IMMUNIZATION THEORY: A SIMPLIFIED EXAMPLE

by

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1. Introduction

Redington (1952) described immunization as "the investment of assets in such a way that existing business is immune to a general change in the rate of interest." In other words, immunization involves a mathematical model which may be used to build an investment portfolio which will minimize the risk of interest rate fluctuation to a firm holding a set of financial liabilities and the investment portfolio. Redington concluded that the essence of the immunization model rests on two rules:

"Rule (1). The mean term of the value of asset-proceeds must equal the mean term of the liability-outgo.

Rule (2). The spread about the mean of the value of the asset-proceeds should be greater than the spread of the value of the liability-outgo."

To obtain a better understanding of what is meant by these rules, we must first define a few terms.

FI(t): Cash flow rate from existing investment assets at time t.

FO(t): Net cash flow rate from insurance operations at time t, i.e., (expenses + claims - premiums). In an individual life insurance operation, FO(t) would be increased by the amount of policy loans and decreased by their repayment.

i: Annual effective interest rate and $\delta = \ln(1+i)$, the equivalent force of interest.

$A(\delta)$: Asset-proceeds valued at force of interest δ. That is,

$$A(\delta) = = \int_0^\infty v^t \; FI(t)dt,$$

where $v = (1+i)^{-1} = e^{-\delta}$.

$L(\delta)$: Liability-outgo valued at force of interest δ. That is,

$$L(\delta) = \int_0^\infty v^t \; FO(t)dt.$$

$S(\delta)$: Surplus valued at force of interest δ,

$$S(\delta) = A(\delta) - L(\delta).$$

To minimize the effect of interest rate change on $S(\delta)$, an investment portfolio is sought such that at the valuation interest rate δ,

$$\frac{dS(\delta)}{d\delta} = 0 \text{ and } \frac{d^2S(\delta)}{d\delta^2} > 0.$$

These two conditions lead to mathematical interpretations of Rules (1) and (2),

$$\int_0^\infty tv^t \; FI(t)dt = \int_0^\infty tv^t \; FO(t)dt,$$
$$\int_0^\infty t^2v^t \; FI(t)dt > \int_0^\infty t^2v^t \; FO(t)dt.$$

Another quantity by which we can study the risk to the company of interest rate variation is the surplus ratio, denoted by

$$R(\delta) = 1 - L(\delta)/A(\delta).$$

The first and second derivatives of the surplus ratio lead to duration and dispersion conditions which also reflect Redington's rules.

We seek to develop an asset portfolio so that $R(\delta)$ is a minimum at the valuation rate of interest appropriate at the valuation date. We start by finding a solution to

$$R'(\delta) = - L'(\delta)/A(\delta) + L(\delta)A'(\delta)/A(\delta)^2 = 0.$$

The solution occurs when

$$- L'(\delta) \; A(\delta) + L(\delta) \; A'(\delta) = 0$$

or

$$A'(\delta)/A(\delta) = L'(\delta)/L(\delta).$$

We define the mean term, or duration, of asset-proceeds by

$$D_A(\delta) = -A'(\delta)/A(\delta),$$

and the mean term, or duration, of liability cash flows by

$$D_L(\delta) = -L'(\delta)/L(\delta).$$

Therefore, the first order condition for a minimum of $R(\delta)$ is

$$D_H(\delta) = D_L(\delta). \tag{1}$$

The second order condition for a minimum of $R(\delta)$ is

$$\frac{d^2}{d\delta^2} R(\delta) > 0.$$

This condition yields

$$\frac{[-A(\delta)L''(\delta) + A''(\delta)L(\delta)]}{A(\delta)^2} + \frac{2A'(\delta)[A(\delta)L'(\delta) - L(\delta)A'(\delta)]}{A(\delta)^3} > 0.$$

