COMMENTS ON SCHEEL'S PROPOSAL FOR SPLITTING LIFE INSURANCE TO ACHIEVE VARIANCE REDUCTION

by

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Introduction

Separation of the savings and protection elements of life insurance, i.e. "buy term and invest the difference", has been advocated by a number of critics of conventional level premium life insurance. Scheel [2] recently argued for such a separation on the grounds that under certain assumptions the separation will achieve a reduction in the total variance of the risk associated with a level premium cash value policy and that cost to the policyholder will thereby be reduced if insurers load premiums to some extent in proportion to the variance of the risk.

The purpose of this note is to point out that the very model which Scheel adopted assumes independence between the protection and savings elements and that no variance reduction will result if the savings and protection elements are truly separated. It will be shown that what Scheel did achieve was variance reduction by virtue of a separation of just the savings elements into independent components. It will be further shown that Scheel's proposal can be extended so as to achieve the optimum amount of variance reduction not only with respect to the savings element but with respect to the protection element as well. In the case of separation into just two parts, the optimal arrangement will turn out to be equal share level premium insurance. Additional variance reduction can be achieved by splitting into more than two independent components, but expense considerations will eventually defeat that strategy.

Scheel's Model

Scheel, building upon earlier results of Hickman [1], considered certain random variables associated with the t^{th} policy year of an annual premium life insurance policy. Letting ${}_tB$ be the death benefit payable at the end of policy year t for a policy with annual premium P, Scheel identified the loss function

$$L[\tilde{s},\tilde{i}_t:t] = {}_tB - (P + {}_{t-1}V)(1+\tilde{i}_t) \quad \text{if the policyholder dies } (\tilde{s}=1), \quad (1a)$$

$$= {}_tV - (P + {}_{t-1}V)(1+\tilde{i}_t) \quad \text{if the policyholder lives } (\tilde{s}=0). \quad (1b)$$

In the model $\tilde{i}_t$ is the investment rate of return over policy year t and is viewed as being a random variable with expected value $E[\tilde{i}_t] = i$. $L(\tilde{s},\tilde{i}:t)$ takes on the value ${}_tB - (P + {}_{t-1}V)(1+\tilde{i}_t)$ with probability q_{t-1} and the value ${}_tV - (P + {}_{t-1}V)(1+\tilde{i}_t)$ with probability $1 - q_{t-1}$. The quantities ${}_{t-1}V$ and ${}_tV$ are the respective reserves at the ends of policy years t-1 and t based on mortality rates q_{t-1} for t = 1, 2, ... and rate of interest i. The usual recursive relationship for terminal reserves is assumed to hold:

$${}_tV = (P + {}_{t-1}V)(1+i) - q_{t-1}\,({}_tB - {}_tV) \quad (2)$$

from which it follows that $E[L(\tilde{s},\tilde{i}:t)] = 0$.

When the mortality random variable $\tilde{s}$ and the rate of return variable $\tilde{i}_t$ are assumed to be independent, then

$$Var[L(\tilde{s},\tilde{i}:t)] = ({}_tB - {}_tV)^2(q_{t-1} - q_{t-1}^2) + (P + {}_{t-1}V)^2\, Var(\tilde{i}_t) \quad (3)$$

as shown by Hickman.

Equations (1a) and (1b) may be combined into a single equation which more clearly identifies the mortality random variable $\tilde{s}$:

$$L(\tilde{s},\tilde{i}:t) = ({}_tB - {}_tV)\tilde{s} + {}_tV - (P + {}_{t-1}V)(1+\tilde{i}_t) \quad (4)$$

Hence, the amount at stake with respect to the mortality risk is ${}_tB - {}_tV$, and the corresponding component of the variance of $L(\tilde{s},\tilde{i}:t)$ is $({}_tB - {}_tV)^2\, Var(\tilde{s})$

$({}_tB - {}_tV)^2 q_{t-1}(1-q_{t-1})$ which readers will recognize as the variance of a random variable which is a constant times a binomially distributed random variable.

Also from equation (4) it is seen that the insurer has available to offset the end of year requirements the beginning of year resources, $P + {}_{t-1}V$, plus investment return on those resources. The investment risk is therefore on the amount $P + {}_{t-1}V$ and the variance for that risk is $(P + {}_{t-1}V)^2 \operatorname{Var}(\tilde{i}_t)$. Thus, the variance of $L(\tilde{s},\tilde{i}:t)$ consists of two terms, one corresponding to the mortality risk and one corresponding to the investment risk. Since they are assumed to be independent, there is no cross product or covariance term.

