A MORE GENERAL PRESENTATION OF PENSION FUNDING

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ABSTRACT

This paper improves Chapter 2 of "Mathematics of Pension Funding"(Part 7 Study Notes). Key formulas are derived in more general fashions than in Study Notes. For three individual cost methods, concise formulations and auxiliary diagrams of normal costs and accrued liabilities are also presented.

I. INTRODUCTION

We shall adopt symbols used in Study Notes. In addition, we use ${}^{s}_{t}Q^{j}_{z}$ to denote that Q is a quantity relating to an individual j at age z determined at time t under salary-increased assuption(SIA). The subscipt t will be omitted, if t is the time at which j's age is z or in the case that Q is independent of t. We also define

$$ {}^{s}_{\Delta t}Q^{j}_{z} = {}^{s}_{t+1}Q^{j}_{z} - {}^{s}_{t}Q^{j}_{z} \,. $$

Let w_j be the entry age of j. If t is the current time of discussion, we define

$$ x_j = w_j + t \,. $$

For convenience, we define

$$ u^{*} = u + 1 $$

for an integral quantity u.

Let $\mathcal{C}$ be a category of employees covered by the plan. Then we define

$$ \sum_{\mathcal{C}} {}^{s}_{t}Q^{j}_{z} = \sum_{j \in \mathcal{C}} {}^{s}_{t}Q^{j}_{z} \,, $$

$$ \sum_{q\mathcal{C}} {}^{s}_{t}Q^{j}_{z} = \sum_{j \in \mathcal{C}} q_{z} \cdot {}^{s}_{t}Q^{j}_{z} \,, $$

and

$$\sum_{\Delta C} {}_t^sQ_z^j = \sum_{j \in C} {}_{\Delta t}^sQ_z^j .$$

We also define

$$Q_t^s = \sum_{\mathcal{O}_t} {}^sQ_{x_j}^j .$$

For convenience, we define

$$\mathcal{O}_{\cap t} = \mathcal{O}_t \cap \mathcal{O}_{t^*}$$

and

$$\sum_{\mathcal{O}_t^*} {}_t^sQ_{x_j^*}^j = \sum_{\mathcal{O}_t + \Delta\mathcal{O}_{\cap t} - \mathcal{J} - R} {}_t^sQ_{x_j^*}^j .$$

Let y be the uniform retirement age. Then we define

$$\ddot{a}_{z|y}^{(12)} = \frac{D_y}{D_z} \ddot{a}_y^{(12)} .$$

We also define

$${}^s\ddot{a}_{z:v}^{-1} = \frac{{}^sD_z}{{}^sN_z - {}^sN_v}$$

and

$${}^s\ddot{a}_{z:\overset{1}{v}}^{-1} = \frac{{}^sD_v}{{}^sN_z - {}^sN_v} .$$

For quantities that we have so far defined, if SIA is not imposed, we simply drop the superscript s. We shall assume that SIA is imposed throughout our discussion. However, all the formulas that we are dealing with also hold true without the superscript s(i.e. without assuming SIA).

We use PVFS to denote the present value of future service so that ${}^{s}PVFS_{x_j}^{j}$ stands for the present value of future salary and $PVFS_{x_j}^{j}$ the present value of future years. Thus

$$ {}^{s}PVFS_{x_j}^{j} = S_{x_j}^{j} \cdot \frac{{}^{s}N_{x_j} - {}^{s}N_y}{{}^{s}D_{x_j}} , $$

$$ {}_{t}^{s}PVFS_{x_j^*}^{j} = S_{x_j}^{j} \cdot \frac{S_{x_j^*}}{S_{x_j}} \cdot \frac{{}^{s}N_{x_j^*} - {}^{s}N_y}{{}^{s}D_{x_j^*}} , $$

and

$$ PVFS_{x_j}^{j} = \frac{N_{x_j} - N_y}{D_{x_j}} . $$

Note that $PVFS_{x_j^*}^{j}$ is the counterpart of both ${}_{t}^{s}PVFS_{x_j^*}^{j}$ and ${}^{s}PVFS_{x_j^*}^{j}$. Note also that

$$ {}_{t}^{s}B_{z}^{j} = \frac{S_{x_j}}{S_{w_j}} \, {}_{t}B_{z}^{j} . $$

We shall make use of the following relations to derive our formulas.

