COUNTERPART EXPOSURE FORMULAS UNDER THE BALDUCCI AND UNIFORM DEATHS ASSUMPTIONS*

H.J. Boom
University of Manitoba

ABSTRACT

To derive exposure formulas of the valuation schedule type on the basis of the Balducci assumption Gershenson [2] and Batten [1] use algebraic elimination, Batten adding rules of thumb for quickly finding the correct coefficients in specific cases. In the first portion of this paper we propose a shorter and, we hope, more elegant method of proof for formulas of this type as well as a short cut for the coefficients of the terms involving deaths.

We then proceed to derive a general exposure formula of the valuation-schedule type that may be used for all mortality studies with an observation period covering an integral number of consecutive years and give a general proof that this formula will lead to results identical to those produced by an individual record exposure formula based on the same assumptions.

Finally, the results obtained are extended by letting the assumption of a uniform distribution of deaths replace Balducci's assumption.

1. INTRODUCTION

To determine estimates of the mortality rate q_X, the probability that a member of a homogeneous group of lives attaining age X will die before age $X+1$, it is rarely practical to observe, during an appropriately chosen period of time, all members in the group who attain age X and are either still alive in the group at age $X+1$ or have died, under observation, before that age and then to divide the number of deaths by the total number observed. One reason is that any contribution to the accuracy of the estimate to be gained from members joining or leaving the group at ages between X and $X+1$ is eliminated.

Because of this, *exposure formulas* have been developed, the *exposure*

*Presented at the 1980 Actuarial Research Conference, August 14-15, 1980, University of British Columbia, Vancouver, B.C., Canada.

being the quantity by which Θ_x, the total number of all deaths observed in the group between ages X and $X+1$ is to be divided to obtain the estimated mortality rate q_X, i.e. the exposure is the denominator of $q_X = \Theta_x / E_x$.

For a mathematical derivation of formulas to calculate exposures it is necessary to make an assumption, or hypothesis, regarding the pattern of mortality at intermediate ages between X and $X+1$. Of such assumptions the one probably most commonly used by actuaries was proposed by Balducci in 1920[(1)]. It has the advantage that the exposure can be obtained from the sum of all time segments actually spent under observation between ages X and $X+1$ by the lives in the group by adding to this sum, for each death, the time remaining from the moment of death until exact age $X+1$ would have been attained. Another way of stating this is that for the purpose of calculating exposures deaths must be considered to continue being "exposed" until they would have attained age $X+1$.

Except for the last section of this paper, all exposure formulas in this paper will be developed on the basis of Balducci's assumption.

Most exposure formulas in practical use can be divided in two families, that of *individual record formulas*, produced from data derived from individual records regarding entry, exit and deaths of the lives in the group, and the family of *valuation schedule formulas* based on death records and periodic censuses, or counts, of the membership. In actuarial work these censuses are often produced as a byproduct of the regular valuation process and this has led to the name for this family of formulas. The existence of the latter type of formulas depends on the condition that the average age of entrants to the group during age intervals in $(X, X+1)$ can be assumed to be equal to the average age of those leaving the group during the same intervals.

2. INDIVIDUAL RECORD FORMULAS

On the basis of the individual records the lives in the group are partitioned, first, by mode of entry, in the two categories of *starters* (those already in the group when the observation period started) and of

---

(1) The assumption is that ${}_{1-t}q_{X+t}$ is linear in t for $0 \le t \le 1$.

new entrants (joining the group during this period) and second, by mode of exit, in the three categories of *enders* (still in the group at the end of the period), of *withdrawals* (leaving the group during the period) and of *deaths* (those dying in the group while under observation).

Each of these five categories is further subdivided by *tabulating rules* into *decks*, labelled by successive integers x. The tabulating rules for deaths are chosen in such a manner that the deaths tabulated "at age x" actually occurred, or may be assumed to occur, at some age between $X = x-\alpha$ and $X+1 = x-\alpha+1$, $0 \leq \alpha < 1$.

For the other four categories we shall assume that the tabulating rules have been constructed so that, for each category, the average age at which the events tabulated at age x occur falls in the age interval $(X, X+1)$. The numbers of lives in the five categories tabulated at age x are denoted by s_x, n_x, e_x, w_x and θ_x for starters, new entrants, enders, withdrawals and deaths respectively. For each of these categories (except the deaths) an *f-factor* is defined which measures the difference between the average age at which the relevant event occurs and the upper limit $X+1$ of the age interval. The *f-factor* for starters is denoted by ${}^{s}f$; we have ${}^{n}f$, ${}^{e}f$ and ${}^{w}f$ for the other categories. An *f-factor* for deaths, ${}^{\theta}f$, may be introduced, in which case we put ${}^{\theta}f = 0$ under the Balducci assumption.

Two functions, the *j-function* and the *f-function* are defined:

$$j_x = s_x - e_x + n_x - w_x - \theta_x \qquad (2.1)$$

$$f_x = {}^{s}f \cdot s_x - {}^{e}f \cdot e_x + {}^{n}f \cdot n_x - {}^{w}f \cdot w_x \qquad (2.2)$$

It can then be shown ([1], 71; [2], 61) that the formula becomes:

$$E_x = \sum^{x-1} j_r + f_x \qquad (2.3)$$

If, as must be the case if an exposure formula of the valuation schedule family is to be applicable, the *f-factors* ${}^{n}f$ and ${}^{w}f$ are equal, we can define the *net migration*, $m_x = n_x - w_x$, with corresponding *f-factor* ${}^{m}f = {}^{n}f$ $(= {}^{w}f)$; we then have:

$$j_x = s_x - e_x + m_x - \theta_x \tag{2.4}$$

$$f_x = {}^{s}f \cdot s_x - {}^{e}f \cdot e_x + {}^{m}f \cdot m_x \tag{2.5}$$

3. VALUATION SCHEDULE FORMULAS

In the two textbooks published on the subject of exposure theory, Gershenson ([3], Chapter 4) and Batten ([1], Chapter 5), valuation schedule formulas are derived, for a limited number of standard cases and instructive examples, by algebraic elimination of the migration terms. An alternative method of derivation (the "method of undetermined coefficients") is also presented ([2], 146; [1], 136). However, for all but the simplest cases, derivations by these methods, particularly the first, tend to be tedious and "rather troublesome to students" (Batten [1], p. 137) and this was our motivation in looking for an improved procedure.

To introduce this procedure, let us consider Gershenson's "Case II" ([3], p. 125; this is the case which Gershenson considered "fairly difficult to reproduce under examination conditions" [2], p. 144) as generalized by Batten([1], p. 142). Here we are studying the mortality in the age interval $(x, x+1)$ during a single calendar year z. Standard demographic notation is used: E_x^z denotes the number of lives attaining exact age x in the calendar year z, P_x^z the number age x last birthday when the observation year z begins and D_x^z the number of deaths occurring during z between exact ages x and $x+1$, subdivided in ${}_{\alpha}D_x^z$, whose last birthday occurred in z and ${}_{\delta}D_x^z$, for whom the last birthday fell in the year $z-1$.

The net migration m_x^z is similarly subdivided in ${}_{\alpha}m_x^z$ and ${}_{\delta}m_x^z$. Making the (standard) assumption that birthdays in z occur on the average on July 1, the middle of the year, we can construct a two-dimensional diagram, in which time runs horizontally and age vertically. Substituting July 1 for the actual birthdays, we see (Figure 3.1) that all events will occur on two diagonal segments, with α-deaths and α-migration along the upper diagonal and δ-deaths and δ-migration along the lower diagonal.

