BALDUCCI AND THE 'UNIFORM DISTRIBUTION OF DEATHS' HYPOTHESES

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ABSTRACT

This paper shows that Balducci hypothesis imposed on a survivorship function l_x over a closed interval $[o, w]$ is dual to the 'Uniform distribution of deaths' hypothesis imposed on the survivorship function l_{w-x}^{-1} over the same interval, where w is such that $l_{w+1} = 0$. This fact enables us to give a simple visualization of interpolation relations concerning ${}_tq_x$ and ${}_hq_{x+t}$.

For the sake of convenience, we shall refer to Balducci and the "Uniform distribution of deaths' hypotheses as B - hypothesis and the U - hypothesis, respectively and impose superscripts B and U on quantities such as ${}_tq_x$ whenever appropriate.

For each x in $[o, w]$, we define

$$l'_x = l_{w-x}^{-1}.$$

Note that w is so chosen that $l_{w+1} = o$. This is to prevent l_{w-x}^{-1} from taking an infinite value and therefore to assure that l'_x is a survivorship function over $[o, w]$.

Let us now impose B-hypothesis on l_x. Then we have, for every integral age x less than w,

$$(1)\quad l_{x+t}^{-1} = l_{x+1}^{-1} - (1-t)(l_{x+1}^{-1} - l_x^{-1}),$$

where t varies between 0 and 1. It then follows that

$$\begin{aligned} l'_{x+t} &= l_{w-(x+t)}^{-1} \\ &= l_{[w-(x+1)] + (1-t)}^{-1} \\ &= l_{[w-(x+1)]+1}^{-1} - [1 - (1-t)] \cdot (l_{[w-(x+1)]+1}^{-1} - l_{w-(x+1)}^{-1}) \\ &= l_{w-x}^{-1} - t(l_{w-x}^{-1} - l_{w-(x+1)}^{-1}) \\ &= l'_x - t(l'_x - l'_{x+1}). \end{aligned}$$

Hence l'_x assumes the U-hypothesis. We can also see from the above derivation that if l_x assumes the U-hypothesis, then l'_x assumes B-hypothesis.

Next, let us impose the U-hypothesis on l_x. Let x be a fixed integer less than w. Since ${}_tq_x^{\mathbf{U}}$ is a linear function of t, we have

$$(2)\quad {}_tq_x^{\mathbf{U}} = t \cdot q_x.$$

This equality can also be seen to be true from the diagram:

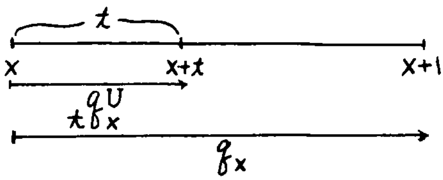

Diagram 1

From the following diagram,

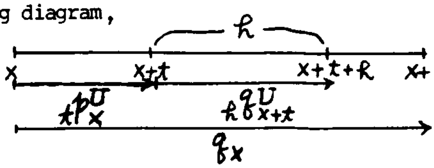

Diagram 2

we see that

$${}_tp_x^U \cdot {}_hq_{x+t}^U = h \cdot q_x .$$

Then it follows from (2) that

(3) $${}_hq_{x+t}^U = \frac{h \cdot q_x}{1 - t \cdot q_x} .$$

By setting h = 1-t in equality (3) we have

$${}_{1-t}q_{x+t}^U = \frac{(1-t) \cdot q_x}{1 - t \cdot q_x} ,$$

which can also be seen to be true from the following diagram.