(1.1) $$ {}_{t}^{s}PVFNC_{z}^{j} = \sum_{v=z}^{y} {}_{t}^{s}NC_{v}^{j} \frac{D_v}{D_z} $$

(1.2) $$ {}_{t}^{s}PVFB_{z}^{j} = {}_{t}^{s}B_{z}^{j} \, \ddot{a}_{z|y}^{(12)} $$

(1.3) $$ {}_{t}^{s}B_{x_j^*}^{j} = {}^{s}B_{x_j}^{j} $$

(1.4) $$ \frac{p_{x_j}}{D_{x_j^*}} = \frac{1+i}{D_{x_j}} . $$

II. KEY FORMULAS FOR A GENERAL METHOD

(2.1) $$Q^s_{t^*} = \sum_{\mathcal{O}_t} p_{x_j}\, {}^s_t Q^j_{x_j^*} + \sum_{\mathcal{O}^*_t} {}^s_t Q^j_{x_j^*} + \sum_{\mathcal{N}} {}^s Q^j_{x_j^*}$$

(2.2) $$PVFNC^s_{t^*} = (PVFNC^s_t - NC^s_t)(1+i) + \sum_{\mathcal{O}^*_t} {}^s_t PVFNC^j_{x_j^*} + \sum_{\mathcal{N}} {}^s PVFNC^j_{x_j}$$

(2.3) $$PVFB^s_{t^*} = PVFB^s_t (1+i) + \sum_{\mathcal{O}^*_t} {}^s_t PVFB^j_{x_j^*} + \sum_{\mathcal{N}} {}^s PVFB^j_{x_j^*}$$

(2.4) $${}^s_t AL^j_{x_j^*} \cdot p_{x_j} = ({}^s AL^j_{.x_j} + {}^s NC^j_{x_j})(1+i)$$

(2.5) $$AL^s_{t^*} = (AL^s_t + NC^s_t)(1+i) + \sum_{\mathcal{O}^*_t} {}^s_t AL^j_{x_j^*} + \sum_{\mathcal{N}} {}^s AL^j_{x_j^*}$$

(2.6) $$UAL^s_{t^*} = UAL^s_t (1+i) + [(iF_t + I_c - I_p) - I] - [(C + I_c) - NC^s_t (1+i)]$$
$$+ \sum_{\Delta \mathcal{O}_{nt}} {}^s_t AL^j_{x_j^*} - \sum_{\mathcal{T}-g\mathcal{O}_t} {}^s_t AL^j_{x_j^*} + [(P + I_p) - \sum_{\mathcal{R}} {}^s_t AL^j_{x_j^*}] + \sum_{\mathcal{N}} {}^s AL^j_{x_j^*}$$

(2.7) $$U^s_{t^*} = U^s_t + \Big[\sum_{\mathcal{O}_{nt}} {}^s PVFS^j_{x_j^*}\Big]^{-1} \Big\{ [(iF_t + I_c - I_p) - I]$$
$$+ \Big[\sum_{\Delta \mathcal{O}_{nt}} {}^s_t PVFB^j_{x_j^*} - U^s_t \sum_{\Delta \mathcal{O}_{nt}} {}^s_t PVFS^j_{x_j^*}\Big]$$
$$- \sum_{\mathcal{T}-g\mathcal{O}_t} ({}^s_t PVFB^j_{x_j^*} - U^s_t \cdot {}^s_t PVFS^j_{x_j^*})$$
$$+ [(P + I_p) - \sum_{\mathcal{R}} {}^s_t PVFB^j_{x_j^*}]$$
$$+ \sum_{\mathcal{N}} ({}^s PVFB^j_{x_j^*} - U^s_t \cdot {}^s PVFS^j_{x_j^*})$$
$$+ G^s_t \Big\},$$

where G stands for Gain.