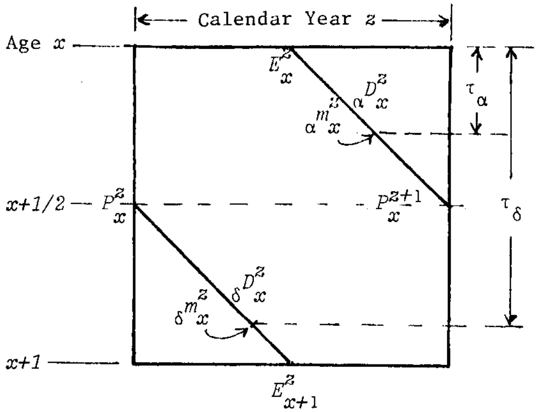


Figure 3.1

It is further assumed that α-migration occurs, on the average, a fraction of a year τ_α and δ-migration a fraction of a year τ_δ after age x is attained.

Our proposed procedure for deriving valuation schedule exposure formulas will now be demonstrated by applying it to this example.

Disregarding deaths, or, rather, assuming that α- and δ-deaths occurred on the average at the same ages as α- and δ-migration, the total number under observation between ages x and $x + \tau_\alpha$ would be constant and amount to E_x^z; similarly the number observed between ages $x + \tau_\alpha$ and $x + 1/2$ would be P_x^{z+1}, between $x + 1/2$ and $x + \tau_\delta$ we would have P_x^z and, from age $x + \tau_\delta$ to $x+1$, E_{x+1}^z lives under observation, resulting in a sum of time segments lived in year z between ages x and $x+1$ of

$$C_x^z = E_x^z \cdot \tau_\alpha + P_x^{z+1} (1/2 - \tau_\alpha) + P_x^z (\tau_\delta - 1/2) + E_{x+1}^z (1 - \tau_\delta)$$

years.

However, because of Balducci's assumption we have to treat each death as if it occurred at age $x+1$ instead of at ages $x + \tau_\alpha$ and $x + \tau_\delta$, so for each α-death we have to add a fraction $1 - \tau_\alpha$ and for each

δ-death a fraction $1 - \tau_\delta$, i.e. for all deaths together we have to add $\mathbf{D}_x^z = {}_\alpha D_x^z\,(1 - \tau_\alpha) + {}_\delta D_x^z\,(1 - \tau_\delta)$ years of exposure, so that the correct exposure for the calendar year z, $\mathbf{E}_x^z$, becomes:

$$\mathbf{E}_x^z = \mathbf{C}_x^z + \mathbf{D}_x^z$$

For an observation period starting with year $z = a$ and ending with year $z = k$ this expression must be summed over all values of z from a to k and the exposure is

$$\mathbf{E}_x = \mathbf{C}_x + \mathbf{D}_x = \sum_a^k \left[\mathbf{C}_x^z + \mathbf{D}_x^z\right] =$$

$$= \sum_a^k \Big[E_x^z \cdot \tau_\alpha + P_x^{z+1}\,(1/2-\tau_\alpha) + P_x^z\,(\tau_\delta-1/2) + E_{x+1}^z\,(1-\tau_\delta) +$$

$$+ {}_\alpha D_x^z\,(1-\tau_\alpha) + {}_\delta D_x^z\,(1-\tau_\delta)\Big]$$

Note that this result can immediately be written down, without any algebra at all, from the basic assumptions and that the entire procedure consists of two simple steps, each of which carries its own justification so that no separate proof is necessary.

I have used this procedure, or a slight variation thereof using essentially the same two steps, in my classes since the mid-sixties, with very good results.

Roach ([5], p. 6) has proposed a somewhat similar technique; however, because of the graph required, it is not quite as simple.

4. GENERAL EXPOSURE FORMULAS OF THE VALUATION SCHEDULE FAMILY

4.1 The great variety of exposure formulas of this family, caused by the many different assumptions that can be made about the average occurrence of birthdays, the tabulation rules for deaths determining the age interval being analysed, the ages at which migration may be assumed to occur and the number of censuses or valuations to be used made it desirable to see if perhaps a general approach would be possible resulting in a single general valuation schedule formula representing the entire family.

As we shall see below, the family is made up of two types of formulas which we shall call *Type I*, or *single diagonal* formulas and *Type II*, or *double diagonal* formulas.

Although we shall see that Type I may be regarded as a special case of Type II, it will be less laborious to develop the Type I formula first and then to proceed to the more general Type II formula. The "generality" of our Type II formula will only be restricted to the extent that we shall only consider formulas applicable to observation periods covering an integral number of consecutive years (but not necessarily calendar years).

4.2 An important factor in our development of general exposure formulas appeared to be the notation used. The "demographic notation" (this is the notation we used in our example in the previous section) is ill suited for our purpose, particularly if the age X in q_X, the mortality rate to be estimated, is not an integer. We have therefore developed a more suitable notation for our purposes, without, however, introducing very drastic changes in the general approach.

(i) *Deaths*. As in section 2 we shall assume that all deaths are tabulated by an integer variable x (the *tabulating age*) in such a way that all deaths tabulated at age x have occurred (or may be assumed to have occurred) in a one year interval $[X, X+1]$, so that $X = x-\alpha$, $0 \leq \alpha < 1$. This interval shall be referred to as the *analysis year of age (x)*, or just simply the *analysis year*.

Events occurring in the analysis year of age *(x)* during the observation year *z* (*z* being one of the consecutive integers from *a* to *k*, identifying the years in the observation period) will be said to occur in the *cell (x, z)*. In a two-dimensional diagram this cell will appear as a square (we saw an example of this in Figure 3.1), the vertical sides of which represent the beginning and end of the observation year *z* and the horizontal sides indicating the ages *X* and *X+1*. For each life in the group there will be a diagonal segment in the diagram representing passage through the cell *(x, z)*. These segments may start on the left vertical or the top horizontal boundary of the cell or in its interior and end in the interior or on the right vertical or lower boundary of the cell.

The number of deaths occurring in the cell *(x, z)* will be denoted by Θ_x^z; $\Theta_x = \sum_a^k \Theta_x^z$ is the total number of deaths recorded in the analysis year *(x)*.

(ii) *Birthdays*. For all observation years we shall assume a common distribution of birthdays, resulting in an "average birthday" which, for each observation year, shall fall a fraction of a year β $(0 \leq \beta < 1)$ after the start of the year, the fraction β being the same for all observation years.

Replacing all birthdays by this average birthday will have the result that all diagonal segments referred to above will now be concentrated on one or two diagonal lines, depending on whether $\alpha = \beta$ or $\alpha \neq \beta$. If $\alpha = \beta$ (see Figure 4.2.1(a)) there is only one diagonal, the main diagonal of the square representing the cell *(x, z)*; the resulting formula will be a *Type I* or *single diagonal* formula. If $\alpha \neq \beta$ we have two diagonal segments and the formula will be a *Type II* or *double* *diagonal* formula (Figure 4.2.1(b), 4.2.1(c)).

If we introduce τ $(0 \le \tau < 1)$ to denote the fraction of a year between the attainment of exact age X and the end of the observation year, we can distinguish two different cases in the Type II situation: if $\alpha < \beta$ (Figure 4.2.1(b)), $\tau = 1 - (\beta - \alpha)$ and the average birthday falls on the upper diagonal; if $\alpha > \beta$ (Figure 4.2.1(c)), $\tau = \alpha - \beta$ and the average birthday falls on the lower diagonal. In the single diagonal case (Type I) $\alpha = \beta$ and we take $\tau = 1$.

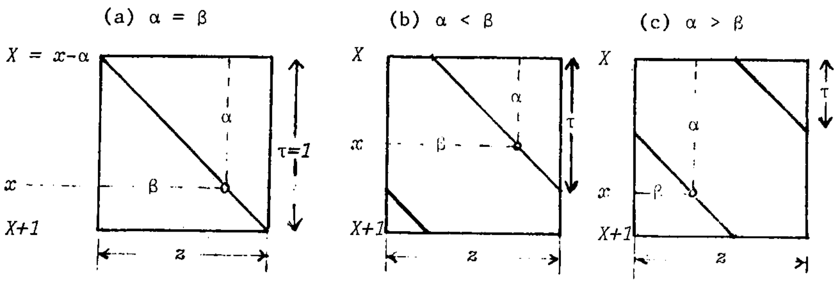


Figure 4.2.1

We shall first (items (iii) to (v)) consider the Type I, or single diagonal case and then (item (vi)) return to the Type II or double diagonal case.