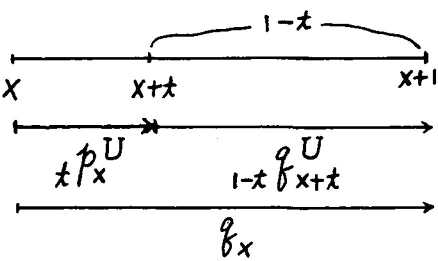

Diagram 3

Finally, let us impose B-hypothesis on l_x. We have shown that l'_x assumes the U-hypothesis. Hence we have

$$(4) \qquad {}_tq_x^B = \frac{l_x - l_{x+t}}{l_x}$$

$$= \frac{l^{-1}_{x+t} - l^{-1}_x}{l^{-1}_{x+t}}$$

$$= \frac{l'_{w-(x+t)} - l'_{w-x}}{l'_{w-(x+t)}}$$

$$= {}_tq'^{U}_{w-(x+t)}$$

$$= {}_tq'^{U}_{[w-(x+1)]+(1-t)} .$$

When t=1, we see that

$$(5) \qquad q'_{w-(x+1)} = q'^{U}_{w-(x+1)} = q^B_x = q_x .$$

Combining (3), (4), and (5), we have

$$(6) \qquad {}_tq_x^B = \frac{t \cdot q_x}{1-(1-t)\cdot q_x} .$$

Similarly, we have

$$(7) \qquad {}_kq^B_{x+t} = \frac{l_{x+t} - l_{x+t+k}}{l_{x+t}}$$

$$= \frac{l^{-1}_{x+t+k} - l^{-1}_{x+t}}{l^{-1}_{x+t+k}}$$

$$= \frac{l'_{w-(x+t+k)} - l'_{w-(x+t)}}{l'_{w-(x+t+k)}}$$

$$= {}_kq'^{U}_{w-(x+t+k)}$$

$$= {}_kq'^{U}_{[w-(x+1)]+(1-t-k)}$$

$$= \frac{k \cdot q'^{U}_{w-(x+1)}}{1-(1-t-k)\cdot q'^{U}_{w-(x+1)}}$$

$$= \frac{k \cdot q_x}{1-(1-t-k)\cdot q_x} .$$

By setting h=1-t in (7), we obtain

$$(8) \qquad {}_{1-t}q^{B}_{x+t} = (1-t)\cdot q_x .$$

Now, let us give a simple visualization of equalities (6), (7), and (8) by using (5). From the following diagram

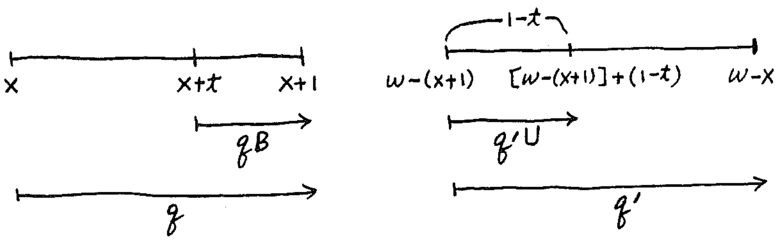

Diagram 4

we see that

$${}_{1-t}q^{B}_{x+t} = {}_{1-t}q'^{U}_{\omega-(x+1)} = (1-t)\cdot q'_{\omega-(x+1)} = (1-t)\cdot q_x .$$

From the following diagram

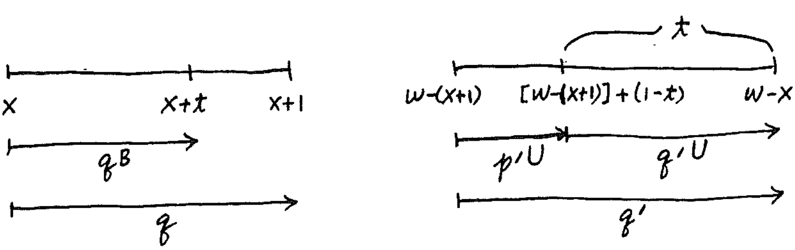

Diagram 5

we see that

$${}_{t}q^{B}_{x} = {}_{t}q'^{U}_{[\omega-(x+1)]+(1-t)} = \frac{t\cdot q'^{U}_{\omega-(x+1)}}{{}_{1-t}p'^{U}_{\omega-(x+1)}} = \frac{t\cdot q_x}{1-(1-t)\cdot q_x}$$

