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CONDITIONAL STOCHASTIC INTEREST RATE MODELS IN LIFE CONTINGENCIES

by

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ABSTRACT

Several previous papers [1, 2, 3, 4] treat the rate of interest as a stochastic process in the determination of values of insurances and annuity functions. None of these papers consider the current interest rate as a starting point in the model. In this paper, autoregressive interest models, which are conditional upon current and past interest rates, are developed. It is shown how these models may be applied to the evaluation of moments of interest, insurance and annuity functions. Numerical results are also given.

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INTRODUCTION

Non-participating insurance products are priced using interest assumptions that reflect anticipated yield rates on a portfolio of assets. Such rates usually reflect in some way both current levels of yield rates as well as some ultimate level that can be reasonably expected to occur. In order to reflect the stochastic nature of yield rates and its associated risk, insurers usually introduce some element of conservation into the assumptions. This usually means a reduction in the expected interest rate by some amount, such as 1%.

This paper is concerned with the quantification of the stochastic nature of yield rates and the resultant effect on interest, insurance and annuity values. The theory developed in this paper recognizes not only that the yield rate on a portfolio of invested assets is stochastic but also that the yield rates in future years are often correlated. That is to say, if the yield rate is high in one year, it will tend to be higher in the next year than if the yield rate is relatively lower. In addition, the theory recognizes the current level of yield rate as a starting point in the stochastic structure of interest rates.

Recent papers by Pollard [3, 4], Boyle [1] and Panjer and Bellhouse [2] have considered the interest rate as a stochastic process. All of these authors use the normal or a related distribution to describe the variation in interest rates from expected values. Boyle [1] assumes that interest rates in successive years are uncorrelated while Pollard [3, 4] assumes that a second order autoregressive stochastic process can be used to model interest rate variability. Panjer and Bellhouse [2] develope general results, based on the moment generating function of the interest rate model, which can be applied to any model. They apply these results to the class of normal processes and more specifically to autoregressive models of interest rate variation.

None of the authors mentioned above use current levels of interest rates as a starting point for the model. This would seem to be the appropriate strategy when pricing insurance and annuity products using realistic assumptions. In this paper results are obtained for discrete autoregressive models conditional on past and current levels of interest.

THE BASIC MODEL

Let i_t and $\delta_t = ln\ (1 + i_t)$ denote the interest rate and force of interest respectively applicable in year t $(t = 1, \ldots, n)$. The value at time 0 of a payment of 1 made at time t is given by $exp\{-\Delta_t\}$, where $\Delta_t = \delta_1 + \delta_2 + \ldots + \delta_t$. This last sum may be expressed as a vector product. Let $\underset{\sim}{\delta} = (\delta_1, \delta_2, \ldots, \delta_n)^T$, where T denotes the transpose, and $\underset{\sim}{1}_t = (1, 1, \ldots, 1, 0, \ldots, 0)^T$ be n-element column vectors where the latter vector contains t 1's and $n-t$ 0's. Then $\Delta_t = \underset{\sim}{1}_t^T \underset{\sim}{\delta}$.

If $\underset{\sim}{\delta}$ is stochastic, Panjer and Bellhouse [2] have shown that the expected values and variances of various interest, insurance and annuity functions depend on the expectations $E[exp\{-\Delta_t\}]$ and $E[exp\{-\Delta_t - \Delta_s\}]$, which are the moment generating functions of Δ_t and $\Delta_t + \Delta_s$ respectively, each evaluated at the point -1. The symbol E denotes the expectation operator with respect to $\underset{\sim}{\delta}$. For example, the annuity certain

$$a_{\overline{n|}} = \sum_{t=1}^{n} exp\{-\Delta_t\}$$

has mean value

$$E[a_{\overline{n|}}] = \sum_{t=1}^{n} E[exp\{-\Delta_t\}]$$

and variance

$$E[a_{\overline{n|}}^2] - E^2[a_{\overline{n|}}] = \sum_{s=1}^{n}\sum_{t=1}^{n} E[exp\{-\Delta_t - \Delta_s\}] - \sum_{t=1}^{n} E[exp\{-\Delta_t\}]^2.$$