(iii) We shall denote the numbers of lives observed in the successive valuations or censuses in the cell (x,z) by $V^z_{x,1}$, $V^z_{x,2}, \ldots$ $V^z_{x,n}$ $(n \ge 1)$; these valuations will divide the analysis year (x) in subintervals and we shall divide the deaths θ^z_x in the cell into subgroups accordingly: we let $\theta^z_{x,j}$ denote the deaths between the valuations $V^z_{x,j}$ and $V^z_{x,j+1}$ $(j=1,\ldots,n-1)$— any deaths in (x,z) preceding $V^z_{x,1}$ or following $V^z_{x,n}$ will be denoted by $\theta^z_{x,0}$ and $\theta^z_{x,n}$ respectively.

We use the same notational principle to divide the net migration m_x^z in the cell (x,z) into subgroups $m_{x,0}^z, \ldots m_{x,n}^z$.

It will be convenient to introduce the symbol $H_{x,j}^z = m_{x,j}^z - \Theta_{x,j}^z$ $(j=0,\ldots n)$ for the *net increase* in numbers of lives observed during the subintervals, so that

$$V_{x,j+1}^z - V_{x,j}^z = H_{x,j}^z \quad (j=1,\ldots n-1) \qquad (4.2.1)$$

(iv) If a valuation occurs *on* a cell boundary, i.e. in a corner point at either end of the diagonal, it may be considered either as the last valuation of the previous cell or the first of the next one. Accordingly, the symbols $V_{x,n}^z$ and $V_{x+1,1}^{z+1}$ may refer to a single valuation; $V_{x,n}^z$ being the number of lives at the end of one year and $V_{x+1,1}^{z+1}$ the same number at the start of the next one; neither migration nor deaths can occur "in between", so the coincidence of $V_{x,n}^z$ and $V_{x+1,1}^{z+1}$ implies that $m_{x,n}^z$ and $m_{x+1,1}^{z+1}$ are both zero, as are $\Theta_{x,n}^z$ and $\Theta_{x+1,1}^{z+1}$ and, hence, $H_{x,n}^z$ and $H_{x+1,1}^{z+1}$.

Whether or not $V_{x,n}^z$ and $V_{x+1,1}^{z+1}$ coincide, we shall always have:

$$V_{x+1,1}^{z+1} - V_{x,n}^z = H_{x+1,0}^{z+1} + H_{x,n}^z \qquad (4.2.2)$$

(v) We shall introduce τ_j $(j=0,\ldots n)$ to indicate that the net migration $m_{x,j}^z$ is assumed to occur, on the average, at age $X + \tau_j$; for τ_0 and τ_n however we define separately $\tau_0=0$ and $\tau_n=1$, thus implying that migration, if any, between the last valuation of a cell and the first one of the next cell will be assumed to occur *on* the cell boundary.

Note that in each time segment between two successive migration points τ_{j-1} and τ_j $(j=1,\ldots n)$ we have a *closed* *group*, i.e. there is no in- or out-movement except by death.

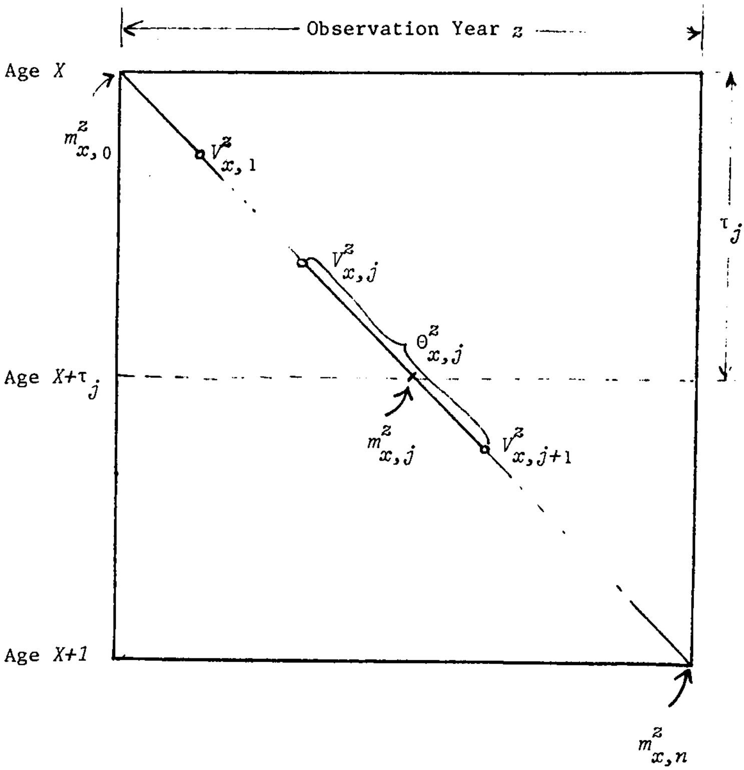


Figure 4.2.2 Type I diagram

(vi) For the Type II, or double diagonal case, we shall use the same symbols as introduced above for the Type I case but we shall use a bar above or below the symbol to indicate whether it refers to the upper or lower diagonal. For instance, the valuations along the upper diagonal will be

$\bar{V}^{z}_{x,1}, \ldots \bar{V}^{z}_{x,n}$ and those along the lower diagonal $\underline{V}^{z}_{x,1}, \ldots \underline{V}^{z}_{x,m}$

(note that, inconsistently, we use n and m rather than $\bar{n}$ and $\underline{n}$, the latter being too cumbersome for use in subscripts).

Similarly we have

$\overline{m}^z_{x,j}$ and $\underline{m}^z_{x,j}$, $\overline{\Theta}^z_{x,j}$ and $\underline{\Theta}^z_{x,j}$, $\overline{H}_{x,j}$ and $\underline{H}_{x,j}$ etc.

In analogy to (4.2.1) we now have

$$\left.\begin{aligned} \overline{V}^z_{x,j+1} - \overline{V}^z_{x,j} &= \overline{H}^z_{x,j} \qquad j = 1,\ldots,n-1 \\ \underline{V}^z_{x,j+1} - \underline{V}^z_{x,j} &= \underline{H}^z_{x,j} \qquad j = 1,\ldots,m-1 \end{aligned}\right\} \tag{4.2.3}$$

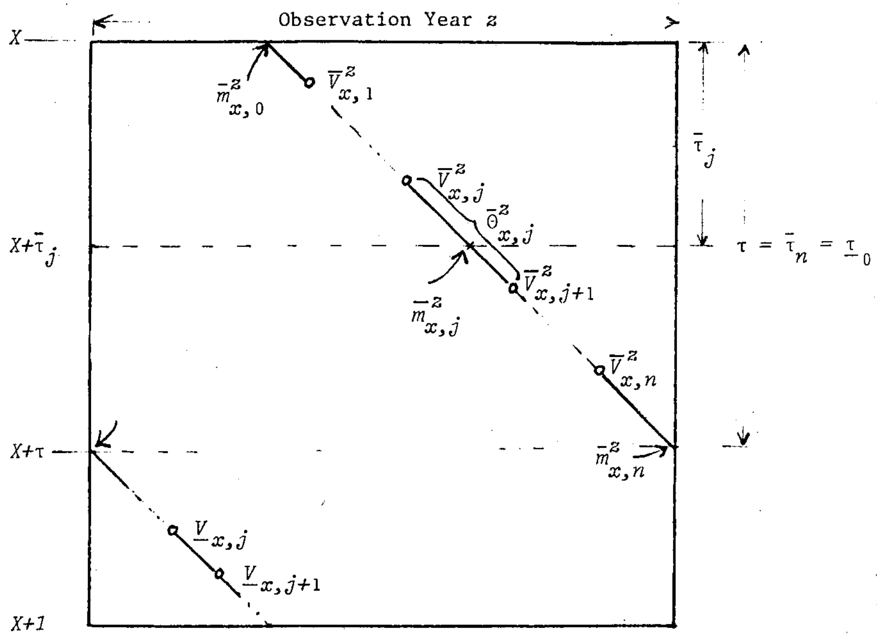


Figure 4.2.3, Type II diagram

Similarly, $\overline{m}^z_{x,j}$ and $\underline{m}^z_{x,j}$ will be assumed to occur, on the average, respectively at ages $X+\overline{\tau}_j$ and $X+\underline{\tau}_j$; we now define separately

$$\overline{\tau}_0 = 0, \quad \overline{\tau}_n = \tau = \underline{\tau}_0, \quad \underline{\tau}_n = 1.$$