The second term of the left hand side of this expression is zero when the first order condition, equation (1), is satisfied. Therefore, the second order condition for $R(\delta)$ to be a minimum is satisfied when

$$A''(\delta)/A(\delta) > L''(\delta)/L(\delta). \tag{2}$$

We will denote $A''(\delta)/A(\delta)$ by ${}_2D_A(\delta)$ and $L''(\delta)/L(\delta)$ by ${}_2D_L(\delta)$. These quantities can be interpreted as second moments, or spread measures, as required by Rule (2), because

$${}_2D_A(\delta) = \int_0^\infty t^2[v^t \; FI(t)]dt/\int_0^\infty v^t \; FI(t)dt$$

and

$${}_2D_L(\delta) = \int_0^\infty t^2[v^t \; FO(t)]dt/\int_0^\infty v^t \; FO(t)dt.$$

When $FO(t) = kFI(t)$, $0 < k \leq 1$, we say that the cash flows are <u>matched</u> and $R(\delta)$ will be constant. In the matched case $D_A(\delta) - {}_LD(\delta) = {}_2D_A(\delta) - {}_2D_L(\delta) = 0$.

2. General Example

In the example it will be assumed that the cash flow rates may be represented by a gamma function,

$$F(t) = k(1 + \delta\ \beta)^{\alpha}\ t^{\alpha-1}\ e^{-t/\beta}\ /\ \Gamma(\alpha)\ \beta^{\alpha}.$$

It is easy to show that

$$\int_0^{\infty} v^t F(t)\,dt = k,$$

$$D(\delta) = \beta\ \alpha\ /\ (1 + \beta\ \delta),$$

$$_2D(\delta) = \beta^2\ \alpha\ (\alpha + 1)\ /\ (1 + \beta\ \delta)^2.$$

If the term structure of interest rates is flat, cash flows rates are certain and independent of the interest rate with gamma type distributions, then by using equations (1) and (2), $R(\delta)$ is a minimum if

$$\beta_A\alpha_A\ /\ (1 + \beta_A\delta) = \beta_L\alpha_L\ /\ (1 + \beta_L\delta)$$

and

$$\beta_A^2\ \alpha_A\ (\alpha_A + 1)\ /\ (1 + \beta_A\delta)^2 > \beta_L^2\ \alpha_L\ (\alpha_L + 1)\ /\ (1 + \beta_L\delta)^2.$$

In these expressions symbols with subscript A are parameters of the asset cash flow rate function, and symbols with subscript L are parameters of the liability cash flow rate function.

3. Specific Examples

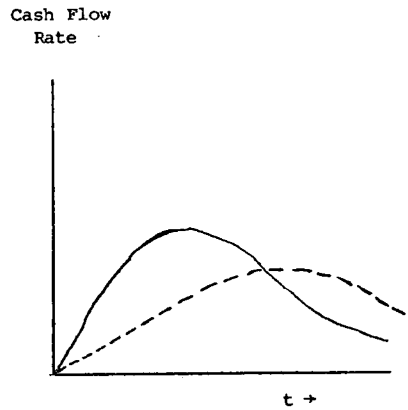
In the following examples the cash flow rates are assumed to be gamma functions to facilitate integration, comparison and observation. The examples are closely related to those in the report of the Society of Actuaries Committee on Valuation and Related Matters (1979). Other assumptions include: (1) cash flows are certain, (2) the recent market rate is $\delta = 7\%$, (3) the lower bound on the feasible range of the force of interest is 3%, (4) the corresponding upper bound is 11%, and (5) the term structure of interest rates is flat.

There are three companies "Long Gamma Company" (with a longer liability than asset duration), "Short Gamma Company" and "Matching Gamma Company." The cash flow structures of these companies correspond to their names and are given by:

Long Gamma Company:

$$FI(t) = 100{,}000 \frac{(1.07)^5}{\Gamma(5)} t^4 e^{-t}$$

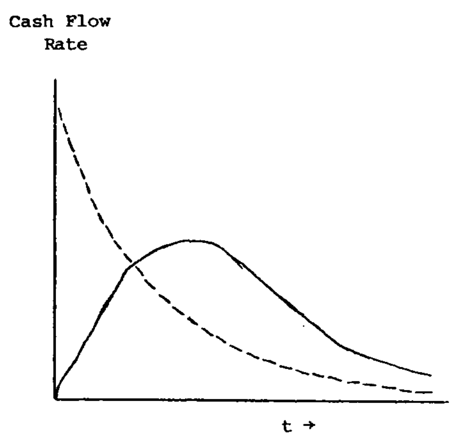
$$FO(t) = 80{,}000 \frac{(1.07)^{10}}{\Gamma(10)} t^9 e^{-t}$$