Scheel's Proposal

Scheel proposes that the loss function be separated into parts, a loss function for a term insurer

$$L_1 = ({}_tB - {}_tV)\tilde{s} - ({}_tB - {}_tV)A'_{t-1:\overline{1|}}(1+\tilde{i}_t), \tag{5}$$

and a loss function for a savings institution

$$L_2 = {}_tV - (P + {}_{t-1}V - ({}_tB - {}_tV)A'_{t-1:\overline{1|}})(1+\tilde{i}_t), \tag{6}$$

where $A'_{t-1:\overline{1|}} = q_{t-1}(1+i)^{-1}$ is the one-year term insurance rate for the t^{th} policy year, and $({}_tB - {}_tV)A'_{t-1:\overline{1|}}$ is the cost of one-year term insurance for the net amount at risk. The two loss functions are then assigned to separate financial institutions so that the total of the loadings for risk for the policy is proportional to $\operatorname{Var}(L_1) + \operatorname{Var}(L_2)$ which Scheel correctly argues is less than $\operatorname{Var}(L)$.

One way to accomplish this ingenious proposal is to have L_1 and L_2 be independent random variables in which case $\operatorname{Var}(L_1 + L_2) = \operatorname{Var}(L_1) + \operatorname{Var}(L_2)$. L_1 and L_2 are independent if, with constant terms ignored, the proposal has the form

$$L = a\tilde{s} - b\tilde{i} \qquad (7)$$

$$L_1 = a\tilde{s} - c\tilde{i}_1 \qquad (8)$$

$$L_2 = \quad - (b - c)\tilde{i}_2 \qquad (9)$$

where $\tilde{i}_1$ and $\tilde{i}_2$ are independent random variables, each having the same distribution as $\tilde{i}$. In the above $a = {}_tB - {}_tV$, $b = P + {}_{t-1}V$ and $c = ({}_tB - {}_tV)A'_{t-1:\overline{1}|}$. Then

$$\mathrm{Var}(L) = a^2\ \mathrm{Var}(\tilde{s}) + b^2\ \mathrm{Var}(\tilde{i}),$$

$$\mathrm{Var}(L_1) = a^2\ \mathrm{Var}(\tilde{s}) + c^2\ \mathrm{Var}(\tilde{i}),\ \text{and}$$

$$\mathrm{Var}(L_2) = \qquad (b - c)^2\ \mathrm{Var}(\tilde{i}).$$

Since $b = c + (b - c)$, it follows that $b^2 = c^2 + (b - c)^2 + 2c(b - c)$. If b and c are both positive and $c < b$, then $c^2 + (b - c)^2 < b^2$. The condition $c < b$ is equivalent to the condition ${}_tV > 0$. Hence, in the case of positive reserves $\mathrm{Var}(L_1) + \mathrm{Var}(L_2) < \mathrm{Var}(L)$. The amount of variance reduction which has been achieved by Scheel's proposal is $2c(b - c)\ \mathrm{Var}(\tilde{i})$, which, as Scheel argues, is the covariance which would exist between L_1 and L_2 if $\tilde{i}_1$ and $\tilde{i}_2$ were the same random variable, i.e. if the policy were not split into two parts assigned to separate financial institutions.

Scheel goes on to argue that since variance is a measure of the risk borne by the insurer, and since higher risk means higher cost to insureds in the form of higher profits for risk bearing or greater contributions to permanent surplus, separation of level premium life insurance across independent insurance and savings institutions will ultimately reduce costs to the insurance buying public.

Comments and Extensions

Scheel describes the covariance that exists in the non-separated case, and which is eliminated via his separation proposal, as covariance between the savings and protection elements of conventional level premium life insurance. Is that a correct description? Examination of equations (5) and (8) reveals that the loss variable for the term insurer contains a term involving $\tilde{i}_t$. That is because under Scheel's proposal the term insurer is given part of the investment risk: the term insurer is given the task of investing the term premiums for the net amount at risk.

Suppose, instead, all of the investment risk is assigned to the savings institution and only insurance risk is assigned to the term insurer by means of the following loss functions:

$$L_1' = ({}_tB - {}_tV)\tilde{s} - ({}_tB - {}_tV)A'_{t-1:\overline{1|}}(1+i), \tag{13}$$

$$L_2' = {}_tV - (P + {}_{t-1}V)(1+\tilde{i}) + ({}_tB - {}_tV)A_{t-1:1}(1+i). \tag{14}$$

L_1' is the loss fucntion for a term insurer which upon receipt of term premiums immediately transfers them to the savings institution for investment. The savings institution guarantees the rate of return i so that $({}_tB - {}_tV)A'_{t-1:\overline{1|}}(1+i)$ is returned to the term insurer at the end of the year. The term insurer now assumes no investment risk so that $Var(L_1') = ({}_tB - {}_tV)^2\ Var(\tilde{s})$. The savings institution now assumes all of the risk of investing the resources, $P + {}_{t-1}V - ({}_tB - {}_tV)A'_{t-1:\overline{1|}}$ from the insured under a buy term and invest the difference arrangement, and $({}_tB - {}_tV)A'_{t-1:\overline{1|}}$ from the term insurer. $Var(L_2') = (P + {}_{t-1}V)^2\ Var(\tilde{i})$. Now the savings and protection elements are truly separated, but alas, there is no variance reduction. Scheel's proposal produced variance reduction only because the investment risk was divided between two financial institutions. To characterize the covariance as being between the savings and protection elements is misleading. $\tilde{s}$ and $\tilde{i}_t$ are assumed independent in the model; there can be no covariance between them. The covariance in question arises only because of the correlation between the two portions into which the investment risk has been divided.