From the following diagram

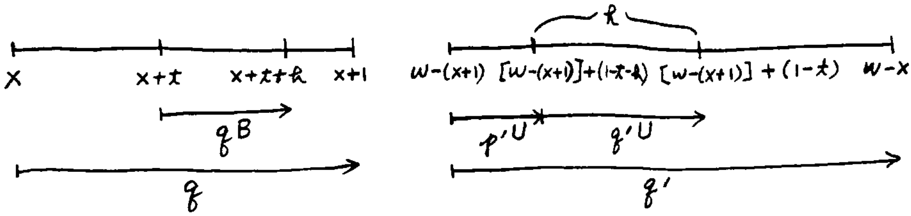

Diagram 6

we see that

$${}_{k}q^{B}_{x+t} = {}_{k}q'^{U}_{[\omega-(x+1)]+(1-t-k)} = \frac{k\cdot q'^{U}_{\omega-(x+1)}}{{}_{1-t-k}p'^{U}_{\omega-(x+1)}} = \frac{k\cdot q_x}{1-(1-t-k)\cdot q_x}$$

For practical purpose, we consolidate Diagram 4 into

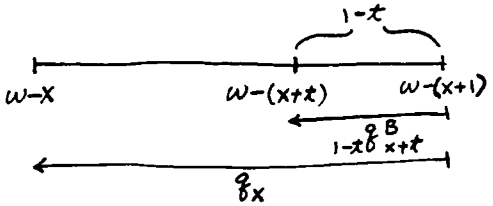

Diagram 7

which is the dual of Diagram 1. From Diagram 7, we see that

$${}_{1-t}q^{B}_{x+t} = (1-t)\cdot q_x\ .$$

Next, we consolidate Diagram 5 into

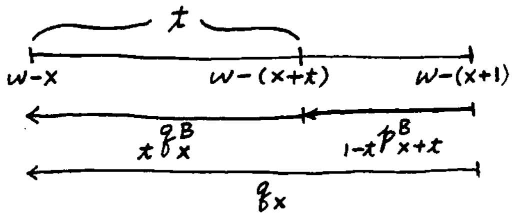

Diagram 8

which is the dual of Diagram 3. From Diagram 8, we see that

$${}_{1-t}p^{B}_{x+t}\cdot {}_{t}q^{B}_{x} = t\cdot q_x\ .$$

It follows that

$$ {}_tq^B_x = \frac{t\cdot q_x}{{}_{1-t}p^B_{x+t}} = \frac{t\cdot q_x}{1-{}_{1-t}q^B_{x+t}} = \frac{t\cdot q_x}{1-(1-t)\cdot q_x} . $$

Finally we consolidate Diagram 6 into

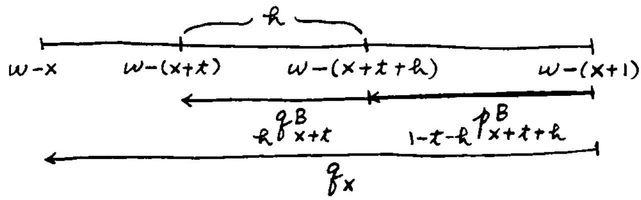

Diagram 9

which is the dual of Diagram 2. From Diagram 9, we see that

$$ {}_{1-t-h}p^B_{x+t+h} \cdot {}_hq^B_{x+t} = h\cdot q_x . $$

It follows that

$$ {}_hq^B_{x+t} = \frac{h\cdot q_x}{{}_{1-(t+h)}p^B_{x+(t+h)}} = \frac{h\cdot q_x}{1-[1-(t+h)]\cdot q_x} = \frac{h\cdot q_x}{1-(1-t-h)\cdot q_x} $$

REFERENCES

[1] C. W. Jordan,Jr.:"ON LIFE CONTINGENCIES", the Society of Actuaries, 1975.

[2] R. W. Batten:"MORTALITY TABLE CONSTRUCTION", Pretice-Hall,Inc., 1978.