Short Gamma Company:

$$FI(t) = 100{,}000 \frac{(1.07)^5}{\Gamma(5)} t^4 e^{-t}$$

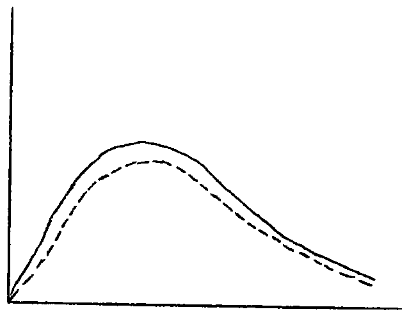
$$FO(t) = 80{,}000\ (1.07)\ e^{-t}$$



Matching Gamma Company:

$$FI(t) = 100{,}000 \frac{(1.07)^5}{\Gamma(5)} t^4 e^{-t}$$

$$FO(t) = 80{,}000 \frac{(1.07)^5}{\Gamma(5)} t^4 e^{-t}$$



________________: FI(t)

- - - - - - - - - -: FO(t)

If cash flows are evaluated at the market rate of 7%, each company has an asset value of A(.07) = 100,000, and each company has a liability value of L(.07) = 80,000. The following table contains summary measures of the cash flows:

	Assets		Liabilities	
	$D_A(.07)$	$_2D_A(.07)$	$D_L(.07)$	$_2D_L(.07)$
Long Gamma	4.67	26.20	9.35	96.08
Short Gamma	4.67	26.20	.93	1.75
Matching Gamma	4.67	26.20	4.67	26.20

Note that none of the three companies satisfy conditions (1) and (2). However, $R(\delta)$ is a constant .2 for Matching Gamma Company. Table 1, and associated Figure 1, show $A(\delta)$ and $L(\delta)$ from the balance sheets and $R(\delta)$, derived from the balance sheets, of the three companies at various forces of interest within the feasible range of .03 to .11.

The C-3 contingency reserve is that fund needed to guard against insolvency due to interest rate change. If 7% is used to value assets and liabilities and the C-3 reserve must guard against insolvency due to interest force variation between .03 and .11, the balance sheets are as follows:

<u>Long Gamma Co.</u> ($R(\delta_0) = .0321$)

Assets	100,000	Insurance Reserve	80,000
		C-3 Reserve	16,790
		Surplus	3,210
	100,000		100,000

<u>Short Gamma Co.</u> ($R(\delta_0) = .0735$)

Assets	100,000	Insurance Reserve	80,000
		C-3 Reserve	12,650
		Surplus	7,350
	100,000		100,000

<u>Matching Gamma Co.</u> ($R(\delta_0) = .20$)

Assets	100,000	Insurance Reserve	80,000
		C-3 Reserve	0
		Surplus	20,000
	100,000		100,000

In computing these balance sheets

$$\text{C-3 Reserve} = [A(.07) - L(.07)] - R(\delta_0)\,A(.07),$$

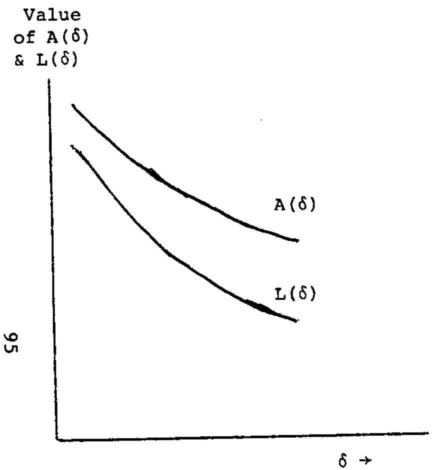
$$= S(.07) - R(\delta_0)\,A(.07).$$

where δ_0 is the force of interest within the feasible interval, .03 to .11 in our example, that will produce a minimum value of $R(\delta)$. These balance sheets follow the suggestion on page 276 of the preliminary report of the Society of Actuaries Committee on Valuation for an explicit C-3 reserve.