If $\underset{\sim}{\delta}$ is a normal process, then

$$E[exp\{-\Delta_t\}] = exp\{-\underset{\sim}{1}_t^T \underset{\sim}{\theta} + \frac{1}{2} \underset{\sim}{1}_t^T \Gamma \underset{\sim}{1}_t\} \tag{1}$$

and

$$E[exp\{-\Delta_t - \Delta_s\}] = exp\{-(\underset{\sim}{1}_t + \underset{\sim}{1}_s)^T \underset{\sim}{\theta} + \frac{1}{2}(\underset{\sim}{1}_t + \underset{\sim}{1}_s)^T \Gamma (\underset{\sim}{1}_t + \underset{\sim}{1}_s)\}, \tag{2}$$

where $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_n)^T$ is the mean vector of $\underset{\sim}{\delta}$ and Γ is the variance-covariance matrix with elements $\gamma_{st} = Cov[\delta_s, \delta_t]$, $s,t = 1,\ldots, n$. Equations (1) and (2) correspond to equations (12) and (13) for discrete time in Panjer and Bellhouse [2]. It remains only to determine $\underset{\sim}{1}_t^T \underset{\sim}{\theta}$, $(\underset{\sim}{1}_t + \underset{\sim}{1}_s)^T \underset{\sim}{\theta}$, $\underset{\sim}{1}_t^T \Gamma \underset{\sim}{1}_t$, and $(\underset{\sim}{1}_t + \underset{\sim}{1}_s)^T \Gamma (\underset{\sim}{1}_t + \underset{\sim}{1}_s)$ when δ_t is conditional on the present and past rates to obtain the conditional analogue to the results of Panjer and Bellhouse [2].

THE CONDITIONAL AUTOREGRESSIVE MODEL

The conditional autoregressive process of order p for δ_t can be written as

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t \tag{3}$$

where $t = 1,\ldots, n$ and $y_t = \delta_t - \theta$, and where $y_0, y_{-1}, \ldots, y_{-p+1}$ are constants and not random variables. The error variables ϵ_t, $t = 1,\ldots, n$ are independent and identically distributed normal random variables each with mean 0 and variance γ^2. Model (3) implies that interest rates in any given year depend on the rates in the previous p years. In most practical situations the values of δ_t at $t = 0,-1,-2,\ldots, -p+1$ will be known, that is, at the current time, $t = 0$, and at the $p-1$ previous periods. The distribution of $\underset{\sim}{y} = (y_1,\ldots, y_n)^T$ and subsequent expectations of functions of the elements of $\underset{\sim}{y}$ is then obtained as conditional on $y_0,y_{-1},\ldots, y_{-p+1}$.

The system of equations (3) for $t = 1,2,\ldots, n > p$ can be written in matrix notation as

$$\underset{\sim}{y} = \Phi\underset{\sim}{y} + \underset{\sim}{e} \qquad (4)$$

where

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 \\ \phi_1 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 \\ \phi_2 & \phi_1 & 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 \\ \phi_3 & \phi_2 & \phi_1 & \ldots & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 \\ \vdots & & & & & & & & & & \\ \phi_p & \phi_{p-1} & \phi_{p-2} & \cdots & \phi_1 & 0 & \ldots & 0 & \ldots & 0 & 0 \\ 0 & \phi_p & \phi_{p-1} & \cdots & \phi_2 & \phi_1 & \cdots & 0 & \ldots & 0 & 0 \\ \vdots & & & & & & & & & & \\ 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & \phi_p & \cdots & \phi_1 & 0 \end{bmatrix} \qquad (5)$$

is an $n \times n$ matrix. The "error" vector $\underset{\sim}{e}$ is comprised of two components, $\underset{\sim}{e} = \underset{\sim}{\phi} + \underset{\sim}{\epsilon}$ where $\underset{\sim}{\epsilon} = (\epsilon_1,\ldots, \epsilon_n)^T$ and

$$
\mathbf{c} = \begin{bmatrix}
\phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-2} & \phi_{p-1} & \phi_p \\
\phi_2 & \phi_3 & \phi_4 & \cdots & \phi_{p-1} & \phi_p & 0 \\
\phi_3 & \phi_4 & \phi_5 & \cdots & \phi_p & 0 & 0 \\
\vdots & & & & & & \\
\phi_p & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix}
\begin{pmatrix} y_0 \\ y_{-1} \\ y_{-2} \\ \vdots \\ y_{-p+1} \end{pmatrix} \tag{6}
$$

The matrix in (6) is $n \times p$. Since $\boldsymbol{\epsilon}$ has mean vector $\mathbf{0}$, the n-element column vector of 0's, and variance-covariance matrix $\gamma^2 I$, where I is the $n \times n$ identity matrix, then $\mathbf{e}$ has mean vector $\mathbf{c}$ and variance-covariance matrix $\gamma^2 I$.

