-0.3	0.1
$r_x^{20} = 0.85$ (all x)	a_{60}	6.1	8.3	10.6	12.8	15.1
		3.1	4.4	5.7	7.0	8.3
		1.6	2.5	3.4	4.2	5.1
		1.0	1.6	2.2	2.8	3.4
	a_{65}	5.7	8.2	10.7	13.2	15.8
		3.1	4.8	6.4	8.0	9.6
		1.8	3.0	4.1	5.2	6.4
		1.2	2.0	2.9	3.7	4.5
	a_{70}	5.3	8.1	10.9	13.8	16.7
		3.2	5.2	7.1	9.1	11.1
		2.0	3.5	5.0	6.4	7.9
		1.4	2.5	3.7	4.9	6.0

TABLE 3

Pensioners (Joint-life Functions : Male/Female)

Percentage addition to PA(90) values of specified function

Improvement basis	Function	Entry Date 1980	1985	1990	1995	2000
$r_x^{20} = 0.95$ (all x)	$a_{65:65}$	-2.9	-2.1	-1.3	-0.6	0.2
		-2.3	-1.8	-1.2	-0.6	0.0
		-1.9	-1.5	-1.0	-0.6	-0.1
		-1.6	-1.3	-0.9	-0.6	-0.2
	$a_{65:60}$	-2.5	-1.8	-1.0	-0.3	0.4
		-2.0	-1.5	-1.0	-0.4	0.1
		-1.7	-1.3	-0.9	-0.4	0.0
		-1.4	-1.1	-0.8	-0.4	-0.1
	$a_{\overline{65:65}}$	-1.4	-0.8	-0.3	0.3	0.9
		-1.0	-0.6	-0.3	0.0	0.4
		-0.7	-0.5	-0.3	-0.1	0.1
		-0.5	-0.3	-0.2	-0.1	0.0
	$a_{\overline{65:60}}$	-1.1	-0.6	-0.1	0.4	0.9
		-0.8	-0.5	-0.2	0.1	0.4
		-0.5	-0.3	-0.2	0.0	0.1
		-0.3	-0.2	-0.2	-0.1	0.0
$r_x^{20} = 0.85$ (all x)	$a_{65:65}$	4.1	6.6	9.2	11.9	14.6
		2.5	4.3	6.2	8.1	9.9
		1.6	3.0	4.4	5.8	7.2
		1.0	2.2	3.3	4.4	5.5
	$a_{65:60}$	4.5	7.0	9.5	12.1	14.7
		2.7	4.4	6.2	7.9	9.7
		1.7	3.0	4.3	5.5	6.8
		1.1	2.1	3.1	4.1	5.1
	$a_{\overline{65:65}}$	5.6	7.6	9.6	11.6	13.6
		2.8	3.9	4.9	6.0	7.1
		1.4	2.0	2.6	3.2	3.8
		0.7	1.1	1.5	1.8	2.2
	$a_{\overline{65:60}}$	5.8	7.5	9.3	11.0	12.8
		2.6	3.5	4.4	5.3	6.1
		1.2	1.7	2.2	2.6	3.0
		0.6	0.9	1.1	1.4	1.6

From these tables we see, for example, that if allowance is made for a 15% reduction in mortality rates every 20 years (i.e. $r^{20} = 0.85$), then for a life attaining age 70 in 1990 the necessary percentage addition to the PA(90) (males) value of a_{70} will be 10.9, 7.1, 5.0 or 3.7 depending on whether the annual interest rate is 0%, 5%, 10% or 15% respectively. On the same mortality improvement basis for the male/female last survivor annuity $a_{\overline{65:60}}$ with entry in 1990 the necessary addition to the published PA(90) annuity value ranges from 9.3% to 1.1% as the annual interest rate varies from 0% to 15%.