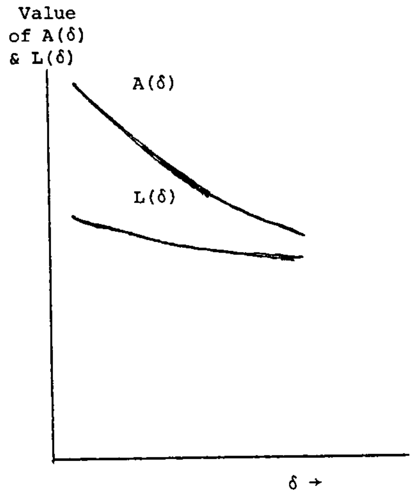
Table 1

	Long Gamma			Short Gamma			Matching Gamma		
δ	A(δ)	L(δ)	R(δ)	A(δ)	L(δ)	R(δ)	A(δ)	L(δ)	R(δ)
3%	120,985	117,099	3.21%	120,985	83,107	31.31%	120,985	96,788	20.00%
5%	109,894	96,612	12.08%	109,894	81,523	25.82%	109,894	87,915	20.00%
7%	100,000	80,000	20.00%	100,000	80,000	20.00%	100,000	80,000	20.00%
9%	91,156	66,476	27.07%	91,156	78,532	13.85%	91,156	72,924	20.00%
11%	83,235	55,434	33.41%	83,235	77,117	7.35%	83,235	66,588	20.00%

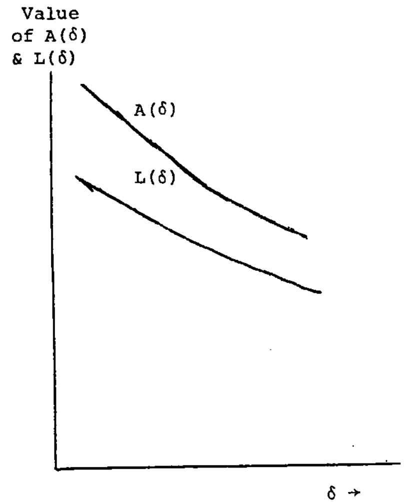
Figure 1



Value of assets and liabilities of Long Gamma Co.



Value of assets and liabilities of Short Gamma Co.



Value of assets and liabilities of Matching Gamma Co.

Under the simplifying assumptions of the example, the surplus in each balance sheet, $R(\delta_0)\ A(.07)$, could be paid out and assets would still equal or exceed liabilities if a worse case interest rate change occurred within the feasible interval.

If liabilities are valued at a special interest rate, the C-3 reserve may be combined with the insurance reserve. In our example with δ_1 denoting the special liability valuation rate to be determined, we have for Long Gamma Company,

$$L(.07) + C\text{-}3 = 96{,}790 = \frac{80{,}000(1.07)^{10} \int_0^\infty e^{-\delta_1 t} t^9 e^{-t} dt}{\Gamma(10)}$$

$$96{,}790 = 80{,}000(1.07)^{10} \ / \ (1 + \delta_1)^{10}$$

$$\delta_1 = .0498$$

For Short Gamma Company's liabilities, a valuation rate to permit insurance reserves and C-3 reserves to be combined does not exist.

4. Limitations

In the business world, immunization theory is not easy. Some of the problems associated with implementing immunization theory in a life insurance company are:

(1) Fixed cash flows is not a very realistic assumption since withdrawal benefits such as policy loans and cash value withdrawals depend on unpredictable economic conditions. Also call bonds, options and other types of assets make cash flows very difficult to estimate.

(2) Some companies are hesitant to use immunization theory because along with protecting against losses, it also protects the company against gains.

(3) For life insurance companies, the weighted average time of insurance cash flows is very long which makes it very difficult to reach an immunized position.

5. References

(1) Redington, F. M. "Review of the Principles of Life-Office Valuation," JIA, Vol. 78 (1952).

(2) Society of Actuaries Committee on Valuation and Related Matters, Preliminary Report, RSA, Vol. 5 (1979).

6. Notes

(1) The main ideas in the note are a small part of an MS thesis written by David C. Wu at the University of Wisconsin in 1980-81. Laura L. Schumacher extended some of the ideas as a special project in 1982. J. C. Hickman supervised both efforts, selected and amalgamated the results.