On rewriting (4) as $(I - \Phi)\,\mathbf{y} = \mathbf{e}$ or $\mathbf{y} = (I - \Phi)^{-1}\mathbf{e}$, and since $\boldsymbol{\epsilon}$ is multivariate normal, then the distribution of $\mathbf{y}$ given $y_0, y_{-1}, \ldots, y_{-p+1}$ is obtained as a multivariate normal with mean vector $(I - \Phi)^{-1}\,\mathbf{c}$ and variance-covariance matrix

$$
\Gamma = (I - \Phi)^{-1}(I - \Phi^T)^{-1}\gamma^2 \tag{7}
$$

Since $\boldsymbol{\delta} = \mathbf{y} + \theta \mathbf{1}_n$ then $\boldsymbol{\delta}$ given $\delta_0, \delta_{-1}, \ldots, \delta_{-p+1}$ has the same distribution as $\mathbf{y}$ but with mean vector

$$
\boldsymbol{\theta} = (I - \Phi)^{-1}\,\mathbf{c} + \theta\,\mathbf{1}_n . \tag{8}
$$

The inverse of $I - \Phi$ and hence $I - \Phi^T$ may be found for any order of the process $p < n$. From (5) it may be noted that $I - \Phi$ is a lower triangular matrix with diagonal elements of value 1. The $(I - \Phi)^{-1}$ is also lower triangular with diagonal elements of value 1. An algorithm for the inversion of a lower triangular matrix may be found in Ralston [5, p. 446]. Denote the (s,t)-th element of $(I-\Phi)^{-1}$ by r_{st}. Then $r_{st} = 0$ for $s < t$ and $r_{tt} = 1$. On applying the algorithm it is found that the elements in row t are the solutions to the set of equations

$$r_{st} - \sum_{j=1}^{m} \phi_j \, r_{s,t+j} = 0, \quad t = s-1, \, s-2, \ldots, \, 1, \tag{9}$$

where $m = min\ (s-t,\ p)$. Equation (9) is a difference equation of at most order p. The values of r_{st} for $s > t$ can be obtained using difference equation techniques. Once $(I-\Phi)^{-1}$ is obtained then $(I-\Phi^T)^{-1}$ is obtained as the transpose of $(I-\Phi)^{-1}$. Then $\underset{\sim}{\theta}$ and Γ may be evaluated.

APPLICATION TO AR(1) AND AR(2) PROCESSES

Autoregressive Processes of Order One - AR(1)

The model is obtained from (3) with $\phi_1 = \phi$ and $\phi_2 = \ldots = \phi_p = 0$, that is

$$\delta_t = \theta + \phi\{\delta_{t-1} - \theta\} + \epsilon_t,$$

where the ϵ's have the same distribution as (3). On applying (9) to find $(I-\Phi)^{-1}$, the recursion relationship

$$r_{s,t} = \phi \, r_{s,t+1}, \quad t = s-1, \, s-2, \ldots, \, 1$$

where $r_{ss} = 1$, is obtained. Hence

$$(I-\Phi)^{-1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ \phi & 1 & 0 & \dots & 0 & 0 \\ \phi^2 & \phi & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \phi^{n-1} & \phi^{n-2} & \phi^{n-3} & \dots & \phi & 1 \end{bmatrix} \qquad (10)$$

For the AR(1) process, $\underset{\sim}{\phi} = (\phi y_0, 0, \dots, 0)^T$ so that

$$\underset{\sim}{\theta} = (\phi y_0, \phi^2 y_0, \phi^3 y_0, \dots, \phi^n y_0)^T + \theta \underset{\sim}{1}_n. \qquad (11)$$

