An Exposure Formula Controversy

The letter entitled "An attempt to convert American Actuaries" by H. Seal appeared in the March, 1981 "Actuary". This letter challenged the method American actuaries use to calculate exposures in estimating mortality rates. There was a short reply by William J. Sohn that was published in the June, 1981 "Actuary". Four other letters were received responding to the original letter. Unfortunately, there was not sufficient space to print these responses in the "Actuary", and therefore they are being printed here. The Broffitt and Klugman, Sohn, and R. Edwards letters discuss formulas for estimating mortality. The final two letters are bibliographical.

AN ATTEMPT TO CONVERT AMERICAN ACTUARIES

by Hilary L. Seal

The history surrounding Gershenson's recipe for deaths among "existings" may be summed up in quotations spanning 72 years:

> 1898: "As the period of observation terminates at the close of a calendar year, the cases 'existing' are necessarily under observation for a portion only of the year of duration [age] then current, and some of the cases of death...during the last calendar year, would, if treated as 'existing', have in like manner completed only a portion of the year of duration [age] current at exit. In strictness, therefore, such cases should contribute to the number exposed to risk, not the full year of duration [age] current at exit, but only that portion of the year which actually fell within the period of observation."
>
> Thomas G. Ackland, J.I.A. 33, 193

> 1943: "...it is evident that observed deaths before age x+1 among the n_{x+k} entrants [at exact age x+k] are to be 'exposed' for the period 1-k and not for the full year."
>
> Ralph E. Edwards,
> T.A.S.A. XLIV, 34

> 1943: ",,,it seems desirable to remark...that the entrants n_{x+k} obviously can be exposed for not more than the period 1-k after entrance (whether they live of die)..."
>
> Hugh H. Wolfenden,
> T.A.S.A. XLIV, 61

> 1945: "...the...exposed to risk...produces exactly the same number as would result from counting...a fraction of a unit for each...person corresponding to the fraction of the year of age during which he was both insured and under observation, *deaths in all cases being treated as if they had occurred at the end of the year of age*. [Italics in original]
>
> Edward W. Marshall,
> T.A.S.A. XLVI, 38

> 1961: "...some students (and even some experts)...think of the exposure...as 'the distance from the point of entry to the end of the observation period, or the end of the unit age interval, whichever end-point occurs sooner'. That this line of thought is inconsistent with $_{1-t}q_{x+t} = (1-t)q$...is easily seen by considering [an individual aged exactly 54.25 at entry who died at exact age 54-5/12 and would have been subject to observational cut-off at age 54.5]:

(a) The quoted line of thought would assign an exposure of one-fourth of a life-year...

(b) The two-step method would assign a potential of three-quarters of a life-year..... Furthermore, no cancelation would be required by the intervention of the end of the observation period, because this employee would not be one of the enders [i.e. existing]. The net exposure is therefore three-quarters of a life-year..."

Harry Gershenson, Measurement of Mortality, pp. 45-46

1970: "...consider a life D born on 1 July 1909, entering assurance on 1 May 1961 and dying on 1 November 1964 [The observation period ended on 31 December 1964]....the amount of risk time during this age interval [55/56]... will be 1 year (notionally, because in calculating $\overset{o}{q}$ [the observed rate of mortality] it is as if all the θ_x [deaths at age x last birthday] were exposed for a full year."

B. Benjamin & H.W. Haycocks, The Analysis of Mortality and Other Actuarial Statistics, pp. 41-42

One's conclusion from all this is that in the 1940's American actuaries came to believe that Ackland had been wrong 45 years earlier, and that their adherence to this viewpoint converted the British later on. In fact, an "existing" entering at age x+a and scheduled to be lost to observation at age x+b (a and b both fractions) can only be "exposed" for b-a of a year at age x last birthday. Whether he lives or dies (the latter happening after exposure has taken place) can surely make no difference.

Ed. Note: We are pleased to have this heretical contribution from our distinguished member now in Switzerland. Mr. Seal's solution in set forth mathematically in his 1977 paper, Multiple Decrements or Competing Risks, in Biometrika 64, 3, pp. 429-39; a reprint entrusted to this editor is available on request. [This note appeared in the "Actuary"]

OPINION UNSHAKEN

Sir:

The common opinion that Hilary L. Seal questions ("An Attempt to Convert American Actuaries," March issue) seems to me to be correct.