For valuations falling on cell boundaries we use the same approach as in the Type I case; now $\underline{V}^z_{x,m}$ and $\overline{V}^z_{x+1,1}$ may refer to a single valuation on a horizontal boundary, similarly

$\overline{V}^{z}_{x,n}$ and $\underline{V}^{z+1}_{x,1}$ may refer to the same valuation on a vertical boundary. In these cases we will have, by the same reasoning as before, $\underline{m}^{z}_{x,m} = \overline{m}^{z}_{x+1,0} = 0$ and $\overline{m}^{z}_{x,n} = \underline{m}^{z+1}_{x,0} = 0$, with corresponding relations for deaths and net increase.

In analogy to (4.2.2) we now have

$$\left.\begin{aligned} \overline{V}^{z}_{x+1,1} - \underline{V}^{z}_{x,m} &= \overline{H}^{z}_{x+1,0} + \underline{H}^{z}_{x,m} \\ \underline{V}^{z+1}_{x,1} - \overline{V}^{z}_{x,n} &= \underline{H}^{z+1}_{x,0} + \overline{H}^{z}_{x,n} \end{aligned}\right\} \qquad (4.2.4)$$

4.3 The general Type I formula can now be derived by following almost verbatim the steps used in our example of Section 3. If $\Theta^{z}_{x,j}$ $(j=0,\ldots n)$ occurred at the same ages $X+\tau_j$ as the corresponding net migration $m^{z}_{x,j}$, the sum of all segments of lifetime observed in the cell (x,z) would be

$$C^{z}_{n} = \sum_{j=1}^{n} V^{z}_{x,j}\ (\tau_j - \tau_{j-1}) .$$

However, under Balducci's assumption we have to add a time segment $1-\tau_j$ for each death $\Theta^{z}_{x,j}$, so that for all deaths together we must add

$$D^{z}_{x} = \sum_{0}^{n} \Theta^{z}_{x,j}\ (1-\tau_j)$$

The correct exposure for the cell (x,z) now is the sum of the "census term" C^{z}_{x} and the "Balducci correction" D^{z}_{x}:

$$E^{z}_{x} = C^{z}_{x} + D^{z}_{x} = \sum_{1}^{n} V^{z}_{x,j}\ (\tau_j - \tau_{j-1}) + \sum_{0}^{n} \Theta^{z}_{x,j}\ (1-\tau_j)$$

and, for the entire observation period, we obtain the general Type I formula:

$$E_{x} = C_{x} + D_{x} = \sum_{a}^{k} \left[\sum_{1}^{n} V^{z}_{x,j}\ (\tau_j - \tau_{j-1}) + \sum_{0}^{n} \Theta^{z}_{x,j}\ (1-\tau_j) \right]$$

or, if we write $V_{x,j} = \sum_{a}^{k} V^{z}_{x,j}$ and $\Theta_{x,j} = \sum_{a}^{k} \Theta^{z}_{x,j}$:

$$\mathbb{E}_x = \mathbb{C}_x + \mathbb{D}_x = \sum_{1}^{n} V_{x,j}\,(\tau_j - \tau_{j-1}) + \sum_{0}^{n} \Theta_{x,j}\,(1-\tau_j) \qquad (4.3.1)$$

$$(\tau_0 = 0,\ \tau_n = \tau = 1)$$

(Note that, regardless of whether or not the deaths $\Theta^{z}_{x,j}$ occur at age $X + \tau_j$, as assumed, any error in the terms $\mathbb{C}^{z}_{x}$ will be exactly compensated by an equal but opposite error in the $\mathbb{D}^{z}_{x}$ terms.)

The reader will now have little difficulty in showing that the general Type II formula is obtained from a census term

$$\boldsymbol{C}_x = \sum_{a}^{k} \left[\sum_{1}^{n} \overline{V}^{z}_{x,j}\,(\bar{\tau}_j - \bar{\tau}_{j-1}) + \sum_{1}^{m} \underline{V}^{z}_{x,j}\,(\underline{\tau}_j - \underline{\tau}_{j-1}) \right]$$

and a correction term

$$\boldsymbol{D}_x = \sum_{a}^{k} \left[\sum_{0}^{n} \bar{\Theta}^{z}_{x,j}\,(1-\bar{\tau}_j) + \sum_{0}^{m} \underline{\Theta}^{z}_{x,j}\,(1-\underline{\tau}_j) \right]$$

using the same notational device as in (4.3.1) above, resulting in the general Type II formula:

$$\left.\begin{aligned} \boldsymbol{E}_x = {} & \sum_{1}^{n} \overline{V}_{x,j}\,(\bar{\tau}_j - \bar{\tau}_{j-1}) + \sum_{1}^{m} \underline{V}_{x,j}\,(\underline{\tau}_j - \underline{\tau}_{j-1}) + \\ & + \sum_{0}^{n} \bar{\Theta}_{x,j}\,(1-\bar{\tau}_j) + \sum_{0}^{m} \underline{\Theta}_{x,j}\,(1-\underline{\tau}_j) \end{aligned}\right\} \qquad (4.3.2)$$

$$(\bar{\tau}_0 = 0,\ \bar{\tau}_n = \tau = \underline{\tau}_0,\ \underline{\tau}_m = 1)$$

It is easy to see that (4.3.1) is a special case of (4.3.2) if we consider the Type I case as the limit of the Type II case as $\alpha \to \beta$, i.e. as $\tau \to 1$. Since $\tau = \underline{\tau}_0 < \ldots < \underline{\tau}_m = 1$, in the limit we must have $\underline{\tau}_0 = \ldots = \underline{\tau}_m = 1$. Hence all factors $\underline{\tau}_j - \underline{\tau}_{j-1}$ and $1-\underline{\tau}_j$ in

(4.3.2) become zero and, dropping the bars in the remaining terms we are left with formula (4.3.1).

5. COUNTERPART EXPOSURE FORMULAS

Two exposure formulas, one a valuation schedule formula and the other an individual record formula, are called *counterpart formulas* if they are based on the same assumptions (Gershenson [3], 129 (ex. 4.4.3); Batten [1], 172).

Although it is obvious that such formulas, when applied to the same set of data, must produce identical results, proving this identity by algebraic methods for various pairs of counterpart formulas has been challenging.

Batten, ([1], Chapter 6) gives various examples of pairs of counterpart formulas, with proofs, but for single years of observation only ([1], 178, 182, 186); one example with a three-year observation period is discussed but no proof is given ([1], 194).

The fact that each of the various examples required its own proof posed the question whether it would not be possible to derive a single pair of counterpart formulas covering all special cases and then produce a single algebraic proof for the fact that these formulas will always produce identical results when applied to the same data.

The general valuation schedule formula has been developed in section (4); for its counterpart we only have to modify the functions (2.4) and (2.5) appearing in (2.3) to recognize the partition of the analysis year

---

As far as I know a proof of this type was first asked on the Part 5 Examination of the Society of Actuaries in November 1966 ([6], 214) and I well remember, after finding what I considered a complete proof, that I was rather disappointed by the "illustrative solution" provided later, which proved the identity for one year of observation only and suggested that the proof for longer observation periods could be obtained from the one given "by addition". It seemed to me that the editors of the illustrative solutions had taken an easy way out - the starters and enders of any single year are not the starters and enders for the entire observation period and handling this appeared to me to be the more difficult part of a complete proof.

