DIVIDED DIFFERENCES BY CONTOUR INTEGRATION

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The Cauchy integral formula

$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-w}\, dz \tag{1}$$

is a powerful tool. It can be applied to elegantly derive various results on divided differences ([9, §1.7], [8, Vol II, §11], [1, §3.6]).

It immediately follows from

$$\frac{1}{x-y}\left(\frac{1}{z-x} - \frac{1}{z-y}\right) = \frac{1}{(z-x)(z-y)}$$

that

$$\underset{y}{\Delta} f(x) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-x)(z-y)}\, dz .$$

By induction we have

$$\underset{x_1,\ldots,x_n}{\Delta^n} f(x_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-x_0)\ldots(z-x_n)}\, dz . \tag{2}$$

In this paper we shall assume that the contour C is a circle centered at zero and large enough to contain all the points $x_0, x_1, \ldots, x_n$ in its interior.

That a divided difference is symmetrical in its arguments [5, p. 105] is obvious from (2). Furthermore, since

$$\frac{1}{(z-x_0)\dots(z-x_n)} = \frac{1}{(z-x_0)(x_0-x_1)\dots(x_0-x_n)} + \dots$$

$$+ \frac{1}{(z-x_n)(x_n-x_0)\dots(x_n-x_{n-1})} ,$$

applying equation (1) n+1 times we have [5, equation (5.11)]:

$$\underset{x_1,x_2,\dots,x_n}{\Delta^n} f(x_0) = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} + \dots$$

$$+ \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} .$$

To derive the Newton divided-difference formula [5, (5.15)] note that

$$\frac{1}{z-x} = \frac{1}{z-x_0} + \frac{x-x_0}{z-x_0}\frac{1}{z-x} ,$$

$$\frac{1}{z-x} = \frac{1}{z-x_1} + \frac{x-x_1}{z-x_1}\frac{1}{z-x} ,$$

$$\dots\dots$$

By repeated substitution for $\frac{1}{z-x}$ we get the identity

$$\frac{1}{z-x} = \frac{1}{z-x_0} + \frac{x-x_0}{z-x_0}\frac{1}{z-x_1} + \frac{(x-x_0)(x-x_1)}{(z-x_0)(z-x_1)}\frac{1}{z-x_2}$$

$$+ \dots + \frac{(x-x_0)(x-x_1)\ \dots\ (x-x_n)}{(z-x_0)(z-x_1)\ \dots\ (z-x_n)}\frac{1}{z-x} .$$

Thus

$$f(x) = f(x_0) + (x-x_0) \underset{x_1}{\Delta} f(x_0) + (x-x_0)(x-x_1) \underset{x_1, x_2}{\Delta^2} f(x_0)$$

$$+ \ldots + (x-x_0)(x-x_1) \cdots (x-x_n) \underset{x_0, \ldots, x_n}{\Delta^{n+1}} f(x).$$

For $m = 0, 1, 2, \ldots,$ let us compute

$$\Delta^n x^m$$

using equation (2). By letting the radius of the contour circle C tend to infinity, we see that for $m < n$,

$$\Delta^n x^m = 0.$$

For the cases where $m \geq n$, consider the generating function

$$g(t) = \sum_{j=0}^{\infty} \left(\underset{x_1, \ldots, x_n}{\Delta^n} x_0^{n+j} \right) t^j .$$

Let the radius of the contour circle be r. For $|t| < 1/r$,

$$g(t) = \frac{1}{2\pi i} \int_C \frac{z^n}{(1-zt)(z-x_0)\ldots(z-x_n)} dz.$$

Put $w = 1/z$; then

$$g(t) = \frac{1}{2\pi i} \int_K \frac{-1}{(w-t)(1-wx_0)\ldots(1-wx_n)} dw,$$

where K is the circle centered at zero with radius $1/r$ and clockwise orientation. Applying equation (1) we immediately obtain

$$g(t) = \frac{1}{(1-tx_0)\ldots(1-tx_n)} . \qquad (3)$$

Expanding (3) we have

$$\underset{x_1,\ldots,x_n}{\Delta^n} x_0^{n+j} = \sum_{\substack{a_0\ge 0,\ldots,a_n\ge 0\\ a_0+a_1+\ldots+a_n=j}} x_0^{a_0}x_1^{a_1}\ldots x_n^{a_n}$$

$$= \sum_{0\le b_1\le b_2\le\ldots\le b_j\le n} x_{b_1}x_{b_2}\ldots x_{b_j}. \qquad (4)$$