III. DERIVATION OF (2.1)

Since

$$\mathcal{A}_{nt} = \mathcal{A}_{t^*} - \mathcal{N} = \mathcal{A}_t - \mathcal{T} - \mathcal{R},$$

we have

$$\begin{aligned}
Q^{s}_{t^*} &= \sum_{\mathcal{A}_{t^*}} {}^{s}Q^{j}_{x_j^*} \\
&= \sum_{\mathcal{A}_t - \mathcal{T} - \mathcal{R}} {}^{s}Q^{j}_{x_j^*} + \sum_{\mathcal{N}} {}^{s}Q^{j}_{x_j^*} \\
&= \sum_{\mathcal{A}_t - \mathcal{T} - \mathcal{R}} {}^{s}_{t}Q^{j}_{x_j^*} + \sum_{\mathcal{A}_{nt}} {}^{s}_{\Delta t}Q^{j}_{x_j^*} + \sum_{\mathcal{N}} {}^{s}Q^{j}_{x_j^*} \\
&= \sum_{\mathcal{A}_t} p_{x_j} \cdot {}^{s}_{t}Q^{j}_{x_j^*} + \sum_{g\mathcal{A}_t - \mathcal{T} - \mathcal{R} + {}_{\Delta}\mathcal{A}_{nt}} {}^{s}_{t}Q^{j}_{x_j^*} + \sum_{\mathcal{N}} {}^{s}Q^{j}_{x_j^*},
\end{aligned}$$

which is (2.1).

IV. DERIVATION OF (2.2), (2.3), AND (2.5)

From (1.1) and (1.4), we have

$$\begin{aligned}
\sum_{\mathcal{A}_t} p_{x_j} \cdot {}^{s}_{t}PVFNC^{j}_{x_j^*} &= \sum_{\mathcal{A}_t} \sum_{z=x_j^*}^{y} {}^{s}_{t}NC^{j}_{z} \cdot D_z \frac{p_{x_j}}{D_{x_j^*}} \\
&= \sum_{\mathcal{A}_t} \Big[\Big(\sum_{z=x_j}^{y} {}^{s}_{t}NC^{j}_{z} D_z \Big) - {}^{s}NC^{j}_{x_j} D_{x_j} \Big] \frac{1+i}{D_{x_j}} \\
&= \sum_{\mathcal{A}_t} \left({}^{s}PVFNC^{j}_{x_j} - {}^{s}NC^{j}_{x_j} \right)(1+i) \\
&= \left(PVFNC^{s}_{t} - NC^{s}_{t} \right)(1+i). \qquad (4.1)
\end{aligned}$$

Hence (2.2) follows from (2.1) and (4.1).

From (1.4), (1.2) and (1.3), we have

$$\sum_{\mathfrak{N}_t} p_{x_j} \cdot {}^{s}_{t}PVFB^{j}_{x_j^*} = \sum_{\mathfrak{N}_t} p_{x_j} \cdot {}^{s}_{t}B^{j}_{x_j^*}\, \ddot{a}^{(12)}_{y} \frac{D_y}{D_{x_j^*}}$$

$$= \sum_{\mathfrak{N}_t} p_{x_j} \cdot {}^{s}B^{j}_{x_j}\, \ddot{a}^{(12)}_{y} \frac{D_y}{D_{x_j^*}}$$

$$= \sum_{\mathfrak{N}_t} {}^{s}B^{j}_{x_j}\, \ddot{a}^{(12)}_{y} \frac{D_y(1+i)}{D_{x_j}}$$

$$= \sum_{\mathfrak{N}_t} {}^{s}B^{j}_{x_j}\, \ddot{a}^{(12)}_{x_j|y}(1+i)$$

$$= \sum_{\mathfrak{N}_t} {}^{s}PVFB^{j}_{x_j}(1+i)$$

$$= PVFB^{s}_{t}\,(1+i). \qquad (4.2)$$

Hence (2.3) follows from (2.1) and (4.2).

Since

$${}^{s}_{t}AL^{j}_{j} = {}^{s}_{t}PVFB^{j}_{j} - {}^{s}_{t}PVFNC^{j}_{j},$$

(2.5) follows from (2.2) and (2.3).

V. DERIVATION OF (2.4) AND (2.5) VIA DIFFERENTIAL EQUATION OF THE CONTINUOUS INDIVIDUAL ACCRUED LIABILITY

We shall take for granted that the differential equation

$$\frac{d\, {}^{s}\overline{AL}^{j}_{x_j}}{dx_j} = {}^{s}\overline{NC}^{j}_{x_j} + (\mu_{x_j} + \delta)\, {}^{s}\overline{AL}^{j}_{x_j}$$

holds true for any cost method. For discrete case, the equation becomes

$${}^{s}_{t}AL^{j}_{x_j^*} - {}^{s}AL^{j}_{x_j} = {}^{s}NC^{j}_{x_j} + \left(\frac{D_{x_j} - D_{x_j^*}}{D_{x_j}} \right) {}^{s}_{t}AL^{j}_{x_j^*} .$$

This is algebraically equivalent to (2.4). Thus (2.5) follows from (2.1) and (2.4).