Let "L" lives enter observation at x + a, "D" of the "L" die before x + b, and the remaining "L - D" leave observation at x + b, where $o \leqq a \leqq b \leqq 1$. Clearly,

$$\frac{D}{L} = {}_{b-a}q_{x+a}$$

If, in general $\quad {}_{1-t}q_{x+t} = (1-t)\,q_x$

then

$${}_{b-a}q_{x+a} = q_x \cdot \frac{b-a}{1-(1-b)q_x}$$

Hence

$$D = q_x \cdot \left[L\,(b-a) + D\,(1-b) \right]$$

$$= q_x \cdot \left[L\,(1-a) - (L-D)(1-b) \right].$$

By induction, we can extend these formulas to the more general situation where there are net migrations at several points of the interval. The first expression shows that deaths are "exposed" to the end of the year of age. The second, and equivalent, expression shows that the "exposure" has meaning only within a specific mortality assumption, Balducci's in this case. General reasoning is insufficient to derive exposure formulas.

William J. Sohn

by

Ralph E. Edwards

As the sole surviving accused named in Hilary L. Seal's AN ATTEMPT TO CONVERT AMERICAN ACTUARIES, in your March issue, I remain recalcitrant. Let's see why.

Where possible I will use mortality table notation. Let l'_x (the prime has no significance) represent a number of lives observed who are exactly age x on a specific date. Let l'_{x+w} of these become disobserved at age x+w, where w is a fraction of a year. The rest of them, d'_x , died in the meantime. Keep in mind that there is no a priori assumption as to the distrubution of deaths over the year of age. Ackland and Seal propose that in this instance the annual mortality rate, q_x , be equal to d'_x divided by E_x , and that $E_x = (w)\ d'_x + (w)\ l'_{x+w}$. This is an 1898 approach.

Around 1920, Balducci proposed that exposure formulas be based on an a priori assumption as to the distribution of deaths over the year of age, this being the assumption that $_{(1-w)}q_{x+w} = (1-w)\ q_x$.

Algebraic manipulation brings about the following:

Balducci E_x for disobserved: $d'_x + (w)\ l'_{x+w}$

Ackland E_x for disobserved: $(w)\ d'_x + (w)\ l'_{x+w}$

These results are nice alternates for disobservation caused by ending the period of observation. However, another disobservation situation is where a policy lapses or there is other withdrawal. Let there be l_x lives which become l_{x+w} lives disobserved by withdrawal, l_{x+1} lives disobserved by leaving the age group, and d_x lives dying before disobservance. Ackland appears to need a subdivision of d_x. He needs it so that a weighting of w can be applied to those who died but would have withdrawn had they lives. As I have stated the problem, this subdivision probably could be achieved by a fairly simple formula, but I have oversimplified. In practice the ranks of l_{x+w} , l_{x+1} and d_x have been swelled by new entrants numbering l_{x+n} where n is less than w. Ackland's approach now becomes exceedingly complex, but Balducci can be applied quite easily.

Seal does not tell us what Ackland actually proposed, and the 1898 J.I.A. just doesn't happen to be lying around the house, but it really is not essential that we know.

What happened after 1920 is that American texts and papers adopted Balducci. In so doing, some authors stated that Balducci gave deaths a full year of exposure in the year of death. Such statements were inexact, and I had the temerity to say so. The point here was that the formulas these writers gave were consistent with Balducci, but Balducci gives a weight of (1-n), and not one, to the deaths which arise before age x+1 among the l_{x+n} lives entering at age x+n.

My discussion, the one quoted by Seal, suggested that there could be an alternate a priori assumption as to the distribution of deaths over the year of age. Where Balducci assumed, ${}_{(1-w)}q_{x+w} = (1-w)\ q_x$ I proposed ${}_{w}q_x = (w)\ q_x$. This had the feature of giving a weight of one, and not (1-n), to the deaths arising from l_{x+n}. I carelessly said that this was interesting and Wolfenden, whose paper I was discussing, really pinned my ears back. An Actuary should never say that a formula is interesting.

In 1978, Thomas N. E. Greville undertook some research. He observed that I appear to have been the first to formulate ${}_{w}q_x = (w)\ q_x$ in reference to exposure formulas. This, in the texts, had come to be known as the uniform distribution assumption, but he proposed to call it the Edwards assumption, or the Edwards uniform distribution assumption.