Provided that $-1 < \phi < 1$ the effect of the initial deviation $y_0 = \delta_0 - \theta$ is reduced exponentially over time. On applying (11), the expressions $\underset{\sim}{1}_t^T \theta$ and $(\underset{\sim}{1}_t + \underset{\sim}{1}_s)^T \underset{\sim}{\theta}$ in (1) and (2) reduce to

$$\underset{\sim}{1}_t^T \underset{\sim}{\theta} = t\theta + \phi(1-\phi^t)(\delta_0 - \theta)/(1-\phi) \qquad (12)$$

and

$$(\underset{\sim}{1}_t + \underset{\sim}{1}_s)^T \underset{\sim}{\theta} = (t+s)\theta + \phi(2-\phi^t-\phi^s)(\delta_0-\theta)/(1-\phi). \qquad (13)$$

In the unconditional AR(1) process, Panjer and Bellhouse [2] obtained $t\theta$ and $(t+s)\theta$ for (12) and (13) respectively. The difference in both cases may be viewed as a correction term to take into account the deviation of the current force of interest from the average force of interest.

The variance-covariance matrix Γ can be obtained from (7) and (10). After some algebra the (s,t)-th element of Γ reduces to

$$Cov(\delta_t, \delta_s) = \gamma^2 \phi^{|t-s|} (1-\phi^{2m})/(1-\phi^2), \qquad (14)$$

where $m = min\ (t,s)$. The asymptotic result, obtained by fixing $t-s$ and letting m become large, is the covariance in the unconditional model obtained from equation (19) of Panjer and Bellhouse [2].

Consider the functions

$$G(x) = \frac{x}{2} + \sum_{u=1}^{x-1} (x-u)\phi^u$$

$$= \frac{x}{2}\,\frac{1+\phi}{1-\phi} - \frac{1-\phi^x}{(1-\phi)^2} \tag{15}$$

and

$$H(x) = \frac{\phi^2}{1-\phi^2}\left[\frac{1-\phi^{2x}}{2} + \sum_{u=1}^{x-1} \phi^u \left(1-\phi^{2(x-u)}\right)\right]$$

$$= \frac{\phi^2}{1-\phi^2}\left[\frac{1-\phi^{2x}}{2} + \frac{\phi}{1-\phi}\left(1-\phi^x\right)\left(1-\phi^{x-1}\right)\right] \tag{16}$$

Then from (14), (15) and (16) and by using arguments similar to those given in Appendix I of Panjer and Bellhouse [2] for the discrete unconditional process the expressions $\underset{\sim}{l}_t^T\,\Gamma\,\underset{\sim}{l}_t$ and $(\underset{\sim}{l}_t + \underset{\sim}{l}_s)^T\,\Gamma\,(\underset{\sim}{l}_t + \underset{\sim}{l}_s)$ in (1) and (2) reduce to

$$\underset{\sim}{l}_t^T\,\Gamma\,\underset{\sim}{l}_t = 2\sigma^2\,\{G(t) - H(t)\} \tag{17}$$

and

$$(\underset{\sim}{l}_t + \underset{\sim}{l}_s)^T\Gamma(\underset{\sim}{l}_t + \underset{\sim}{l}_s) = 2\sigma^2[2\{G(t) - H(t)\} + 2\{G(s) - H(s)\}$$

$$- \{G(|t-s|) - \phi^{2m}H(|t-s|)\}], \tag{18}$$

where $\sigma^2 = \gamma^2/(1-\phi^2)$ and $m = min\ (s,t)$. Panjer and Bellhouse [2] obtained $2\sigma^2 G(t)$ and $2\sigma^2\{2G(t) + 2G(s) - G(|t-s|)\}$ for (17) and (18) respectively. The difference in each case shows the reduction in variance obtained by using the known current value δ_0.

Autoregressive Processes of Order Two - AR(2)

The model is a special case of (3) with $p = 2$ or

$$\delta_t = \theta + \phi_1\{\delta_{t-1} - \theta\} + \theta_2\{\delta_{t-2} - \theta\} + \epsilon_t,$$

where, again, the ϵ's have the same distribution as (3). The recursion relationship (9) to find $(I - \Phi)^{-1}$ becomes

$$r_{st} - \phi_1 r_{s,t+1} - \phi_1 r_{s,t+2} = 0, \quad t = s-2,\ s-3, \ldots,\ 1 \tag{19}$$

with initial conditions $r_{ss} = 1$ and $r_{s,s-1} = \phi_1$. Again, the elements $r_{st} = 0$ for $t < s$. The solution to (19) is