(x) by the successive valuations.

The term m_x in (2.4) is built up from $m^z_{x,j}$; if we put

$$m^z_x = \sum_0^n m^z_{x,j}, \quad m_{x,j} = \sum_a^k m^z_{x,j} \tag{5.1}$$

we can write

$$m_x = \sum_a^k \sum_0^m m^z_{x,j} = \sum_a^k m^z_x = \sum_0^m m_{x,j} \tag{5.2}$$

for Type I and

$$m_x = \sum_a^k \left[\sum_0^n \overline{m}^z_{x,j} + \sum_0^m \underline{m}^z_{x,j}\right] = \sum_a^k (\overline{m}^z_x + \underline{m}^z_x) = \sum_0^n \overline{m}_{x,j} + \sum_0^m \underline{m}_{x,j} \tag{5.3}$$

for the Type II case. For deaths $\Theta^z_{x,j}$ (and net increases $H^z_{x,j}$) expressions of the same form will hold.

Any of the expressions (5.2) or (5.3) or the corresponding expressions in $\Theta^z_{x,j}$ and $H^z_{x,j}$ may be substituted in the j_x function of (2.4); for example, in the Type I case:

$$j_x = s_x - e_x + \sum_0^n (m_{x,j} - \Theta_{x,j}) \tag{5.4}$$

For the f-function of (2.5) we note that, in the Type II case, ${}^sf = {}^ef = 1-\tau$; ${}^mf$ has no direct equivalent since each migration term has its own f-factor $1-\overline{\tau}_j$ or $1-\underline{\tau}_j$ so (2.5) may now be written as, for example (for Type II):

$$f_x = (1-\tau)(s_x-e_x) + \sum_0^n (1-\overline{\tau}_j)\,\overline{m}_{x,j} + \sum_0^m (1-\underline{\tau}_j)\,\underline{m}_{x,j} \tag{5.5}$$

Since $\overline{\tau}_0 = 0$, $\overline{\tau}_n = \underline{\tau}_0 = \tau$, $\underline{\tau}_m = 1$ the last two terms may be replaced by

$$\overline{m}_{x,0} + \sum_1^{n-1} (1-\overline{\tau}_j)\,\overline{m}_{x,j} + (1-\tau)(\overline{m}_{x,n} + \underline{m}_{x,0}) + \sum_1^{m-1} (1-\underline{\tau}_j)\,\underline{m}_{x,j} \tag{5.6}$$

Using (5.2)-(5.6), and corresponding expressions for Θ and H of the same form, the reader may now write various forms of the general individual record formula by appropriate substitutions for j_x and f_x (2.3):

$$E_x = \sum^{x-1} j_r + f_x$$

A convenient form for starting the algebraic proof that the general valuation schedule formulas (4.3.1) and (4.3.2) are equivalent to the corresponding individual record formulas is obtained when we write the latter as:

$$E_x = \sum^{x-1} \left[s_r - e_r + \sum_0^n (m_{r,j} - \Theta_{r,j}) \right] + s_x - e_x + m_{x,0} + \sum_1^{n-1} m_{x,j}(1-\tau_j) \quad (5.7)$$

in the single diagonal or Type I case, and

$$E_x = \sum^{x-1} \left[s_r - e_r + \sum_0^{n-1} (\bar{m}_{r,j} - \bar{\Theta}_{r,j}) + \sum_0^{m} (\underline{m}_{r,j} - \underline{\Theta}_{r,j}) \right]$$

$$+ (s_x - e_x)(1 - \tau) + \bar{m}_{x,0} + \sum_1^{n-1} \bar{m}_{x,j}(1 - \bar{\tau}_j) \quad (5.8)$$

$$+ (\bar{m}_{x,n} + \underline{m}_{x,0})(1 - \tau) + \sum_1^{m-1} \underline{m}_{x,j} (1 - \underline{\tau}_j)$$

for the double diagonal, or Type II, case.

For these proofs the two following pairs of relations, linking the starters and enders with the first and last valuation, will be required:

$$\left.\begin{array}{lll} \text{Type I:} & s_x = V^a_{x,1} - H^a_{x,0}, & e_x = V^k_{x-1,n} + H^k_{x-1,n} \\ \text{Type II:} & s_x = \underline{V}^a_{x,1} - \underline{H}^a_{x,0}, & e_x = \bar{V}^k_{x,n} + \bar{H}^k_{x,n} \end{array}\right\} \quad (5.9)$$

They follow directly from a consideration of the diagrams in figures 4.2.2 and 4.2.3 for $z = a$ (starters) and $z = k$ (enders).

A proof showing that the general Type II valuation schedule formula, when applied to the same set of data, will give results identical to those obtained from its counterpart individual record formula consists, in principle, of the same steps as the corresponding proof for the Type I case. It is only slightly more complicated because the starters and enders appear as $(1 - \tau)(s_x - e_x)$ rather than just plain $s_x - e_x$ and there is an extra term, $(1 - \tau)(\overline{\Theta}_{x,n} + \underline{\Theta}_{x,0})$ to consider - otherwise everything is the same except that, because of our upper and lower bar symbols, the expressions become about twice as long.

For this reason we shall give the proof for the Type I case only (see Appendix), leaving the one for Type II to the interested reader.

6. THE ASSUMPTION OF A UNIFORM DISTRIBUTION OF DEATHS (UDD)

In recent years this assumption has attracted attention in connection with the construction of exposure formulas of the valuation schedule family. The treatment given earlier by Gershenson [3] has been extended by Greville [4] and Batten, [1] and [3].

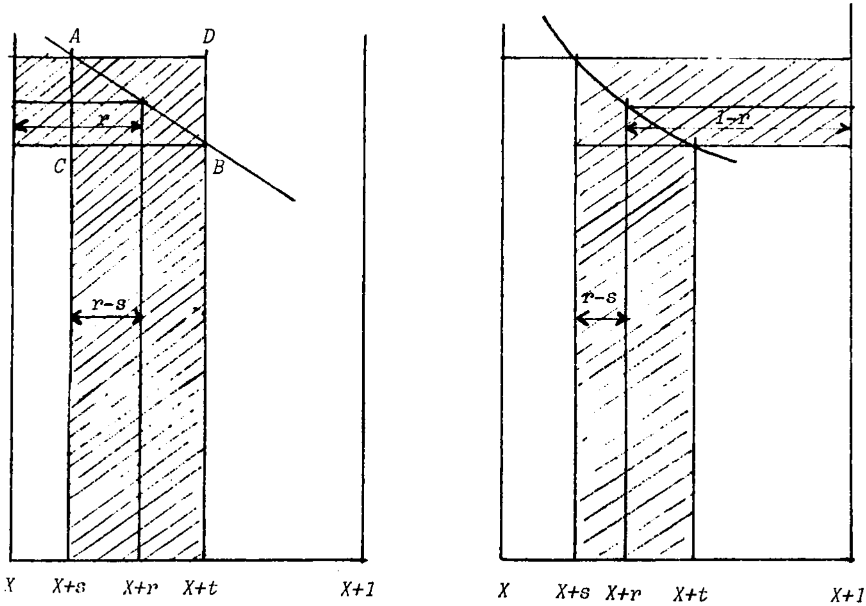
The principal difficulty encountered in developing practical formulas from the UDD assumption appeared to arise from the necessity to consider "unobserved" deaths (i.e. deaths occurring *after* withdrawal).