The next question which naturally arises is whether or not there is a better way to divide the policy if division into two parts so as to reduce variance is the objective. Given

$$L = a\tilde{s} - b\tilde{i}, \tag{15}$$

$$L_1 = d\tilde{s}_1 - c\tilde{i}_1, \tag{16}$$

and

$$L_2 = (a - d)\tilde{s}_2 - (b - c)\tilde{i}_2, \tag{17}$$

a little calculus shows that the values of c and d which achieve the minimum variance are $c = b/2$ and $d = a/2$. In other words each financial institution should be given half of the insurance risk and half of the investment risk, i.e. the equal share level premium separation. Further analysis will show that if separation into more than two independent portions is to be attempted, then equal shares of both insurance and investment risk will be the mathematical solution which will minimize variance. Of course, expense considerations, heretofore ignored, will make separation into very many pieces impractical.

There is a technical difficulty in connection with the foregoing which needs to be examined. Although one might accept the idea that $\tilde{i}_1$ and $\tilde{i}_2$ can be independent random variables corresponding to investment operations in separate financial institutions, the situation in regard to $\tilde{s}_1$ and $\tilde{s}_2$ is more difficult. For a given insured $\tilde{s}_1$ and $\tilde{s}_2$, even if placed with separate insurers, cannot be independent. Nevertheless, one might argue that $Var(L_1) + Var(L_2)$ is still the proper quantity to minimize because each institution loads in proportion to the variance of the risk it assumes.

Standard Deviation Loading

The entire discussion to this point has been based upon variance reduction as the objective. There are, however, those who argue that, rather than the variance, loading for risk ought to be in proportion to the standard deviation of the risk variable. Robinson [3] has suggested that in the latter case the optimal strategy is to place all of the risk with one financial institution. That suggestion will now be examined.

As before let $L = as - bi$ correspond to the non-separated policy, and let

$$L_1 = ds_1 - ci_1,$$

and
$$L_2 = (a - d)s_2 - (b - c)i_2$$

be the loss functions assigned to independent financial institutions. Employing the notation $\sigma_1^2 = \mathrm{Var}(\tilde{s})$ and $\sigma_2^2 = \mathrm{Var}(\tilde{i})$, one may express the problem as

$$\text{minimize } F(c,d) = \sqrt{d^2\sigma_1^2 + c^2\sigma_2^2} + \sqrt{(a - d)^2\sigma_1^2 + (b - c)^2\sigma_2^2} \quad (18)$$

subject to the constraints $0 \leq c \leq b$ and $0 \leq d \leq a$.

Consider first the special case where $d = a$, i.e. the case where all of the mortality risk is assigned to the first institution. Then

$$F(c,a) = \sqrt{a^2\sigma_1^2 + c^2\sigma_2^2} + (b - c)\sigma_2.$$

$$\frac{dF(c,a)}{dc} = c\sigma_2^2 \,/\, \sqrt{a^2\sigma_1^2 + c^2\sigma_2^2} - \sigma_2$$

which is negative for $0 < c < b$. Therefore, setting $c = b$ minimizes $F(c,a)$. Robinson is correct in this case: if all of the mortality risk is assigned to one institution, then all of the investment risk should be placed with that same institution if the objective is to minimize the sum of the standard deviations. The case $d = 0$ similarly leads to $c = 0$, and both risks are assigned entirely to the second institution.

Finally, for $0 < d < a$, it can be shown that $\frac{dF(c,d)}{dc} = 0$ if $c = bd/a$. That means that the proportion of the investment risk assigned to an institution should be the same as the proportion of the mortality risk assigned to it. Furthermore, if $d = a\theta$ and $c = b\theta$, then

$$F(b\theta, a\theta) = \sqrt{a^2\theta^2\sigma_1^2 + b^2\theta^2\sigma_2^2} + \sqrt{a^2(1-\theta)^2\sigma_1^2 + b^2(1-\theta)^2\sigma_2^2}$$

$$= [\theta + (1-\theta)]\sqrt{a^2\sigma_1^2 + b^2\sigma_2^2}$$

which is independent of θ. In other words it does not make any difference what proportion of the two risks is assigned to one financial institution as long as it is the same proportion of both risks.

Conclusion

There may be arguments in support of buying term and investing the difference, but variance and standard deviation minimization principles give greater support to keeping mortality and investment risks under one roof.

References

[1] Hickman, James C. "Notes on Individual Risk Theory and Released from Risk Reserves," ARCH, 1975.1, no. 2.

[2] Scheel, William C. "Variance Reduction Properties of a Split of the Savings and Protection Elements of Life Insurance," ARCH, 1977.1, no. 8.

[3] Robinson, James M. Private communication with the author and William C. Scheel.