$$r_{st} = \alpha_1\psi_1^{s-t} + \alpha_2\psi_2^{s-t} \quad t = s, s-1, \ldots,\ 1 \tag{20}$$

where ψ_1^{-1} and ψ_2^{-1} are the roots of the characteristic equation

$$\phi(r) = 1 - \phi_1 r - \phi_2 r^2 .$$

Upon noting that

$$\psi_1 + \psi_2 = \phi_1 \tag{21}$$

and since $r_{ss} = 1$ and $r_{s,s-1} = \phi_1$ the coefficients reduce to

$$\alpha_1 = \psi_1/(\psi_1 - \psi_2) \quad \text{and} \quad \alpha_2 = -\psi_2/(\psi_1 - \psi_2). \tag{22}$$

For the AR(2) process $\underset{\sim}{\phi} = (\phi_1 y_0 + \phi_2 y_{-1},\ \phi_2 y_0, 0, \ldots,\ 0)^T$. From (20) and (22) and noting that

$$\psi_1\psi_2 = -\phi_1 \tag{23}$$

the i-th $(i = 1,\ldots, n)$ entry of $\underset{\sim}{\theta}$ reduces to

$$\theta + \{\psi_1^{i+1} - \psi_2^{i+1}\}y_0 / (\psi_1 - \psi_2) - \psi_1\psi_2\{\psi_1^i - \psi_2^i\}y_{-1} / (\psi_1 - \psi_2). \tag{24}$$

On applying (24), the expressions $\underset{\sim}{1}_t^T \underset{\sim}{\theta}$ and $(\underset{\sim}{1}_t + \underset{\sim}{1}_s)^T \underset{\sim}{\theta}$ in (1) and (2) reduce to

$$\underset{\sim}{1}_t^T \underset{\sim}{\theta} = t\,\theta + \{\psi_1^2(1-\psi_2)(1-\psi_1^t)(y_0 + \psi_2 y_{-1})$$

$$- \psi_2^2(1-\psi_1)(1-\psi_2^t)(y_0+\psi_1 y_{-1})\} / \{(1-\psi_1)(1-\psi_2)(\psi_1 - \psi_2)\}.$$

and

$$(\underset{\sim}{1}_t + \underset{\sim}{1}_s)^T \underset{\sim}{\theta} = (t+s)\theta + \{\psi_1^2(1-\psi_2)(2-\psi_1^t - \psi_2^s)(y_0 + \psi_2 y_{-1})$$

$$- \psi_2^2(1-\psi_1)(2-\psi_2^t-\psi_2^s)(y_0+\psi_1 y_{-1})\}/\{(1-\psi_1)(1-\psi_2)(\psi_1-\psi_2)\}.$$

The variance-covariance matrix Γ may be obtained from (7), (20) and (22). After some algebra the (s,t)-th element of Γ reduces to

$$\sigma^2[\lambda\psi_1^{|t-s|}\{\psi_1(1-\psi_1^{2m}) - \psi_2(1-\psi_1^m\psi_2^m)\} + (1-\lambda)\psi_2^{|t-s|}\{\psi_1(1-\psi_1^m\psi_2^m)$$

$$- \psi_2(1-\psi_2^{2m})\}] / (\psi_1 - \psi_2), \tag{25}$$

where $m = min(s,t)$,

$$\lambda = \psi_1(1-\psi_1^2) / \{(\psi_1 - \psi_2)(1+\psi_1\psi_2)\},$$

as in Panjer and Bellhouse [2], and

$$\sigma^2 = \gamma^2(1+\psi_1\psi_2)/\{(1-\psi_1\psi_2)(1-\psi_1^2)(1-\psi_2^2)\}. \tag{26}$$

On using (21) and (23) it may be noted that (26) is the variance of δ_t in the unconditional model studied by Panjer and Bellhouse [2]. The asymptotic result, obtained by fixing $t-s$ in (25) and letting m become large, agrees with the covariance of δ_t and δ_s in the unconditional model obtained from equation (21) in Panjer and Bellhouse [2].