8. A spline graduation of the pensioners' data

If the age range spanned by the available data is large, it may prove impossible to obtain a relatively simple formula which provides a satisfactory curve-fitting tool. In such circumstances the use of splines with a few well-chosen knots may be the best method of graduation. On the other hand, if the age range is somewhat limited (as happens for the pensioner and annuitant data described above) a properly chosen formula may yield on excellent graduation. As we have discussed in §6 above, this was the case for the PA(90) and a(90) tables, for which graduations were made by a formula with only 2 parameters.

Nevertheless it is of interest to compare the various methods, particularly to ascertain whether or not the spline technique can lead to a better graduation. Accordingly we consider briefly again the 'lives' data for male pensioners (for which the crude rates of mortality are plotted in figure 2). Since the age range is only 50.5 to 99.5, before attempting a fit by a spline (a piecewise cubic), we graduate the rates by the minimum χ^2 method, using one cubic formula, valid over the entire age range.

Initially, therefore, we consider an equation of the form

$$q_x = \sum_i \alpha_i C_{i-1} \left(\frac{x-70}{50}\right) , \qquad (8.1)$$

where, as before, C_r denotes the r^{th} order Chebyshev polynomial. (We use this particular expression - rather than a more obvious polynomial form - simply to be consistent with our earlier notation.)

It proved impossible to obtain an acceptable graduation with a linear or quadratic form of 8.1 above but a satisfactory fit was given by the cubic formula, for which $\alpha_1 = 0.239697$, $\alpha_2 = 0.450892$, $\alpha_3 = 0.190900$, and $\alpha_4 = 0.083751$. The value of χ^2 for this graduation is 78.01. (Recall that the minimum-χ^2 two-parameter fit by equation 6.2 above gave $\chi^2 = 77.32$, while the corresponding maximum likelihood fit had $\chi^2 = 77.51$.) Given that our cubic has 4 parameters and does not produce a smaller χ^2 value than either of these earlier formulae, we conclude that a simple cubic is not an especially good graduation method in this case.

Finally we consider, as an alternative to the simple cubic, a cubic spline graduation with one internal knot. Using the criterion suggested in §4 above, we find the optimal knot position to be $x_1 = 73.77$. Not surprisingly, when the pattern of the crude rates is examined, this knot position is close to the mid-point of the age interval. For this one-knot spline graduation the value of χ^2 is 74.56 - a deduction of 5.45 from the value given by the simple cubic. This reduction in χ^2 is not particularly significant and close examination of the detailed graduations shows that for the data in question the spline graduation is really no better than that by the single cubic formula. A graph of the spline with one knot is given in figure 3.

The above remarks simply confirm that the most opportune use of a spline graduation may be when the age range is great or when the data are in some sense unusual - in either of which situations it may prove impossible to find a reasonably simple graduation formula.

We hope that the above discussion, although basically simple, will be of some general interest.