This has significance to us because the Ackland approach can readily be transformed to be an example of the Edwards assumption. Seal does not explain adequately, but I think he wants us to use Edwards in part and Balducci for the rest.

The Edwards assumption can be reworked into an expression for E_x. What results, however, is that you are solving, in the end, for q_x

and the equation for E_x contains the q_x you seek. The appropriate solution in this kind of situation is iteration, but without modern computers that was awkward and even today nobody would bother.

I suspect that readers (if any remain) understand my personal frustration at finding my name only very tenuously associated with such a prominent name as Balducci through a formula that nobody had reason to use. The consequence was that Greville's paper sent me back to the drawing board, where I came up with another solution.

It is: $$q_x E_x = D_x - \frac{(\Delta_x)^2}{E_x}\left(\frac{n\,(1-n)\,l_{x+n}}{E_x - n\Delta_x} - \frac{w\,(1-w)\,l_{x+w}}{E_x - w\Delta_x}\right)$$

where $\Delta_x = q_x E_x$. This monstrous proofreader's horror is worse, at first glance, than what I started with, but further study provides rewards. The first conclusion is that it can be worked rather readily by iteration even on a programmable pocket calculator and nothing is required which is not readily available from the usual Balducci solution. A second observation (offered with my apologies if any student is ever asked to state both the Edwards assumption and the Edwards approximation) is to approximate Δ_x by substituting D_x to get

$$q_x = \frac{1}{E_x} \text{ times } D_x\left(1 - \frac{D_x}{E_x}\left(\frac{n\,(1-n)\,l_{x+n}}{E_x - n\,D_x} - \frac{w\,(1-w)\,l_{x+w}}{E_x - w\,D_x}\right)\right)$$

(In exposure formula notation, n_x and w_x would replace the l_{x+n} and l_{x+w} I have used in this explanation. D_x represents the observed deaths.)

It is the third conclusion that brings us back to Seal. Both the Edwards and Balducci approaches end up with the same denominator E_x. Perhaps E_x is the fundamental function and the variable, depending on one's assumption of the distribution of deaths, is the numerator. As actuaries we may have been looking for the horse to fit our rider when we should have been looking for the adjusted deaths rider to fit our exposed-to-risk horse, I wonder if Mr. Seal can provide the numerator his approach develops and explain any virtues it may have.

Method of Moments Derivation
of the Balducci Exposure Formula

We are of the opinion that the exposure formulas of both Seal (The Actuary, March 1981) and Sohn (The Actuary, June, 1981) are justifiable. While their estimators differ, they may both be derived by modifications to the method of moments procedure.

Let the i^{th} of N lines enter observation at age $x + a_i$ and leave either by death prior to age $x + b_i$ or by withdrawal (alive) at age $x + b_i$, where $0 \le a_i \le b_i \le 1$. If D is the number of observed deaths, the method of moments estimator is obtained by setting $D = E(D) = \Sigma_{b_i - a_i} q_{x+a_i}$, expressing $_{b_i-a_i}q_{x+a_i}$ in terms of q_x, and solving the resulting equation for q_x.

Sohn proposes the Balducci assumption, obtaining

$D = \Sigma \dfrac{(b_i - a_i)\tilde{q}_x}{1-(1-b_i)\tilde{q}_x}$, where $\tilde{q}_x$ denotes the estimator of q_x.

Contrary to his statement, this equation is not easily solved, unless the b_i are all equal. To derive the exposure formula as given in Batten (Mortality Table Construction, 1978) using the Balducci assumption write

$$E(D_i) = {}_{b_i-a_i}q_{x+a_i} = {}_{1-a_i}q_{x+a_i} - {}_{b_i-a_i}p_{x+a_i} \; {}_{1-b_i}q_{x+b_i}$$

$$= (1-a_i)q_x - (EW_i)(1-b_i)q_x$$

where $D_i = 1 - W_i$ is 1 if the i^{th} life dies, and 0 otherwise. Summing,

$$E(D) = [\Sigma(1-a_i) - \Sigma(EW_i)(1-b_i)]q_x.$$

The modification which produces the usual estimator is to replace EW_i by W_i. The resulting estimating equation is

$$D = [\Sigma(1-a_i) - \Sigma W_i(1-b_i)]\tilde{q}_x$$

and the exposure is seen to be of the potential minus cancelled form. Note that $[\Sigma(1-a_i) - \Sigma W_i(1-b_i)]q_x$ is not the expected number of deaths (see also Hoem (Arch, 1980)).