Considering a closed group of lives between ages $X + s$ and $X + t$ $(1 \leq s < t \leq 1)$ under the UDD assumption, we can obtain the correct exposure by allocating a full year of exposure to each "observed" death (i.e. those lives in the group dying before age $X + t$); for the lives surviving to age $X + t$ however, the exposure depends on whether or not they will die after leaving the observed group at age $X + t$ but before age $X + 1$ — if they do, thus becoming "unobserved deaths", their contribution is zero, but if they don't it is $t - s$ for each such life (Greville [4], 45 or Gershenson [3], solution to ex. 6.5(c), p. 185).

The problem is, of course, that the data collected for a mortality investigation do not normally include any information as to how many of the lives ceasing to be observed at age $X + t$ will then die before age $X + 1$.

As a result, most formulas were developed under the assumption that there is no migration or, at best, migration is restricted to the boundaries of each cell (Batten [1], 154).

In considering these closed groups, Greville [Ibid.] also notes an interesting duality between the treatment of deaths under the Balducci assumption (exposures to be extended after death to age $X + 1$) and the assumption of UDD (exposures to be extended backwards, positively for observed deaths, negatively for unobserved ones, to age X); however, his formulas are also restricted to situations with limited migration. Shiu [6] investigated this duality further; he found that the same



(a) UDD Exposure (b) Balducci Exposure

Figure 6.1

exposures determined for the closed groups between ages $X + s$ and $X + t$ by the rules given above could be reproduced by a modification of these rules which entirely avoids the problem of unobserved deaths and, moreover, presents the duality involved even more clearly.

Adapted to our notation, this modification allocates an amount of exposure equal to the actual time spent under observation in the group, but for each (observed) death an adjustment is added: under the Balducci assumption this adjustment equals the length of time *from the moment of death to age* $X + 1$, under the UDD assumption it is the length of time *from age* X *to the moment of death* (See Table 1).

Table 1. *Individual exposures for deaths at age* $X + r$ *in closed groups observed between ages* $X + s$ *and* $X + t$ $(0 \le s < t \le 1,\ s \le r \le t)$

Assumption	Actual Lifetime in Closed Group	Adjustment	Total Exposure
Balducci	$r - s$	$1 - r$	$1 - s$
U.D.D.	$r - s$	r	$2r - s$

Actually, Shiu's formulation was slightly different; an analysis year from birthday to birthday was considered, for which "the exposure of an observed death is equal to the time from the age of entry to his next birthday under the Balducci assumption but under the UDD assumption it is the time from his last birthday to the end of the observation period or the next migration point". It is easy to see that our present formulation in terms of individual adjustments is equivalent. The segments $r - s$ in Figure 6.1(a) are counted twice and the two triangles *ABC* and *ABD* are equal.

To develop usable valuation schedule formulas based on the UDD assumption we have to divide the deaths $\theta_{x,j}$ between two successive valuations in those before migration and those after migration. Omitting the subscript x for simplicity, we have, for the Type I case:

$$\theta_j = {}^{a}\theta_j + {}^{b}\theta_j$$

for the deaths between V_j and V_{j+1}, with ${}^{b}\theta_j$ denoting the number of deaths occurring before age $X + \tau_j$ and ${}^{a}\theta_j$ the number of deaths after age $X + \tau_j$ $(j = 1, \ldots n-1)$. We put ${}^{b}\theta_0 = {}^{a}\theta_n = 0$; ${}^{a}\theta_0$ is the number of deaths between $\tau_0 = 0$ and V_1, ${}^{b}\theta_n$ is that between V_n and $\tau_n = 1$. We will

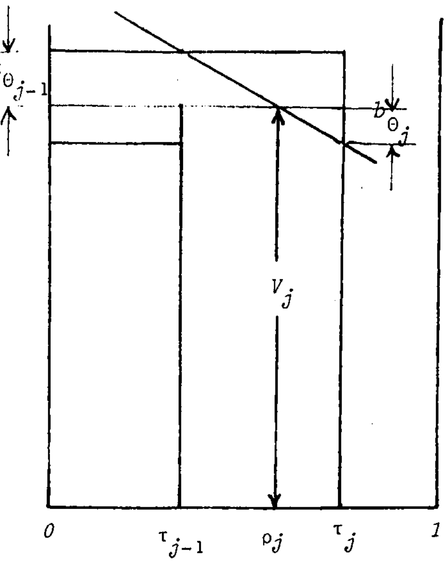


Figure 6.2

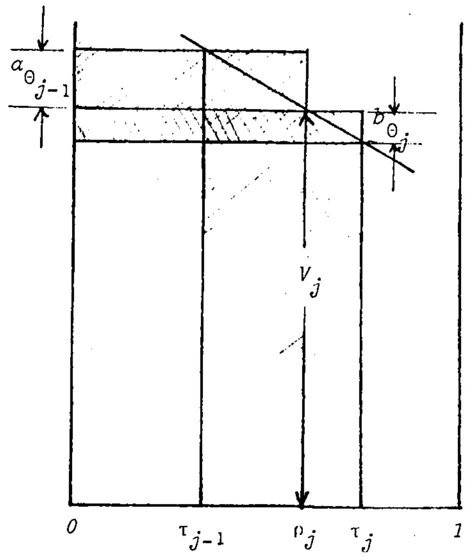


Figure 6.3

further define ρ_j so that the valuation V_j takes place at age $X + \rho_j$.

(i) Considering each closed group between ages $X + \tau_{j-1}$ and $X + \tau_j$ as a whole (see Figure 6.2) we see that the exposure can be obtained by the addition of three rectangles, one with width $\tau_j - \tau_{j-1}$ and height V_j, one with width τ_{j-1} and height ${}^b\theta_j$ and finally one with width τ_j and height ${}^a\theta_{j-1}$; summing for all these closed groups we obtain:

$$ {}^U E_x = \sum_1^n \left[V_{x,j}\,(\tau_j - \tau_{j-1}) + {}^a\theta_{x,j-1}\tau_j + {}^b\theta_{x,j}\,\tau_{j-1} \right] \qquad (6.1) $$

(ii) Alternatively, we can consider the closed groups from $X + \tau_{j-1}$ to $X + \rho_j$, with deaths ${}^a\theta_{j-1}$, and from $X + \rho_j$ to $X + \tau_j$, with deaths ${}^{b}\theta_j$, separately (see Figure 6.3). Now we have two rectangles for each closed group; in the first group these have, respectively, width $\rho_j - \tau_{j-1}$ and height V_j for one and width ρ_j and height ${}^{a}\theta_{j-1}$ for the other, in the second group they have width $\tau_j - \rho_j$ and height V_j for the one and width ρ_j and height ${}^{b}\theta_j$ for the other, i.e.

$$\text{for first group:} \quad V_j(\rho_j-\tau_{j-1}) + {}^{a}\theta_{j-1}\cdot\rho_j$$

$$\text{for second group:} \quad \underline{V_j(\tau_j-\rho_j) + {}^{b}\theta_j\cdot\rho_j}$$

$$\text{Total} = \quad V_j(\tau_j-\tau_{j-1}) + ({}^{a}\theta_{j-1}+{}^{b}\theta_j)\cdot\rho_j$$

This leads to a second form of the general UDD Type I formula:

$$^{U}\boldsymbol{E}_x = \sum_1^n \left[V_{x,j}(\tau_j-\tau_{j-1}) + ({}^{a}\theta_{x,j-1}+{}^{b}\theta_{x,j})\rho_j\right] \tag{6.2}$$

The corresponding Type II formula is, of course:

$$^{U}\boldsymbol{E}_x = \sum_1^n \Big[\overline{V}_{x,j}(\overline{\tau}_j-\overline{\tau}_{j-1}) + \underline{V}_{x,j}(\underline{\tau}_j-\underline{\tau}_{j-1}) + ({}^{a}\overline{\theta}_{x,j-1}+{}^{b}\overline{\theta}_{x,j})\overline{\rho}_j + ({}^{a}\underline{\theta}_{x,j-1}+{}^{b}\underline{\theta}_{x,j})\underline{\rho}_j\Big] \tag{6.3}$$

In either case the exposure can be written as the sum of a census term, $\boldsymbol{C}_x$, which is the same as in the Balducci case, and a death term $^{U}\boldsymbol{D}_x$; for Type I formulas these are:

$$\boldsymbol{C}_x = \sum_1^n \left[V_{x,j}(\tau_j-\tau_{j-1})\right]$$

$$^{U}\boldsymbol{D}_x = \sum_1^n \left[{}^{a}\theta_{x,j-1}\tau_j + {}^{b}\theta_{x,j}\tau_{j-1}\right] = \sum_1^n ({}^{a}\theta_{j-1}+{}^{b}\theta_j)\rho_j\,, \tag{6.4}$$

the extension to Type II being obvious.