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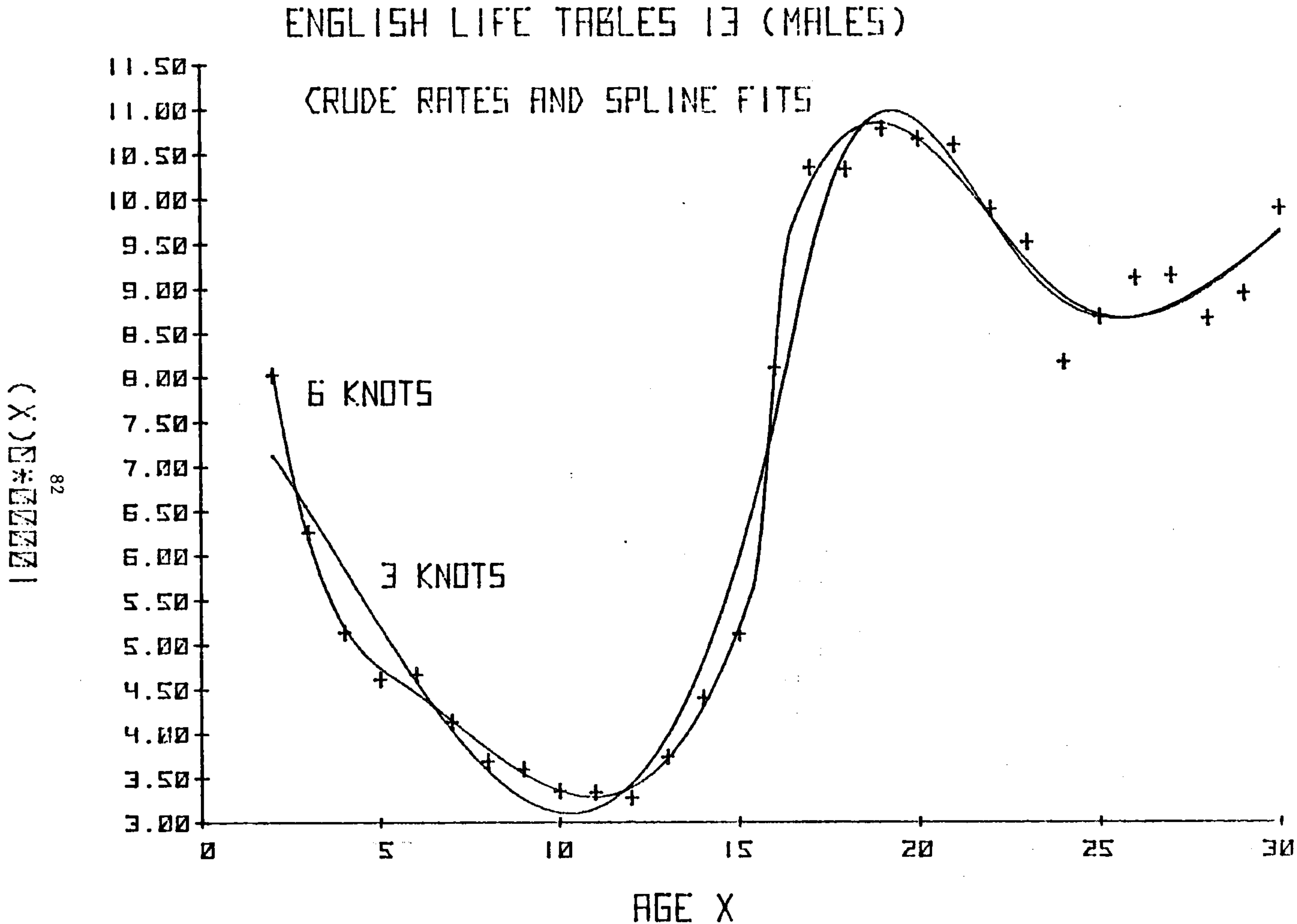
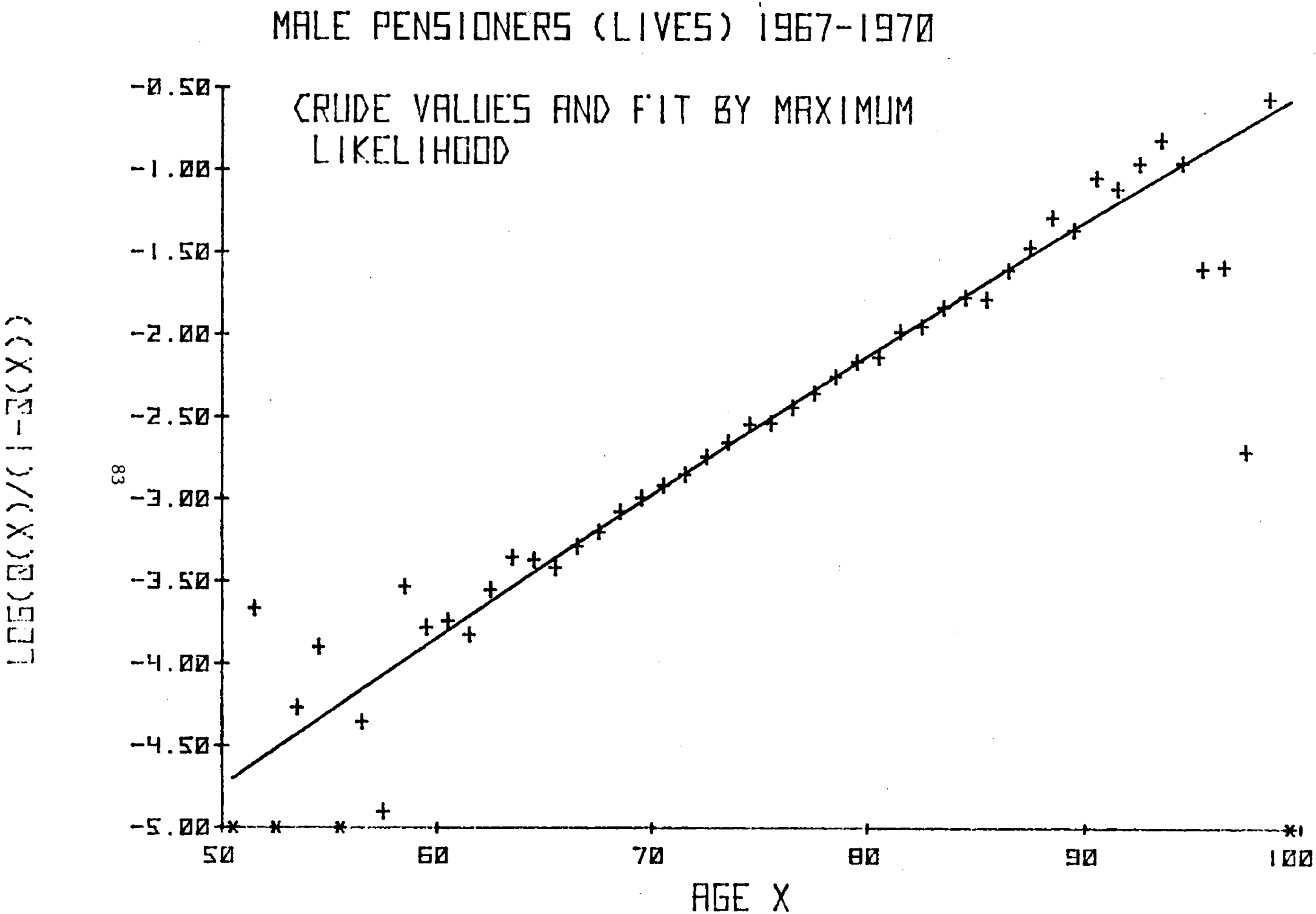
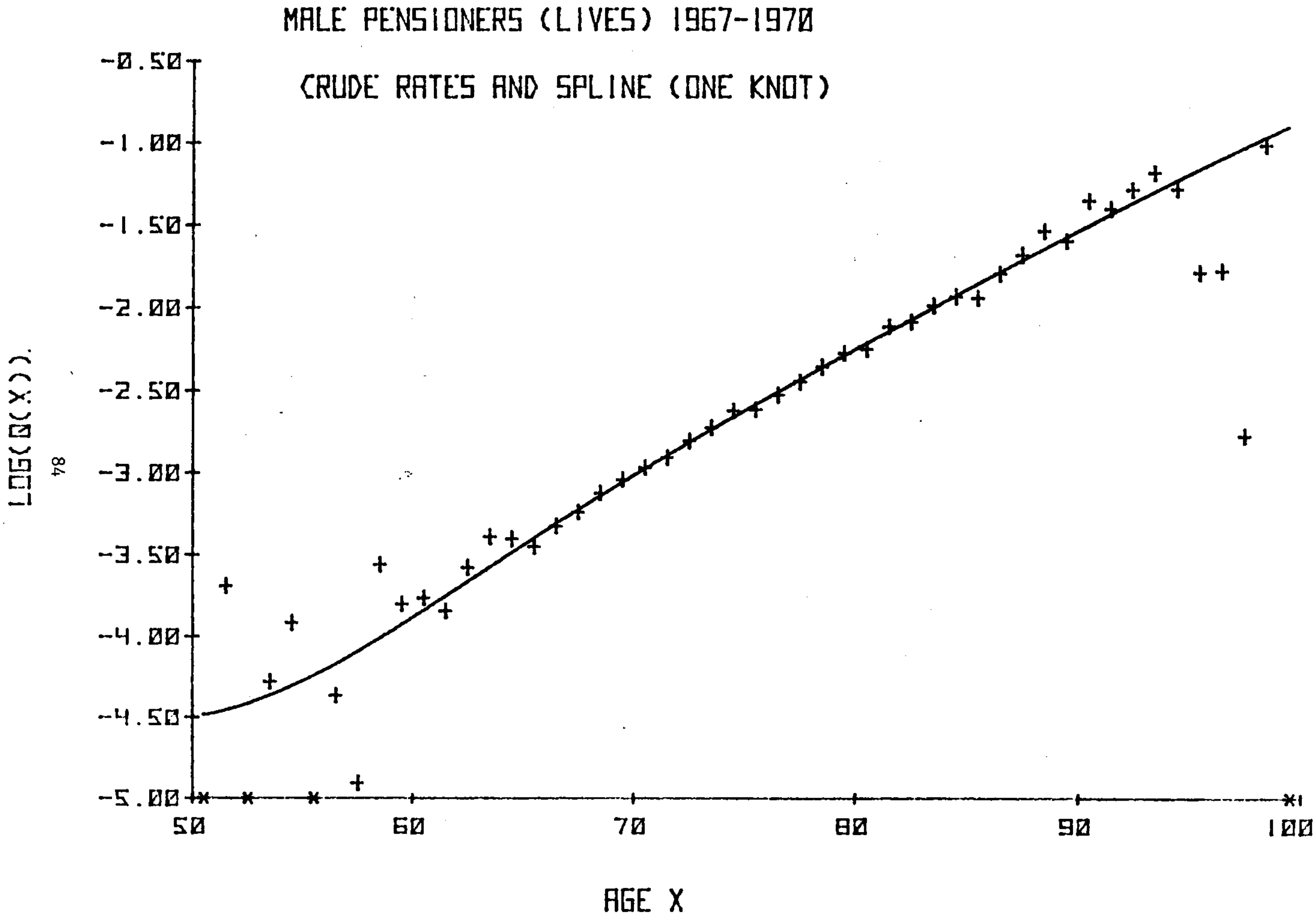

ENGLISH LIFE TABLES 13 (MALES)
CRUDE RATES AND SPLINE FITS
6 KNOTS
3 KNOTS
1000*Q(X)
AGE X
11.50
11.00
10.50
10.00
9.50
9.00
8.50
8.00
7.50
7.00
6.50
6.00
5.50
5.00
4.50
4.00
3.50
3.00
0
5
10
15
20
25
30

(Figure 1)


MALE PENSIONERS (LIVES) 1967-1970
CRUDE VALUES AND FIT BY MAXIMUM LIKELIHOOD
-0.50
-1.00
-1.50
-2.00
-2.50
-3.00
-3.50
-4.00
-4.50
-5.00
50
60
70
80
90
100
AGE X
LOG(Q(X)/(1-Q(X)))

83

(Figure 2)


MALE PENSIONERS (LIVES) 1967-1970
CRUDE RATES AND SPLINE (ONE KNOT)
-0.50
-1.00
-1.50
-2.00
-2.50
-3.00
-3.50
-4.00
-4.50
-5.00
LOG(Q(X)).
50
60
70
80
90
100
AGE X

(Figure 3)

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