Consider the functions

$$G_i(x) = \frac{x}{2} + \sum_{u=1}^{x-1} (x-u)\psi_i^u, \quad i = 1,2, \tag{27}$$

and

$$H_{ij}(x) = \frac{\psi_i\psi_j}{1-\psi_i\psi_j} \left\{\frac{1-(\psi_i\psi_j)^x}{2} + \sum_{u=1}^{x-1} \psi_i^u \, [1-(\psi_i\psi_j)^{x-u}]\right\}, \tag{28}$$

$$i = 1,2;\ j = 1,2.$$

Then from (26), (27) and (28), and using the same arguments as for the AR(1) process, the expressions $\underset{\sim}{1}_t^T \, \Gamma \, \underset{\sim}{1}_t$ and $(\underset{\sim}{1}_t + \underset{\sim}{1}_s)^T \, \Gamma \, (\underset{\sim}{1}_t + \underset{\sim}{1}_s)$ in (1) and (2) reduce to

$$\underset{\sim}{1}_t^T \, \Gamma \, \underset{\sim}{1}_t = 2\sigma^2\{\lambda G_1(t) + (1-\lambda)G_2(t) - [\lambda\psi_1 H_{11}(t) - \lambda\psi_2 H_{12}(t)$$

$$+ (1-\lambda)\psi_1 H_{21}(t) - (1-\lambda)\psi_2 H_{22}(t)] / (\psi_1 - \psi_2)\}$$

and

$$(1_t + 1_s)^T \Gamma (1_t + 1_s) = 2\sigma^2\{2G(t) + 2G(s) - G(|t-s|) - [\lambda\psi_1 F_{11} - \lambda\psi_2 F_{12} + (1-\lambda)\psi_1 F_{21}$$

$$- (1-\lambda)\psi F_{22}] / (\psi_1 - \psi_2)\} \qquad (29)$$

where in (29)

$$G(x) = \lambda G_1(x) + (1-\lambda)G_2(x).$$

and

$$F_{ij} = 2H_{ij}(s) + 2H_{ij}(t) - (\psi_i\psi_j)^m H_{ij}(|t-s|),$$

NUMERICAL ILLUSTRATIONS

Tables 1, 2 and 3 give the mean present value of an annuity certain, a whole life annuity and a whole life insurance respectively. Each table is based on the conditional autoregressive model of order 1 with a value of θ of 6%, values of σ of 1% and 10%, values of $\emptyset$ of 0, $.25$, $.50$ and $.75$ and values of δ_0, the starting values of 4%, 6% and 8%.

From these tables it is noted that the expected present values increase as the variability as measured by the standard deviation σ increases, increases as the degree of dependency, $\emptyset$, increases and decreases as the current rate, δ_0, increases.

Table 4 gives the net annual premium for an ordinary life insurance policy under the above assumptions. The values are obtained by dividing the values in Table 3 by those in Table 2 increased by an amount of 1.

It can be seen that the net annual premium increases as σ increases, as $\emptyset$ increases and as δ_0 decreases. Changes in the starting value δ_0, do not affect the net annual premium as much as it did the net single premiums in

Tables 1, 2 and 3. The effect of δ_0 may be more pronounced for other types of policies. The value of $\emptyset$ seems to be much more important than the value of σ or δ_0. This demonstrates that the correlation of yield rates in successive years is an important variable in the determination of interest, insurance and annuity values.

The reader is cautioned that those observations apply only to the ordinary life plan. The effects of the various parameters may be more or less pronounced for other plans.

TABLE 1: Mean present value of an n-year annuity certain

θ	σ	ϕ	δ_0	n = 10	20	30	40	50
.06	.01	0	any	7.298	11.306	13.506	14.714	15.378
		.25	.04	7.346	11.382	13.599	14.816	15.485
			.06	7.299	11.308	13.511	14.720	15.384
			.08	7.253	11.235	13.423	14.624	15.284
		.50	.04	7.430	11.523	13.773	15.010	15.690
			.06	7.300	11.313	13.518	14.730	15.397
			.08	7.173	11.106	13.268	14.456	15.109
		.75	.04	7.607	11.870	14.220	15.514	16.227
			.06	7.302	11.321	13.534	14.753	15.424
			.08	7.010	10.799	12.884	14.032	14.664
	.10	0	any	7.482	11.799	14.290	15.727	16.556
		.25	.04	7.622	12.169	14.881	16.499	17.464
			.06	7.573	12.090	14.784	16.391	17.350
			.08	7.525	12.012	14.688	16.284	17.236
		.50	.04	7.841	12.836	16.021	18.052	19.347
			.06	7.704	12.600	15.722	17.712	18.982
			.08	7.569	12.368	15.428	17.379	18.624
		.75	.04	8.178	14.320	19.092	22.809	25.703
			.06	7.844	13.635	18.129	21.629	24.354
			.08	7.526	12.984	17.217	20.513	23.080