To recapitulate:

(i) we may consider each life as contributing an amount of exposure equal to the time interval in which it can actually be observed, while living, during the analysis year of age $[X, X+1]$;

(ii) in addition, for each death observed during this interval a corrective adjustment is to be made under either the Balducci or the UDD assumption and these adjustments are complimentary;

(iii) the Balducci adjustment is equal to the portion of the analysis year *following* the death, whereas the UDD adjustment equals the portion *preceding* the death (see Figure 6.4 - and note how this again brings out the duality between the two assumptions).

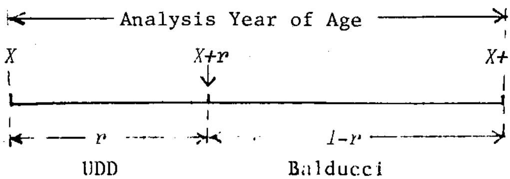


Figure 6.4. Complementarity of UDD and Balducci adjustments for death at age $X+r$

We can now write down UDD exposure formulas of the valuation schedule type with the same ease as the corresponding Balducci formulas.

Using either assumption, after the average-migration points have been chosen, we start with giving each life counted at any of the valuations an exposure equal to the time interval from the last preceding to the next following migration point; this gives us the C_x-term.

We then divide the analysis year $[X, X+1]$ in segments each with a valuation or census point at one end and a migration point at the other: for the Balducci exposure the deaths in each of these segments are considered to occur at the *migration* end and for UDD exposures at the *census* or *valuation* end. The appropriate adjustments, as in (iii) above, then form the D_x-term, giving each death an additional exposure from the migration point to the $X+1$ end of the analysis year of age in the Balducci case, and from the X end of the analysis year to the census-point in the UDD case.

We shall illustrate this by considering again the example of Section 3 (now generalized a little further by letting the α-period, i.e. the interval from the average birthday to the end of the observation year, be a fraction τ of a year rather than ½ year, so that P_x^z and P_x^{z+1} occur at age $x+\tau$).

For the C_x^z term we obtain immediately, under either assumption:

$$C_x^z = E_x^z \cdot \tau_\alpha + P_x^{z+1}\,(\tau-\tau_\alpha) + P_x^z\,(\tau_\delta-\tau) + E_{x+1}^z\,(1-\tau_\delta). \tag{6.5}$$

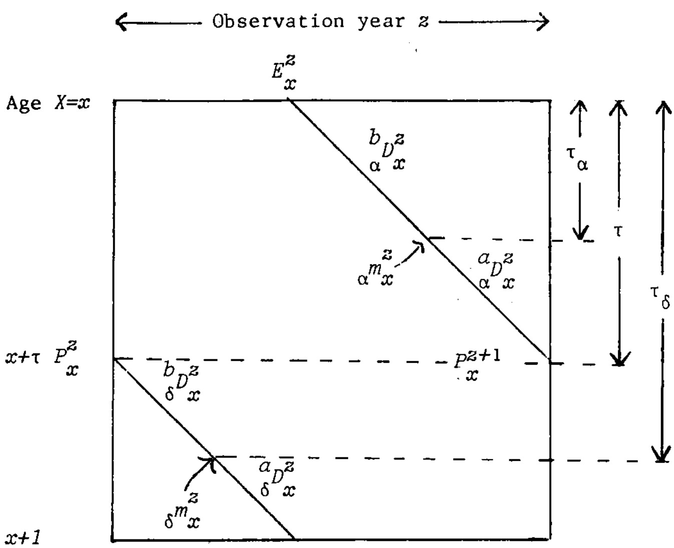


Figure 6.5 .

For the D_x^z term we divide the α-deaths in ${}_\alpha^a D_x^z$ and ${}_\alpha^b D_x^z$, the former denoting the deaths after and the latter those before the migration point: ${}_\alpha D_x^z = {}_\alpha^b D_x^z + {}_\alpha^a D_x^z$ and, similarly, ${}_\delta D_x^z = {}_\delta^b D_x^z + {}_\delta^a D_x^z$.

Under the Balducci assumption the deaths are deemed to occur at ages $x+\tau_\alpha$ and $x+\tau_\delta$, the migration points, and under the UDD at ages $x, x+\tau$ and $x+1$, the census points, so we have:

$$ {}^{B}\boldsymbol{D}_x^z = {}_{\alpha}D_x^z\,(1-\tau_{\alpha}) + {}_{\delta}D_x^z\,(1-\tau_{\delta}) \qquad (6.6) $$

and

$$ {}^{U}\boldsymbol{D}_x^z = ({}_{\alpha}^{a}D_x^z + {}_{\delta}^{b}D_x^z)\,\tau + {}_{\delta}^{a}D_x^z \qquad (6.7) $$

Adding (6.5) and 6.6) gives the Balducci exposure ${}^{B}\boldsymbol{E}_x^z$, adding (6.5) and (6.7) gives the corresponding UDD exposure ${}^{U}\boldsymbol{E}_x^z$. Note that ${}^{B}\boldsymbol{E}_x^z$ obtained here agrees with the last formula of Sec. 3 (p. 6).

If we let $\tau_{\alpha} \to 0$ and $\tau_{\delta} \to 1$ we obtain the first pair of formulas in Batten's Table 5.6 ([1], p. 159), by letting $\tau_{\alpha} \to \tau$ and $\tau_{\delta} \to \tau$ the second set appears.

The reader will have no trouble in writing down the formula's of Batten's Tables 5.7 and 5.8 ([1], pp. 160-161) with equal ease (see also [6], pp. 4-5).

It is to be noted that UDD formulas obtained as described here are all of the form (6.2) or 6.3); to obtain formulas of the form (6.1) we would have to "average" the deaths by letting ${}^{a}\theta_{x,j-1}$ occur at the next following migration point (age $X+\tau_j$) and ${}^{b}\theta_{x,j}$ at the last preceding one (age $X+\tau_{j-1}$, see Figure 6.6). Although, of course, there can be no doubt as to the correctness of this procedure, it does, on the surface, appear rather far-fetched.

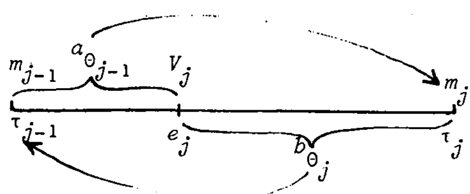


Figure 6.6

If we consider the difference between the Balducci adjustment

$$ {}^{B}\boldsymbol{D}_x = \sum_{0}^{n} \theta_j(1-\tau_j) = \sum_{1}^{n} {}^{a}\theta_{j-1}(1-\tau_{j-1}) + \sum_{1}^{n} {}^{b}\theta_j(1-\tau_j) $$

and the first form of the UDD adjustment in (6.4), i.e. the difference

$$^{U}\mathbf{D}_x - {}^{B}\mathbb{D}_x = \sum_{1}^{n} ({}^{a}\theta_{j-1} + {}^{b}\theta_j)(\tau_j + \tau_{j-1}) - \sum_{0}^{n} \theta_j \quad ,$$

it is clear that adding this difference (or its Type II counterpart) to the Balducci formulas (5.7) and (5.8) will result in individual card formulas valid under the UDD assumption. For instance, (5.7) becomes:

$$^{U}\mathbb{E}_x = \sum^{x-1} \left[s_r - e_r + \sum_{0}^{n} (m_{r,j} - \theta_{r,j}) \right]$$

$$+ s_x - e_x + \sum_{0}^{n} m_{x,j}(1-\tau_j)$$

$$- \theta_x + \sum_{1}^{n} ({}^{a}\theta_{x,j-1} + {}^{b}\theta_{x,j})(\tau_j + \tau_{j-1}).$$

7. CONCLUSION

If we consider exposure formulas from a practical point of view, nothing much of what Batten wrote in the last paragraph of his 1978 paper ([2]) has changed. Assumptions with respect to average birthdays, migration points and issue dates will still be of much greater influence on the accuracy of the results of mortality investigations than the assumptions specifying the instantaneous behaviour of mortality during the analysis year.

