On a note by R. Garfield concerning the separation of symbols

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In a recent note Ralph Garfield [1] brought up a problem which seems to exist in using the method of separation of symbols. In contrast with his conviction we believe that the operator relations proved by operational calculus are practically useful, of course if they are interpreted in the correct sense.

In the method of separation of symbols the operators are manipulated as algebraic quantities, independently of the function on which they are operating. Although this technique is very powerful for the derivation of the formulas themselves it does not furnish an expression for the relevant remainder term. This makes that one has to be very cautious in the applications by suitably restricting the class of operands so that the result of the operation will have the correct meaning.

In this context it should be noticed that the operators can be split up in two classes whether they reduce the degree of a polynomial or not. For the operators which we will use in the sequel this leads to the following classification

a) delta operators

forward difference operator : $\Delta f(x) = f(x+h) - f(x)$

backward difference operator : $\nabla f(x) = f(x) - f(x-h)$

central difference operator : $\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$

derivative operator : $Df(x) = \frac{df(x)}{dx}$

b) not delta operators

shifting operator : $Ef(x) = f(x+h)$

From this classification it is clear why formulas for numerical differentiation are generally expressed in terms of the delta operators Δ, ∇ and δ. Indeed, when the formulas

$$D = \frac{1}{h} \ln(1 + \Delta) = \frac{1}{h} [\Delta - \frac{1}{2} \Delta^2 + \frac{1}{3} \Delta^3 - \dots] \tag{1}$$

$$D = - \frac{1}{h} \ln(1 - \nabla) = \frac{1}{h} [\nabla + \frac{1}{2} \nabla^2 + \frac{1}{3} \nabla^3 + \dots] \tag{2}$$

$$D = \frac{2}{h} \sinh^{-1} \frac{\delta}{2} = \frac{1}{h} [\delta - \frac{1}{24} \delta^3 + \frac{3}{640} \delta^5 - \dots] \tag{3}$$

are applied to a polynomial the corresponding series is finite so that the expansion of operation symbols is justified. However, these formulas do not depend on whether the operand is a polynomial or not (except the remainder term). Therefore the expansion into infinite series will also have meaning for non polynomials provided that the corresponding series is convergent (more precisely if the remainder term converges to zero.)

As an example we apply formula (1) to a^x

$$Da^x = \frac{1}{h} \sum_{n=1}^{\infty} (-1)^{n-1} a^x (a^h - 1)^n$$

where h is choosen so that $0 < a^h \leqslant 2$.
This gives

$$Da^x = \frac{1}{h} a^x \ln a^h = a^x \ln a$$

which is of course the desired result.

Now we consider - following S.G. Kellison [3, problem 33 p. 167] - the relationship between the operators D and E. By the technique of separation of symbols we obtain

$$\begin{aligned} D &= \frac{1}{h} \ln E \\ &= \frac{1}{h} [\ln(1+E) - \ln(1+E^{-1})] \\ &= \frac{1}{h} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{E^n - E^{-n}}{n} \end{aligned} \tag{4}$$

In his note R. Garfield [1] was surprised that this expansion is not valid for polynomials. In fact this has nothing surprising since the operator E does not reduce the degree of a polynomial. Apart from the trivial case f(x) = constant, the expansion (4) is in general an infinite series which makes that the convergence properties have to be examined. In a more general way it can be noticed that the problem of R. Garfield is closely related to the problem of convergence by adding an rearranging series.

In order to show that the expansion (4) makes sense we determine a class of operands to which (4) can be applied. We will show the formula to be valid for the class of functions

$$f(x) = \int_{-\pi}^{\pi} e^{itx} g(t)dt$$

By applying (4) we obtain

$$Df(x) = \frac{1}{h} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{E^n - E^{-n}}{n} \int_{-\pi}^{\pi} e^{itx} g(t)dt$$

$$= \frac{1}{h} \int_{-\pi}^{\pi} g(t)dt \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} [e^{it(x+nh)} - e^{it(x-nh)}]$$

$$= \frac{2i}{h} \int_{-\pi}^{\pi} e^{itx} g(t)dt \sum_{n=1}^{\infty}(-1)^{n-1} \frac{\sin tnh}{n}$$

Making use of the well known series expansion (see e.g. [2, p. 38])

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin an}{n} = \frac{a}{2} \qquad -\pi < a < \pi$$

we find the desired result

$$Df(x) = \int_{-\pi}^{\pi} it\, e^{itx} g(t)dt$$

From all this we can conclude that the operator relations proved by the technique of separation of symbols are practically useful, provided that they are applied to the appropriate class of operands.

References

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