For unit credit and individual level premium methods, since ${}^{s}_{\Delta t}O^{j}_{x_j^*} = 0$, (2.4) and (2.5) become

$$p_{x_j} \cdot {}^{s}AL^{j}_{x_j^*} = \left({}^{s}AL^{j}_{x_j} + {}^{s}NC^{j}_{x_j} \right)(1+i)$$

and

$$AL^{s}_{t^*} = \left(AL^{s}_{t} + NC^{s}_{t} \right)(1+i) + \sum_{j \in \mathcal{A}_t - \mathcal{T} - \mathcal{R} + \mathcal{N}} {}^{s}AL^{j}_{x_j^*} .$$

VI. DERIVATION OF (2.4) AND (2.5) VIA RETROACTIVE DEFINITION OF THE INDIVIDUAL ACCRUED LIABILITY

For any cost method, we can always write

$${}^{s}_{t}AL^{j}_{z} = \sum_{v=w_j}^{z-1} {}^{s}_{t}NC^{j}_{v} \frac{D_v}{D_z} .$$

Then

$$\begin{aligned} p_{x_j} \cdot {}^{s}_{t}AL^{j}_{x_j^*} &= p_{x_j} \sum_{z=w_j}^{x_j} {}^{s}_{t}NC^{j}_{z} \frac{D_z}{D_{x_j^*}} \\ &= \left[\left(\sum_{z=w_j}^{x_j-1} {}^{s}_{t}NC^{j}_{z} D_z \right) + {}^{s}NC^{j}_{x_j} D_{x_j} \right] \frac{p_{x_j}}{D_{x_j}} \\ &= \left({}^{s}AL^{j}_{x_j} D_{x_j} + {}^{s}NC^{j}_{x_j} D_{x_j} \right) \cdot \frac{1+i}{D_{x_j}} \\ &= \left({}^{s}AL^{j}_{x_j} + {}^{s}NC^{j}_{x_j} \right)(1+i), \end{aligned}$$

which is (2.4). Again, (2.5) follows from (2.1) and (2.4).

VII. DERIVATION OF (2.6) AND (2.7)

Since

$$F_{t^*} = F_t + I + C - P$$

and

$$UAL_t^s = AL_t^s - F_t ,$$

(2.6) follows immediately from (2.5). Note that the second, third, fifth, and sixth terms of (2.6) are deviations of actual experience from the expected.

For any cost method, we can write

$${}^{s}PVFNC_{x_j}^{j} = U_t^s \cdot {}^{s}PVFS_{x_j}^{j}$$

and

$${}_{t}^{s}PVFNC_{x_j^*}^{j} = U_t^s \cdot {}_{t}^{s}PVFS_{x_j^*}^{j} .$$

Then, from (2.6), we have

$$\begin{aligned} UAL_{t^*}^s = {} & UAL_t^s(1+i) + [(iF_t + I_c - I_p) - I] - [(C+I_c) - NC_t^s(1+i)] \\ & + \Big[\sum_{\Delta\mathcal{A}_{nt}} {}_{t}^{s}PVFB_{x_j^*}^{j} - \Big(U_{t^*}^s \sum_{\mathcal{A}_{nt}} {}^{s}PVFS_{x_j^*}^{j} - U_t^s \sum_{\mathcal{A}_{nt}} {}_{t}^{s}PVFS_{x_j^*}^{j}\Big)\Big] \\ & - \sum_{\mathcal{J}-\{\mathcal{A}_t} \left({}_{t}^{s}PVFB_{x_j^*}^{j} - U_t^s \cdot {}_{t}^{s}PVFS_{x_j^*}^{j} \right) \\ & + \Big[(P + I_p) - \sum_{\mathcal{R}} {}_{t}^{s}PVFB_{x_j^*}^{j}\Big] \\ & + \sum_{\mathcal{N}} \left({}^{s}PVFB_{x_j^*}^{j} - U_t^s \cdot {}^{s}PVFS_{x_j^*}^{j} \right). \end{aligned}$$

Since

$$G_t^s = (UAL_t^s + NC_t^s)(1+i) - (C + I_c) - UAL_{t^*}^s ,$$

(2.7) follows immediately.