Seal (Conference of Actuaries in Public Practice, 1961-62) uses the approximation ${}_{b_i-a_i}q_{x+a_i} = (b_i - a_i)q_x$ to obtain $ED = q_x\ \Sigma(b_i-a_i)$. This approximation may be obtained from any of the three assumptions in Batten (1978) by writing ${}_{b-a}q_{x+a}$ as a power series in q_x and dropping all terms of degree greater than one (reasonable for small q_x). It should be noted that there is no survivorship function l_x which produces ${}_{b-a}q_{x+a} = (b-a)\ q_x$ for all $0 \leq a \leq b \leq 1$, so $(b_i-a_i)q_x$ must be viewed only as an approximation to ED_i.

As both formulas can be justified by the method of moments, their relative merits should not be debated on the basis of their derivations or of any interpretations (or lack thereof) of the exposure formulas. More important is how accurate they are in estimating q_x.

July 17, 1981 James Broffitt and Stuart Klugman

Letter to the Editor

Hilary Seal's "An Attempt to Convert American Actuaries" was both educational and enjoyable to read. Although I have nothing to add to his ideas, his cited paper "Multiple Decrements or Competing Risks" in Biometrika 64, 3, pp. 429-39 does prompt some remarks. The major theme of that article was to review the history both of actuarial and statistical papers on that subject.

It is very valuable to remind actuaries of the overlap of their discipline with other scientific areas. Let me add to what Hilary has provided in those two articles, and elsewhere. Probably every beginning actuarial student hears that actuarial techniques are applied in medical investigations, and elsewhere. However, it is unusual to be given any references. In 1977, Dr. Charles Sampson of the Eli Lilly Research Laboratories asked me to prepare a talk for the Midwest Biopharmaceutical Statistics Workshop which would apply actuarial techniques to a toxicology study. In the course of this research, I was fascinated with the references which seemed to unfold. As a result of this research and talk, I prepared a paper, "Life Table Techniques Applied to Experiments in Carcinogenesis, and Other Investigations", which now appears in ARCH 1979.1. Parts of the paper are so elementary that they could have been deleted except they serve as the connecting links to the references. Readers of The Actuary might be interested in some of those references. In addition, there are excellent papers authored by Professor Seal, and Professor Jim Hickman which relate to that area. Since 1977, I have become aware of quite a few other papers, and books involving actuarial techniques in a clinical trials or pharmaceutical research setting. One reference I would mention is "Survival Models and Data Analysis" by Regina C. Elandt-Johnson and Norman L. Johnson, John Wiley and Sons, New York, 1980.

I thank Professor Seal for two more excellent contributions to our education, and literature.

April 2, 1981 John A. Beekman

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April 15, 1981

Mr. E. J. Moorhead
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Dear Jack:

In connection with Ralph Edwards' letter to me, of which you received a copy, I have located the journal article in which Balducci's 1920 letter was published.

The letter appears on Pages 184-186 of Volume LII of the Journal of the Institute of Actuaries. It appears from this letter that the mathematical assumption commonly referred to as the Balducci Hypothesis might actually be attributable to someone else, as Balducci's letter makes no claim of originality.

Of course, I am in direct disagreement with Seal's remarks. I am anxious to read Ralph's response. I find it difficult to respond myself in any way other than to restate the exposition on this subject in Chapter Two of MORTALITY TABLE CONSTRUCTION. Only Mr. Seal's last paragraph is his own, and that paragraph gives absolutely no justification for his acceptance of Ackland and his rejection of the other authors. I must admit curiosity as to why he made no mention of my text. Whether or not he agrees with it, it treats the matter in much greater detail than any of the other references.

I suppose, then, it would be appropriate to request from you a copy of Seal's 1977 paper which you mentioned in the Editor's Note. Perhaps I will later send a formal response as a Letter to the Editor.

I'm looking forward to seeing you in Asheville.

Sincerely,

Bob

Robert W. Batten
Professor of Actuarial Science
Head of Actuarial Science Program

RWB/eg

cc: Mr. Ralph E. Edwards