Equation (3) has been derived by different methods in [2, §III.8], [9, §1.31] and [13, §3]. For $j = 1$, equation (4) is [5, p.121, #7] and [7, p.34, #33]. Equation (4) can also be derived by means of determinants; see [12] or [10, Theorem 2.51]. An alternative expression for (4) is

$$\sum_{\substack{c_1,c_2,\ldots,c_j\ge 0\\ c_1+2c_2+\ldots+jc_j=j}} \frac{S_1^{c_1}}{1^{c_1}c_1!}\,\frac{S_2^{c_2}}{2^{c_2}c_2!}\ldots\frac{S_j^{c_j}}{j^{c_j}c_j!}, \qquad (5)$$

where

$$S_k = x_0^k + x_1^k + \ldots + x_n^k .$$

Expression (5) is obtained using the identity

$$g(t) = e^{\ln g(t)} .$$

For details, see [7, pp.91-92]; however, we remark that such a technique has found applications in individual risk theory [6, §II].

We now compute

$$\Delta^n \; x^{-1}$$

by contour integration. The following result is sometimes called *Cauchy's integral formula for an unbounded domain* [8, Vol. I, p.318, #14.14]:

> Let L be a closed rectifiable Jordan curve, traversed in the counter-clockwise direction. If h is a function analytic in the exterior of L, E(L), then for each $w \in E(L)$
>
> $$\frac{1}{2\pi i}\int_L \frac{h(z)}{z-w}\,dz = -h(w) + \lim_{z\to\infty} h(z).$$

Consider

$$h(z) = \frac{1}{(z-x_0)(z-x_1)\dots(z-x_n)}.$$

Assume L contains the points $x_0, x_1, \dots, x_n$ in its interior but not the origin. Thus by equation (2)

$$\underset{x_1,\dots,x_n}{\Delta^n} \frac{1}{x_0} = \frac{1}{2\pi i}\int_L \frac{1}{z(z-x_0)\dots(z-x_n)}\,dz$$

$$= -h(0) + 0$$

$$= (-1)^n / x_0 x_1 \dots x_n.$$

This result generalizes Example 5.2 on page 106 of [5] .

An elegant application of formula (2) arises when two or more of the points of collocation coincide. (Cf. [9, §1.8], [11] and [4, p. 57].) Since

$$\left(\frac{\partial}{\partial x}\right)^n \frac{1}{z-x} = \frac{n!}{(z-x)^{n+1}} ,$$

we immediately have

$$\underset{x,\ldots,x}{\Delta^n} f(x) = \frac{f^{(n)}(x)}{n!} ,$$

which is [5, (5.19)]. Similarly, it follows from

$$\left(\frac{\partial}{\partial x_1}\right)^k \frac{1}{(z-x_0)(z-x_1)\ldots(z-x_n)} = \frac{k!}{(z-x_0)(z-x_1)^{k+1}(z-x_2)\ldots(z-x_n)}$$

that

$$\left(\frac{\partial}{\partial x_1}\right)^k \underset{x_1,\ldots,x_n}{\Delta^n} f(x_0) = k! \underset{\underbrace{x_1,\ldots,x_1}_{k+1},x_2,\ldots,x_n}{\Delta^{n+k}} f(x_0) .$$

Thus, problems such as No. 21 on page 122 of [5] become trivial. The example considered in [3] is

$$\underset{x,x,x,y,y,}{\Delta^5} f(z)$$

which simplifies to

$$\frac{1}{2} \frac{\partial}{\partial y} \left(\frac{\partial}{\partial x}\right)^2 \underset{x,y}{\Delta^2} f(z).$$

Divided differences can also be expressed as multiple integrals. The following formula is due to Hermite ([9, p.10], [13, p.17]):

$$\underset{x_1,\ldots,x_n}{\Delta^n} f(x_0) = \int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n\, f^{(n)}(u),$$

where $u = (1-t_1)x_0 + (t_1-t_2)x_1 + \ldots + (t_{n-1}-t_n)x_{n-1} + t_n x_n.$

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