TABLE 2: Mean present value of a life annuity using 1958 CSO Mortality Tables

θ	σ	ϕ	δ_0	Age 0	10	20	30	40
.06	.01	0	any	15.437	15.290	14.858	14.162	12.983
		.25	.04	15.545	15.398	14.961	14.260	13.072
			.06	15.544	15.297	14.864	14.168	12.987
			.08	15.343	15.198	14.767	14.075	12.903
		.50	.04	15.752	15.601	15.158	14.446	13.240
			.06	15.457	15.310	14.876	14.177	12.995
			.08	15.169	15.024	14.598	13.914	12.755
		.75	.04	16.297	16.138	15.673	14.927	13.667
			.06	15.489	15.340	14.902	14.199	13.012
			.08	14.723	14.584	14.171	13.509	12.390
.06	.10	0	any	16.731	16.519	15.980	15.140	13.773
		.25	.04	17.744	17.475	16.847	15.889	14.373
			.06	17.629	17.361	16.737	15.785	14.279
			.08	17.514	17.248	16.628	15.682	14.186
		.50	.04	19.929	19.504	18.651	17.411	15.555
			.06	19.552	19.136	18.299	17.084	15.265
			.08	19.182	18.775	17.955	16.763	14.980
		.75	.04	28.447	26.988	24.883	22.276	19.003
			.06	26.937	25.564	23.582	21.126	18.042
			.08	25.511	24.218	22.351	20.038	17.133

θ	σ	ϕ	δ_0	50	60	70	80	90
.06	.01	0	any	11.235	8.962	6.431	4.073	2.174
		.25	.04	11.311	9.022	6.473	4.099	2.187
			.06	11.238	8.964	6.432	4.074	2.174
			.08	11.165	8.906	6.391	4.048	2.161
		.50	.04	11.453	9.131	6.547	4.142	2.206
			.06	11.243	8.967	6.434	4.074	2.174
			.08	11.038	8.806	6.322	4.007	2.142
		.75	.04	11.804	9.389	6.709	4.225	2.238
			.06	11.254	8.973	6.436	4.075	2.174
			.08	10.732	8.577	6.175	3.930	2.112
.06	.10	0	any	11.810	9.327	6.627	4.160	2.202
		.25	.04	12.244	9.602	6.776	4.227	2.226
			.06	12.165	9.541	6.733	4.200	2.212
			.08	12.086	9.479	6.690	4.174	2.199
		.50	.04	13.067	10.098	7.026	4.329	2.256
			.06	12.826	9.915	6.903	4.257	2.222
			.08	12.589	9.736	6.782	4.187	2.189
		.75	.04	15.215	11.227	7.506	4.486	2.290
			.06	14.472	10.708	7.189	4.323	2.224
			.08	13.767	10.216	6.888	4.166	2.160

TABLE 3: Mean present value (net single premium) of a whole life insurance using 1958 CSO Mortality Tables

						Age		
θ	σ	ϕ	δ_0	0	10	20	30	40
.06	.01	0	any	43.58	52.09	77.27	117.75	186.36
		.25	.04	43.90	52.49	77.87	118.64	187.73
			.06	43.62	52.15	77.35	117.86	186.49
			.08	43.34	51.80	76.84	117.08	185.26
		.50	.04	44.50	53.29	79.04	120.41	190.45
			.06	43.70	52.26	77.51	118.07	186.74
			.08	42.92	51.25	76.00	115.77	183.11
		.75	.04	46.17	55.62	82.46	125.58	198.29
			.06	43.92	52.56	77.92	118.60	187.37
			.08	41.80	49.68	73.64	112.01	177.06
	.10	0	any	51.15	62.48	91.34	136.29	209.45
		.25	.04	57.70	71.26	102.91	151.13	227.48
			.06	57.32	70.79	102.23	150.13	225.97
			.08	56.96	70.32	101.55	149.14	224.48
		.50	.04	74.68	93.34	130.89	185.47	267.13
			.06	73.28	91.51	128.33	181.83	261.91
			.08	71.91	89.72	125.83	178.27	256.79
		.75	.04	184.66	220.24	272.37	336.90	418.37
			.06	174.36	207.61	256.78	317.63	394.67
			.08	164.64	195.71	242.10	299.48	372.34