Balducci's assumption still appears to be the easier for developing exposure formulas and the formulas it produces are simpler. However, the assumption of a uniform distribution of deaths appears a more realistic one to make: not only is linearity of ${}_tq_x$ or l_{x+t}, a more natural assumption than that of ${}_{1-t}q_{x+t}$ and $1/l_{x+t}$, but it also leads to a force of mortality which decreases over the analysis year of age, in agreement with the pattern in by far the larger part of the mortality table and in contrast to Balducci's assumption for which the opposite is the case.

The main disadvantage of the UDD assumption in connection with exposure formulas has always been the problem of "unobserved deaths", but now that this onus has been lifted by Shiu's paper [6], it is gratifying to see that pairs of counterpart formulas of a very general form can also be constructed under this assumption and that they do not have to be limited to cases with restricted migration.

APPENDIX

We shall write E_x^i for the exposure obtained by the general individual card formula and E^v for the Type I valuation schedule conterpart formula.

For the proof that $E_x^v = E_x^i$ we use (4.3.1) for E^v:

$$E_x^v = C_x + D_x ,$$

$$C_x = \sum_1^n V_{x,j}(\tau_j - \tau_{j-1}), \qquad D_x = \sum_0^n \Theta_{x,j}(1-\tau_j)$$

For E_x^i we substitute $H_{x,j} = \sum_a^k H^z_{x,j} = m_{x,j} - \Theta_{x,j}$ in (5.7):

$$E_x^i = \sum^{x-1} (s_r - e_r + \sum_0^n H_{r,j}) + s_x - e_x + H_{x,0} + \sum_1^{n-1} H_{x,j}(1-\tau_j) + \Theta_{x,0} + \sum^{n-1} \Theta_{x,j}(1-\tau_j) \quad .$$

The terms in the last line equal D_x, so the remaining terms can be written as

$$E_x^i - D_x = \sum^{x-1} \left[s_r - e_r + H_{r,0} + H_{r,n} + \sum_1^{n-1} H_{r,j} \right] + s_x - e_x + H_{x,0} + \sum_1^{n-1} H_{x,j} - \sum_1^{n-1} H_{x,j}\tau_j = R + S \quad ,$$

with $R = \sum^x \left[s_r - e_r + H_{r,0} + H_{r-1,n} \right]$

and $S = \sum^{n-1} \sum^x H_{r,j} - \sum^{n-1} H_{x,j}\tau_j \quad .$

We first consider R. Substitution of s_r and e_r from (5.9) gives:

$$R = \sum^{x} \left[V^{a}_{r,1} - V^{k}_{r-1,n} + H_{r,0} - H^{a}_{r,0} + H_{r-1,n} - H^{k}_{r-1,n} \right] .$$

But $H_{r,0} - H^{a}_{r,0} = \sum_{a}^{k} H^{z}_{r,0} - H^{a}_{r,0} = \sum_{a}^{k-1} H^{z+1}_{r,0}$ and, similarly,

$$H_{r-1,n} - H^{k}_{r-1,n} = \sum_{a}^{k-1} H^{z}_{r-1,n} ,$$

so the last four terms in the summand become

$$\sum_{a}^{k-1} (H^{z+1}_{r,0} + H^{z}_{r-1,n}) = \sum_{a}^{k-1} (V^{z+1}_{r,1} - V^{z}_{r-1,n}) ,$$

the last expression being obtained from (4.2.2). Thus R becomes

$$R = \sum^{x} \left[V^{a}_{r,1} + \sum_{a}^{k-1} (V^{z+1}_{r,1} - V^{z}_{r-1,n}) - V^{k}_{r-1,n} \right]$$

$$= \sum^{x} \sum_{a}^{k} (V^{z}_{r,1} - V^{z}_{r-1,n})$$

$$= \sum^{x} (V_{r,1} - V_{r-1,n}) .$$

Returning to S we can write

$$S = \sum^{x} \sum_{1}^{n-1} \left[H_{r,j} - (H_{r,j} - H_{r-1,j})\tau_{j} \right]$$

From (4.2.1): $H_{r,j} = \sum_{a}^{k} H^{z}_{r,j} = V_{r,j+1} - V_{r,j}$ so

$$S = \sum^{x} \sum_{1}^{n-1} \left[V_{r,j+1} - V_{r,j} - \{(V_{r,j+1} - V_{r,j}) - (V_{r-1,j+1} - V_{r-1,j})\}\tau_{j} \right]$$

$$= \sum^{x} (V_{r,n} - V_{r,1}) - \sum_{1}^{n-1} (V_{x,j+1} - V_{x,j})\tau_{j} .$$

Now, putting R and S together again:

$$\boldsymbol{E}_x^i - \boldsymbol{D}_x = R + S$$

$$= \sum^{x} (V_{r,1} - V_{r-1,n}) + \sum^{x} (V_{r,n} - V_{r,1}) - \sum_1^{n-1} (V_{x,j+1} - V_{x,j})\tau_j$$

$$= \sum^{x} (V_{r,n} - V_{r-1,n}) - \sum_2^{n} V_{x,j}\, \tau_{j-1} + \sum_1^{n-1} V_{x,j}\, \tau_j \quad .$$

But $\sum^{x} (V_{r,n} - V_{r-1,n}) = V_{x,n} = V_{x,n} \cdot \tau_n$ (since $\tau_n = 1$) and, because

for $j = 1$, $\tau_{j-1} = \tau_0 = 0$, we have $\sum_2^{n} V_{x,j}\, \tau_{j-1} = \sum_1^{n} V_{x,j}\, \tau_{j-1}$,

we finally have

$$\boldsymbol{E}_x^i - \boldsymbol{D}_x = \sum_1^{n-1} V_{x,j}\, \tau_j + V_{x,n}\, \tau_n - \sum_1^{n-1} V_{x,j}\, \tau_{j-1}$$

$$= \sum_1^{n} V_{x,j}\, (\tau_j - \tau_{j-1}) = \boldsymbol{C}_x \quad ,$$

$$\therefore \quad \boldsymbol{E}_x^i = \boldsymbol{C}_x + \boldsymbol{D}_x = \boldsymbol{E}_x^v \quad \text{, q.e.d.}$$

REFERENCES

[1] R.W. Batten, Mortality Table Construction, Prentice Hall, Englewood Cliffs, New Jersey, 1978.

[2] __________, Some Further Comments on the Uniform Death Assumption, ARCH 1978.2, Society of Actuaries, 57-64.

[3] H. Gershenson, Measurement of Mortality, Society of Actuaries, Chicago, 1961.

[4] T.N.E. Greville, Estimation of the Rate of Mortality in the Presence of In-and-Out Movement, ARCH, 1978.2, Society of Actuaries, 41-56.

[5] W.R. Roach, A Graphical Technique for Writing Exposure Formulas, The Actuary, March, 1978, Society of Actuaries, 6-7.

[6] E.S.W. Shiu, Duality between the Uniform Distribution of Deaths and the Balducci Assumptions, ARCH, this issue.

[7] Society of Actuaries, 1967 Yearbook, Chicago, 1967, 214.