ON THE DIFFERENCE BETWEEN BALDUCCI AND U.D.D.

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The selection of a law of mortality is fundamental to mortality table construction. Crucial to this choice, is the fact that for most ages x, the difference between the values of ${}_{1-t}q_{x+t}$ for $0 \le t \le 1$ under the various assumptions is negligible. Unfortunately, the standard texts do not treat this point fully. In [1], page 190, it is pointed out that the U.D.D. curve is the chord of the Balducci curve for $0 \le t \le 1$ and therefore, the two curves should reasonably close to each other. In [2], page 16, this problem is explicitly discussed, but the resolution consists some numerical examples in the exercises of Chapter 1 and a numerical example in the introduction to Chapter 2. The purpose of this note is to give an inequality which is true for all t $(0 \le t \le 1)$ and x. Further relations between U.D.D. and Balducci are developed and discussed.

Theorem 1: ${}_{1-t}q_{x+t}^{\text{U.D.D.}} - {}_{1-t}q_{x+t}^{\text{Balducci}} \le q_x^2$ for all $0 \le t \le 1$ and all x.

Proof: $${}_{1-t}q_{x+t}^{\text{U.D.D.}} = \frac{(1-t)\cdot q_x}{1-t\cdot q_x} = \frac{{}_{1-t}q_{x+t}^{\text{Balducci}}}{1-t\cdot q_x}$$

Thus, $${}_{1-t}q_{x+t}^{\text{U.D.D.}} - {}_{1-t}q_{x+t}^{\text{Balducci}} = t\cdot q_x \cdot {}_{1-t}q_{x+t}^{\text{U.D.D.}} \le t\cdot q_x^2 \le q_x^2.$$

This inequality is sufficient for many pedagogical purposes; the next theorem sharpens this inequality and lays the foundation for some related results.

Theorem 2: $${}_{1-t}q_{x+t}^{\text{U.D.D.}} - {}_{1-t}q_{x+t}^{\text{Balducci}} \le \frac{q_x^2}{(1+\sqrt{p_x})^2}$$ for $0 \le t \le 1$ and all x.

Remark: Roughly, for most of the life table $\sqrt{p_x} \doteq 1$ and the upper bound approaches $\frac{1}{4}\cdot q_x^2$.

Proof: Let $\lambda = \ell_{x+t}^{U.D.D.} - \ell_{x+t}^{Balducci} = \ell_x - t \cdot d_x - \frac{\ell_x \ell_{x+1}}{\ell_{x+1} + t \cdot d_x}$

Then, $\frac{d\lambda}{dt} = 0 = -d_x + \frac{\ell_x \cdot \ell_{x+1} \cdot d_x}{(\ell_{x+1} + T \cdot d_x)^2}$, where $T = t_{MAX}$.

$$(\ell_{x+1} + T \cdot d_x)^2 = \ell_x \cdot \ell_{x+1} \qquad (1)$$

and, $$T = \frac{-\ell_{x+1} + \sqrt{\ell_x \ell_{x+1}}}{d_x} \qquad (2)$$

Insert equations (1) and (2) into the equation for λ:

$$\lambda_{MAX} = \ell_x + \ell_{x+1} - 2\sqrt{\ell_x \ell_{x+1}} = (\sqrt{\ell_x} - \sqrt{\ell_{x+1}})^2 \qquad (3)$$

Hence, $\lambda \le (\sqrt{\ell_x} - \sqrt{\ell_{x+1}})^2 = \ell_x (1 - \sqrt{p_x})^2 = \frac{\ell_x \, q_x^2}{(1+\sqrt{p_x})^2}$

Thus, $$\ell_{x+t}^{U.D.D.} - \ell_{x+t}^{Balducci} \le \frac{\ell_x q_x^2}{(1+\sqrt{p_x})^2} \qquad (4)$$

or $${}_tq_x^{U.D.D.} - {}_tq_x^{Balducci} \le \frac{q_x^2}{(1+\sqrt{p_x})^2}$$

Since this result is true for all t in the interval (0,1),

$${}_{1-t}q_x^{U.D.D.} - {}_{1-t}q_x^{Balducci} \le \frac{q_x^2}{(1+\sqrt{p_x})^2}$$

But ${}_{1-t}q_x^{U.D.D.} = {}_{1-t}q_{x+t}^{Balducci}$ and ${}_{1-t}q_x^{Balducci} = {}_{1-t}q_{x+t}^{U.D.D.}$; hence,

$${}_{1-t}q_{x+t}^{Balducci} - {}_{1-t}q_{x+t}^{U.D.D.} \le \frac{q_x^2}{(1+\sqrt{p_x})^2} .$$

q.e.d.

The proof of theorem 2 yields several auxillary results which give added insight into the difference between the $\ell_{x+t}^{U.D.D.}$ and $\ell_{x+t}^{Balducci}$ curves.

Theorem 3: $\ell_{x+t}^{U.D.D.} - \ell_{x+t}^{Balducci} \le (\sqrt{\ell_x} - \sqrt{\ell_{x+1}})^2$ and the maximum difference is actually attained at t = T.

Proof: equation 3

Theorem 4: For fixed ℓ_x, $0 < T < \frac{1}{2}$ and T is a strictly decreasing function of d_x.

Proof: Apply the classical arithmetic-geometric inequality to equation (2):

$$T \le \frac{-\ell_{x+1} + \frac{\ell_x + \ell_{x+1}}{2}}{d_x} = \frac{1}{2} .$$

Further, from the classical arithmetic-geometric inequality, it follows that T is a strictly increasing function of ℓ_{x+1} and therefore, a strictly decreasing function of d_x.

q.e.d.

REFERENCES

[1] Gershenson, H. Measurement of Mortality Society of Actuaries 1961

[2] Batten, R. W. Mortality Table Construction Prentice Hall 1978