θ	σ	ϕ	δ_0	50	60	70	80	90
.06	.01	0	any	288.08	420.33	567.60	704.80	815.33
		.25	.04	290.15	423.25	571.42	709.38	820.39
			.06	288.24	420.48	567.73	704.89	815.38
			.08	286.33	417.73	564.06	700.43	810.41
		.50	.04	294.18	428.86	578.49	717.35	828.35
			.06	288.52	420.75	567.94	705.03	815.44
			.08	282.96	412.80	557.59	692.92	802.74
		.75	.04	305.48	443.74	595.64	734.26	842.35
			.06	289.17	421.31	568.32	705.20	815.48
			.08	273.75	400.08	542.35	677.44	789.59
	.10	0	any	314.49	447.34	591.82	723.87	828.62
		.25	.04	334.67	467.68	609.99	738.34	839.06
			.06	332.46	464.61	606.04	733.66	833.94
			.08	330.27	461.57	602.12	729.00	828.84
		.50	.04	376.66	507.39	642.77	761.82	853.63
			.06	369.36	497.73	630.96	748.63	840.25
			.08	362.21	488.26	619.37	735.68	827.09
		.75	.04	513.72	616.29	716.17	802.92	872.34
			.06	485.37	584.01	682.14	770.26	844.09
			.08	458.61	553.48	649.84	739.09	816.91

TABLE 4: Net annual premium of a $100 ordinary life insurance policy using 1958 CSO Mortality Tables

θ	σ	ϕ	δ_0	Age 0	10	20	30	40
.06	.01	0	any	2.65	3.20	4.87	7.77	13.33
		.25	.04	2.65	3.20	4.88	7.77	13.07
			.06	2.64	3.20	4.88	7.77	13.33
			.08	2.65	3.20	4.87	7.77	12.90
		.50	.04	2.66	3.21	4.89	7.80	13.37
			.06	2.66	3.20	4.88	7.78	13.34
			.08	2.65	3.20	4.87	7.76	13.31
		.75	.04	2.67	3.25	4.95	7.88	13.52
			.06	2.66	3.22	4.90	7.80	13.37
			.08	2.66	3.19	4.85	7.72	13.22
	.10	0	any	2.88	3.57	5.38	8.44	14.18
		.25	.04	3.08	3.86	5.77	8.95	14.80
			.06	3.08	3.86	5.76	8.94	14.79
			.08	3.08	3.85	5.76	8.94	14.78
		.50	.04	3.57	4.55	6.66	10.07	16.14
			.06	3.57	4.54	6.65	10.05	16.10
			.08	3.56	4.54	6.64	10.04	16.07
		.75	.04	6.27	7.87	10.52	14.47	20.92
			.06	6.24	7.82	10.45	14.36	20.73
			.08	6.21	7.76	10.37	14.24	20.53

θ	σ	ϕ	δ_0	50	60	70	80	90
.06	.01	0	any	23.55	42.19	76.38	138.93	256.88
		.25	.04	23.57	42.23	76.46	139.12	256.42
			.06	23.55	42.20	76.39	138.92	256.89
			.08	23.54	42.17	76.32	138.17	256.38
		.50	.04	23.62	42.33	76.65	139.51	258.37
			.06	23.57	42.21	76.40	138.95	256.91
			.08	23.51	42.10	76.15	138.39	255.49
		.75	.04	23.86	42.71	77.27	140.53	260.15
			.06	23.60	42.25	76.43	138.96	256.93
			.08	23.33	41.78	75.59	137.41	253.72
	.10	0	any	24.55	43.32	77.60	140.28	258.78
		.25	.04	25.27	44.11	78.45	141.26	260.09
			.06	25.25	44.08	78.37	141.09	259.54
			.08	25.24	44.05	78.30	140.90	259.09
		.50	.04	26.78	45.72	80.09	142.96	262.17
			.06	26.71	45.60	79.84	142.41	260.79
			.08	26.65	45.48	79.59	141.83	259.36
		.75	.04	31.68	50.40	84.20	146.28	265.15
			.06	31.37	49.88	83.30	144.70	261.81
			.08	31.06	49.35	82.38	143.07	258.52

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