Since $PVFS^{j}_{x_j*}$ is the counterpart of both ${}^{S}_{t}PVFS^{j}_{x_j*}$ and ${}^{S}PVFS^{j}_{x_j*}$, if SIA is not imposed, (2.7) becomes

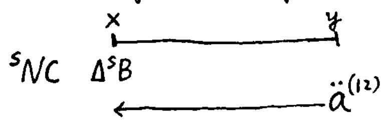
$$\begin{aligned} U_{t*} = U_t + \Big[\sum_{\mathcal{A}_{nt}} PVFS^{j}_{x_j*}\Big]^{-1} \Big\{ & [(iF_t + I_c - I_p) - I] \\ & + \sum_{\Delta\mathcal{A}_{nt}} {}_{t}PVFB^{j}_{x_j*} \\ & - \sum_{\mathcal{T}-g\mathcal{A}_t} ({}_{t}PVFB^{j}_{x_j*} - U_t \cdot {}_{t}PVFS^{j}_{x_j} \\ & + [(P + I_p) - \sum_{\mathcal{R}} {}_{t}PVFB^{j}_{x_j*}] \\ & + \sum_{\mathcal{N}} (PVFB^{j}_{x_j*} - U_t \cdot PVFS^{j}_{x_j*} \\ & + G_t \Big\}. \end{aligned}$$

For frozen initial liability and aggregate methods, G^{S}_{t} and G_t are zero.

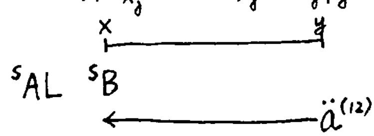
VIII. NORMAL COSTS AND ACCRUED LIABILITIES FOR INDIVIDUAL COST METHODS

1. Unit Credit Method

$${}^{s}NC^{j}_{x_j} = (\Delta {}^{s}B^{j}_{x_j})\, \ddot{a}^{(12)}_{x_j|y}$$

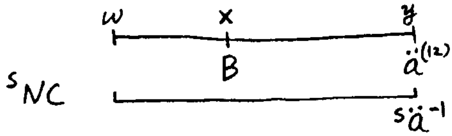


$${}^{s}AL^{j}_{x_j} = {}^{s}B^{j}_{x_j}\, \ddot{a}^{(12)}_{x_j|y}$$

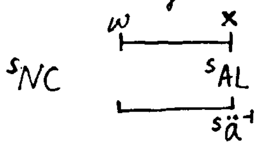


2. Entry Age Normal Cost Method

$$ {}^{s}NC_{x_j}^{j} = {}_{t}B_y^{j}\, \ddot{a}_y^{(12)}\, {}^{s}\ddot{a}_{w_j:\overset{1}{y}}^{-1} $$

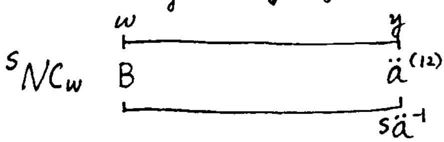


$$ {}^{s}NC_{x_j}^{j} = {}^{s}AL_{x_j}^{j}\, {}^{s}\ddot{a}_{w_j:\overset{1}{x_j}}^{-1} $$

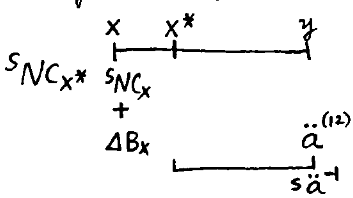


3. Individual Level Premium Method

$$ {}^{s}NC_{w_j}^{j} = B_{w_j}^{j}\, \ddot{a}_y^{(12)}\, {}^{s}\ddot{a}_{w_j:\overset{1}{y}}^{-1} $$



$$ {}^{s}NC_{x_j^*}^{j} = {}^{s}NC_{x_j}^{j} + (\Delta B_{x_j}^{j})\, \ddot{a}_y^{(12)}\, {}^{s}\ddot{a}_{x_j^*:\overset{1}{y}}^{-1} $$



IX. REFERENCE

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